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► **To cite this version:**

Na Wang, Sébastien Boyaval, Yuxi Hu. Global solutions and uniform convergence stability for compressible Navier-Stokes equations with oldroyd-type constitutive law. 2024. hal-04619590

HAL Id: hal-04619590

<https://enpc.hal.science/hal-04619590v1>

Preprint submitted on 20 Jun 2024

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GLOBAL SOLUTIONS AND UNIFORM CONVERGENCE STABILITY FOR COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH OLDROYD-TYPE CONSTITUTIVE LAW

NA WANG, SEBASTIEN BOYAVAL AND YUXI HU

ABSTRACT. We consider one dimensional isentropic compressible Navier-Stokes equations with Oldroyd-type constitutive law. By establishing uniform a priori estimates (with respect to relaxation time), we show global existence of smooth solutions with small initial data. Moreover, we get global-in-time convergence of the system towards the classical isentropic compressible Navier-Stokes equations.

Keywords: Oldroyd-type constitutive law; global solutions; relaxation limit

2020 MSC: 35 L 03, 35 Q 35, 76 N 10

1. INTRODUCTION

In this paper, we investigate the Cauchy problem for a model of hyperbolic approximation of compressible Navier-Stokes equations. The equations are described as follows

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x + p(\rho)_x = S_x, \\ \tau(S_t + uS_x - 2aSu_x) + S = \mu u_x, \end{cases} \quad (1.1)$$

where $(t, x) \in (0, +\infty) \times \mathbb{R}$. Here, ρ , u , S denote fluid density, velocity and stress, respectively. $\mu > 0$ is the viscosity constant and $a \in [-1, 1]$ a kinematic constant. The pressure p is assumed to satisfy the usual γ -law, $p(\rho) = A\rho^\gamma$ where $\gamma > 1$ denotes the adiabatic index and A is any positive constant. Without loss of generality, we assume $A = 1$ in the sequel.

The constitutive relation (1.1)₃ can be viewed as a one-dimensional version of Oldroyd-type model describing viscoelastic fluids (see [2, 9, 14, 18, 25] and the references therein) as

$$\tau(S_t + u \cdot \nabla S + g_a(S, \nabla u)) + S = \mu_1(\nabla u + \nabla u^T - \frac{2}{n} \operatorname{div} u I_n) + \mu_2 \operatorname{div} u I_n \quad (1.2)$$

where $g_a(S, \nabla u) := SW(u) - W(u)S - a(D(u)S + SD(u))$, $-1 \leq a \leq 1$, and $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, $W(u) = \frac{1}{2}(\nabla u - (\nabla u)^T)$. n is the space dimension. I_n denotes the n -th order identity matrix, μ_1 and μ_2 are shear and bulk viscosities, respectively. For $a = 1, -1, 0$, the equation (1.2) is the so-called Upper-Convected, Lower-Convected and Co-rotational Maxwell model, respectively. The positive parameter τ is the relaxation time describing the time lag in the response of the stress tensor to velocity gradient. The constitutive equation (1.2) has wide application in the field of complex fluids, such as macromolecular or polymeric fluids, where the relaxation parameter τ is

Na Wang, School of Applied Science, Beijing Information Science and Technology University, Beijing, 100192, P.R. China, wn_math@126.com

Sebastien Boyaval, Laboratoire d'Hydraulique Saint-Venant (LHSV), Ecole des Ponts ParisTech, EDF R&D, Chatou & Matherials, Inria, Paris – France, sebastien.boyaval@enpc.fr

Yuxi Hu, Department of Mathematics, China University of Mining and Technology, Beijing, 100083, P.R. China, yxhu86@163.com

comparable to macro-time scale. In fact, even in simple fluid, water for example, the "time lag" does exist, but is very short from 1 ps to 1 ns, see [13, 19]. However, Pelton et. al [15] showed that such "time lag" can not be neglected, even for simple fluid, in the experiments of high-frequency (20GHZ) vibration of nanoscale mechanical devices immersed in water-glycerol mixtures. It showed that, see also [3], the equation (1.1)₃ provides a general formalism with which to characterize the fluid-structure interaction of nano-scale mechanical devices vibrating in simple fluids.

Note that the compressible Navier-Stokes equations coupled with linearized constitutive relation (1.2) have been widely studied both in 1-d and multi-d cases. Yong [24] first studied the 3-d isentropic Navier-Stokes equations with revised Maxwell law (in its linear form) and obtained a local well-posedness theory, plus a local relaxation limit. This results were extended by Hu and Racke [4] to a non-isentropic case, and then by Peng [16] to yet more general cases. The blow-up phenomenon was studied by Hu and Wang [8, 23] and Bärnin [1]. As for the nonlinear versions of (1.2), there are only few special results due to the complex mathematical structure. In the particular one-dimensional case, by assuming $a = 0$, Hu and Racke [5] obtained the global-in-time existence of smooth solutions for a non-isentropic system, and they showed the local relaxation limit. See also [17] for a global weak relaxation limit. But to our knowledge, even in the particular one-dimensional case, there are no rigorous result showing convergence to the compressible Navier-Stokes equations when $a \neq 0$. The aim of this paper is to fill this gap.

For convenience, we rewrite the system (1.1) in Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = S_x, \\ \tau(S_t - \frac{2aS}{v}u_x) + S = \mu \frac{u_x}{v}, \end{cases} \quad (1.3)$$

where $v = \frac{1}{\rho}$ denotes the specific volume per unit mass.

We are interested in the Cauchy problem to system (1.3) for the functions

$$(v, u, S) : [0, +\infty) \times \mathbb{R} \rightarrow (0, \infty) \times \mathbb{R} \times \mathbb{R}$$

with initial conditions

$$(v, u, S)(0, x) = (v_0, u_0, S_0)(x). \quad (1.4)$$

Note that when $\tau = 0$, the system (1.3) is reduced to classical isentropic Navier-Stokes equations

$$\begin{cases} v_t = u_x, \\ u_t + p(v)_x = \left(\frac{\mu u_x}{v}\right)_x, \end{cases} \quad (1.5)$$

for which the global large solution (away from vacuum) was already known, see [10]. But the methods there can not be applied to the relaxed system due to the essential change of structure, i.e., from hyperbolic-parabolic to pure hyperbolic system. On the other hand, it has been show that, see [8, 1], solutions to the relaxed system may blowup in finite time for some large data. Therefore, a global defined smooth solutions should not be expected for large data.

Let us introduce some notations. We denote $W^{m,p} \equiv W^{m,p}(\mathbb{R})$, $0 \leq m \leq \infty$, $1 \leq p \leq \infty$ the usual Sobolev space with norm $\|\cdot\|_{W^{m,p}}$, H^m and L^p stand for $W^{m,2}$ resp. $W^{0,p}$. We standardly denote $\|(f, g)\|_{W^{m,p}}^2 = \|f\|_{W^{m,p}}^2 + \|g\|_{W^{m,p}}^2$ the (squared) norm of a set of elements in $W^{m,p}$ (like $f \in W^{m,p}$, $g \in W^{m,p}$ here). We recall the continuous embedding $W^{2,2} \subset C^1(\mathbb{R})$.

Our main results are stated as follows.

Theorem 1.1. *Given $a \in [-1, 1]$, $\mu > 0$ and $\tau \in (0, 1)$, there exists a constant $C_0 > 0$ such that for any $\epsilon_0 \in \left(0, \min\{1, \frac{1}{2C}\} \min\{\delta, \frac{1}{(2C)^2}\}\right)$ where δ and C are the constants defined in Prop. 2.2 below, the initial value problem (1.3)-(1.4) with initial conditions satisfying*

$$E_0 := \|(v_0 - 1, u_0, \sqrt{\tau}S_0)\|_{H^2}^2 < \epsilon_0 \quad (1.6)$$

has a unique global solution $(v, u, S) \in C^1([0, \infty) \times \mathbb{R})$ satisfying

$$(v - 1, u, S) \in C([0, \infty), H^2(\mathbb{R})) \cap C^1([0, \infty), H^1(\mathbb{R})) \quad (1.7)$$

$$\sup_{0 \leq t < \infty} \|(v - 1, u, \sqrt{\tau}S)(t, \cdot)\|_{H^2}^2 + \int_0^\infty (\|(v_x, u_x)(t, \cdot)\|_{H^1}^2 + \|S(t, \cdot)\|_{H^2}^2) dt \leq C_0 E_0 \quad (1.8)$$

where $C_0 > 0$ is independent of τ and of the initial data.

Based on uniform estimates of solutions, we have the following convergence theorem.

Theorem 1.2. (Global weak convergence). *Given $a \in [-1, 1]$ and $\mu > 0$, let (v^τ, u^τ, S^τ) be global solutions obtained in Theorem 1.1 for relaxation parameters $\tau \in (0, 1)$ and initial states $(v_0^\tau, u_0^\tau, S_0^\tau)$. Then there exists $(v^0, u^0) \in L^\infty(\mathbb{R}^+; H^2(\mathbb{R})) \cap C^0(\mathbb{R}^+; H^1(\mathbb{R})) \cap C^1(\mathbb{R}^+; L^2(\mathbb{R}))$ such that, as $\tau \rightarrow 0$*

$$(v^\tau, u^\tau) \rightharpoonup (v^0, u^0) \quad \text{weakly} - * \quad \text{in} \quad L^\infty(\mathbb{R}^+; H^2(\mathbb{R})), \quad (1.9)$$

up to subsequences, where (v^0, u^0) is solution to the one-dimensional isentropic compressible Navier-Stokes equations (1.5) with initial value (v_0^0, u_0^0) weak limit of (v_0^τ, u_0^τ) . In (1.5), $\mu \frac{u_0^0}{v_0^0}$ coincides, for almost every (a.e.) $t > 0$, with $S^0 \in L^2(\mathbb{R}^+; H^2(\mathbb{R})) \cap L^\infty(\mathbb{R}^+; H^1(\mathbb{R}))$ the weak limit of S^τ .

The paper is organized as follows. In Section 2 we show the local existence of smooth solutions to system (1.4) and establish uniform a priori estimates for the obtained solutions. In Section 3, we show the global existence of smooth solutions by usual bootstrap methods and we justify the limit $\tau \rightarrow 0$ by compactness arguments.

2. LOCAL EXISTENCE AND UNIFORM A PRIORI ESTIMATES

In this part, we first present a local existence theorem and we then give uniform a priori estimates for the obtained solutions. Note that (1.3) rewrites in the symmetric hyperbolic form

$$A^0(W)W_t + A^1(W)W_x + B(W)W = 0$$

when the vector-valued function $W = [v, u, S]$ is sufficiently regular, using symmetric matrices

$$A^0(W) = \begin{pmatrix} -p'(v) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\tau v}{2\tau a S + \mu} \end{pmatrix}, A^1(W) = \begin{pmatrix} 0 & p'(v) & 0 \\ p'(v) & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, B(W) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{v}{2\tau a S + \mu} \end{pmatrix}$$

where $A^0(W)$ is *positive* symmetric provided $v = \frac{1}{\rho} > 0$, $2\tau a S + \mu > 0$. Then, we have the following local existence theorem, see [11, 20, 22].

Theorem 2.1. *Let $a \in [-1, 1]$ and $\mu > 0$ be fixed. Then, for any $\tau \in (0, 1)$ and any initial data*

$$(v_0 - 1, u_0, \sqrt{\tau}S_0) \in H^2(\mathbb{R}) \quad (2.1)$$

with $\min_x 2\tau a S_0(x) + \mu > 0$ and $\min_x v_0(x) > 0$, the initial value problem (1.3)-(1.4) has a unique solution (v, u, S) on the time interval $t \in [0, T]$ for some $T > 0$, with

$$(v - 1, u, \sqrt{\tau}S) \in C^0([0, T]; H^2(\mathbb{R})) \cap C^1([0, T], H^1(\mathbb{R})) \quad (2.2)$$

and $2\tau a S(t, x) + \mu > 0$, $v(t, x) > 0$ for $(t, x) \in [0, T] \times \mathbb{R}$.

To next prove Theorem 1.1, the key point is an a priori estimate when τ and ϵ_0 are small enough. To that aim, denoting $v(s, \cdot), u(s, \cdot), S(s, \cdot)$ values in $H^2(\mathbb{R})$ at $s \in [0, T]$ we introduce

$$E(t) = \sup_{0 \leq s \leq t} \|(v - 1, u, \sqrt{\tau}S)(s, \cdot)\|_{H^2}^2, \quad \mathcal{D}(t) = \|(v_x, u_x)(t, \cdot)\|_{H^1}^2 + \|S(t, \cdot)\|_{H^2}^2.$$

The a priori estimate result is stated as follows, given $a \in [-1, 1]$ and $\mu > 0$ fixed.

Proposition 2.2. *Let $(v - 1, u, S) \in C^0([0, T], H^2) \cap C^1([0, T], H^1)$ be a local solution given by Theorem 2.1 for some $T > 0$. A constant $\delta > 0$ independent of T and $\tau \in (0, 1)$ exists such that, if*

$$E(T) < \delta \quad (2.3)$$

then the solution (v, u, S) satisfies

$$E(t) + \int_0^t \mathcal{D}(s) ds \leq C(E_0 + E^{\frac{1}{2}}(t) \int_0^t \mathcal{D}(s) ds), \quad \forall t \in (0, T) \quad (2.4)$$

where C is a constant independent of T , initial data and τ .

We show that Proposition 2.2 holds using a series of lemmas where C denotes a universal constant independent of τ , the initial data and the time span.

From now on we only consider small δ e.g. $\delta \leq \min\{\frac{1}{16}, \frac{\mu}{4}\}$. This is a non-void assumption: one can always require E_0 small enough such that $E(T)$ is as small as necessary for some $T > 0$. Indeed, first, it then holds on the one hand

$$\begin{aligned} \|(v - 1, u, \sqrt{\tau}S)(t, \cdot)\|_{L^\infty} &\leq \|(v - 1, u, \sqrt{\tau}S)(t, \cdot)\|_{H^1} \leq E(t)^{\frac{1}{2}} \leq \frac{1}{4} \quad \forall t \in [0, T], \\ \frac{3}{4} &\leq \|v\|_{L^\infty} \leq \frac{5}{4}. \end{aligned}$$

Second, since we require $\tau < 1$, it also holds

$$\frac{\mu}{2} \leq \|2\tau aS + \mu\|_{L^\infty} \leq \frac{3\mu}{2}.$$

Remark 2.1. *Prop. 2.2 is the reason why we require a bound above for τ : without bound above on τ and μ fixed, it is not obvious how to bound $2\tau aS + \mu$ below.*

In addition, recalling that by Taylor's formula it holds for some $\xi \in (1, v)$

$$\frac{1}{\gamma - 1}(v^{1-\gamma} - 1) + v - 1 = \frac{1}{\gamma - 1}(v^{1-\gamma} - 1 - (1 - \gamma)(v - 1)) = \gamma(v - 1)^2 \xi^{-\gamma-1},$$

it also results from the above assumption that there exist $c_1 > c_0 > 0$ such that for any $(v - 1) \in H^1$

$$c_0 \int_{\mathbb{R}} (v - 1)^2 dx \leq \int_{\mathbb{R}} \left(\frac{v^{1-\gamma} - 1}{\gamma - 1} + v - 1 \right) dx \leq c_1 \int_{\mathbb{R}} (v - 1)^2 dx. \quad (2.5)$$

Let us now start by L^2 estimates of solutions.

Lemma 2.3. *Fix $a \in [-1, 1]$, $\mu > 0$ and $\tau \in (0, 1)$. Let $(v - 1, u, S) \in C^0([0, T], H^2) \cap C^1([0, T], H^1)$ be a local solution given by Theorem 2.1 for some $T > 0$, with $\delta > E(T)$ small. There is a constant C independent of T , initial data and τ such that for all times $t \in [0, T]$*

$$\int_{\mathbb{R}} \left(\frac{u^2}{2} + \frac{\tau v}{2\mu} S^2 + \frac{v^{1-\gamma} - 1}{\gamma - 1} + v - 1 \right) (t, \cdot) dx + \int_0^t \int_{\mathbb{R}} \frac{v}{\mu} S^2 dx \leq C(E_0 + E(t)^{\frac{1}{2}} \int_0^t \mathcal{D}(s) ds). \quad (2.6)$$

Proof. Multiplying equation (1.3)₂ by u , and (1.3)₃ by $\frac{v}{\mu} S$, we get after summing the two equations, and integration with respect to x over \mathbb{R} , on noting $\int_{\mathbb{R}} (uS_x + Su_x) dx = 0$

$$\frac{d}{dt} \int_{\mathbb{R}} \left(\frac{1}{2} u^2 + \frac{\tau v}{2\mu} S^2 \right) dx + \int_{\mathbb{R}} \frac{v}{\mu} S^2 dx + \int_{\mathbb{R}} p_x u dx = \frac{(4a + 1)\tau}{2\mu} \int_{\mathbb{R}} u_x S^2 dx$$

which holds in the distributional sense $\mathcal{D}'(0, T)$. Next, using equation (1.3)₁, one gets

$$\begin{aligned} \int_{\mathbb{R}} (p(v))_x u dx &= - \int_{\mathbb{R}} p(v) u_x dx = - \int_{\mathbb{R}} (p(v) + 1) u_x dx = - \int_{\mathbb{R}} (p(v) + 1) v_t dx \\ &= \frac{d}{dt} \int_{\mathbb{R}} \left(\frac{v^{1-\gamma} - 1}{\gamma - 1} + v - 1 \right) dx \end{aligned}$$

therefore

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \left\{ \frac{1}{2} u^2 + \frac{\tau v}{2\mu} S^2 + \frac{v^{1-\gamma} - 1}{\gamma - 1} + v - 1 \right\} dx + \int_{\mathbb{R}} \frac{v}{\mu} S^2 dx &\leq C \|u_x\|_{L^\infty} \|\sqrt{\tau} S\|_{L^2}^2 \\ &\leq CE(t)^{\frac{1}{2}} \mathcal{D}(t) \end{aligned}$$

which can be integrated over $[0^+, t]$ and bounded above with (2.5) to yield the result (2.6). \square

Next, we give the higher-order energy estimates.

Lemma 2.4. *Fix $a \in [-1, 1]$, $\mu > 0$ and $\tau \in (0, 1)$. Let $(v-1, u, S) \in C^0([0, T], H^2) \cap C^1([0, T], H^1)$ be a local solution given by Theorem 2.1 for some $T > 0$, with $\delta > E(T)$ small.*

There is a constant C independent of T , initial data and τ such that for all times $t \in [0, T]$

$$\begin{aligned} &\sum_{\alpha=1}^2 \int_{\mathbb{R}} \left(\frac{1}{2} (\partial_x^\alpha u)^2 + \frac{v}{2\mu} \tau (\partial_x^\alpha S)^2 - p'(v) (\partial_x^\alpha v)^2 \right) (t, \cdot) dx + \sum_{\alpha=1}^2 \int_0^t \int_{\mathbb{R}} \frac{v}{\mu} (\partial_x^\alpha S)^2 dx dt \\ &\leq C(E_0 + E^{\frac{1}{2}}(t)) \int_0^t \mathcal{D}(s) ds. \end{aligned} \quad (2.7)$$

Proof. Derivating equations (1.3) once or twice (take ∂_x^α , $\alpha \in \{1, 2\}$) we get

$$\begin{cases} \partial_t \partial_x^\alpha v = \partial_x^{\alpha+1} u, \\ \partial_t \partial_x^\alpha u + p'(v) \partial_x^{\alpha+1} v = \partial_x^{\alpha+1} S + f, \\ \tau \partial_t \partial_x^\alpha S + \partial_x^\alpha S = \left(\frac{2a\tau S + \mu}{v} \right) \partial_x^{\alpha+1} u + g, \end{cases} \quad (2.8)$$

where we have denoted

$$f := \partial_x^\alpha (p'(v) v_x) - p'(v) \partial_x^{\alpha+1} v \equiv \partial_x^\alpha (p'(v)) v_x \quad g := \partial_x^\alpha \left(\frac{2a\tau S + \mu}{v} \right) u_x.$$

Multiplying the above equations by $-p'(v) \partial_x^\alpha v$, $\partial_x^\alpha u$ and $\frac{v}{2a\tau S + \mu} \partial_x^\alpha S$, respectively, next summing them and integrating over $x \in \mathbb{R}$, one gets in the distributional sense on $t \in (0, T)$

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}} \left(\frac{1}{2} (-p'(v)) (\partial_x^\alpha v)^2 + \frac{1}{2} (\partial_x^\alpha u)^2 + \frac{v}{2(2a\tau S + \mu)} \tau (\partial_x^\alpha S)^2 \right) dx + \int_{\mathbb{R}} \frac{v}{2a\tau S + \mu} (\partial_x^\alpha S)^2 dx \\ &= \int_{\mathbb{R}} \left(-\frac{1}{2} p''(v) v_t (\partial_x^\alpha v)^2 + \tau \left(\frac{v}{2(2a\tau S + \mu)} \right)_t (\partial_x^\alpha S)^2 \right) dx + \int_{\mathbb{R}} p'(v) (\partial_x^{\alpha+1} u \cdot \partial_x^\alpha v + \partial_x^{\alpha+1} v \cdot \partial_x^\alpha u) dx \\ &+ \int_{\mathbb{R}} (\partial_x^{\alpha+1} S \cdot \partial_x^\alpha u + \partial_x^{\alpha+1} u \cdot \partial_x^\alpha S) dx + \int_{\mathbb{R}} (f \cdot \partial_x^\alpha u + g \cdot \frac{v}{2a\tau S + \mu} \partial_x^\alpha S) dx \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We estimate I_i for $1 \leq i \leq 4$ separately. First, using equations (1.3)₁ and (1.3)₃, we get

$$\begin{aligned} \left\| \tau \left(\frac{v}{2a\tau S + \mu} \right)_t \right\|_{L^\infty} &= \left\| \frac{\tau}{2a\tau S + \mu} v_t - \frac{2a\tau v}{(2a\tau S + \mu)^2} \tau S_t \right\|_{L^\infty} \\ &= \left\| \frac{\tau}{2a\tau S + \mu} u_x - \frac{2a\tau v}{(2a\tau S + \mu)^2} \left(\frac{2a\tau S + \mu}{v} u_x - S \right) \right\|_{L^\infty} \\ &\leq CE^{\frac{1}{2}}(t) \end{aligned}$$

and $\|p''(v)\|_{L^\infty} \leq CE^{\frac{1}{2}}(t)$ on recalling the bounds on $2a\tau S + \mu$, v and $\|u_x\|_{L^\infty} \leq \|u\|_{H_2}$, hence $I_1 \leq CE^{\frac{1}{2}} \mathcal{D}(t)$. Next, on integrating by part, we get

$$I_2 = - \int_{\mathbb{R}} p''(v) v_x \partial_x^\alpha u \partial_x^\alpha v dx \leq \|p''(v) v_x\|_{L^\infty} (\|\partial_x^\alpha u\|_{L^2}^2 + \|\partial_x^\alpha v\|_{L^2}^2) \leq CE^{\frac{1}{2}}(t) \mathcal{D}(t)$$

with Cauchy-Schwarz and Young inequalities, and $I_3 = 0$. Last, we estimate I_4 . We have

$$\begin{aligned} \|f\|_{L^2} &\leq \|v_x\|_{L^\infty} \|\partial_x^\alpha(p'(v))\|_{L^2} \leq CE^{\frac{1}{2}}(t)\mathcal{D}^{\frac{1}{2}}(t) \\ \|g\|_{L^2} &\leq \|\partial_x u\|_{L^\infty} \|\partial_x^\alpha\left(\frac{2a\tau S + \mu}{v}\right)\|_{L^2} \leq CE^{\frac{1}{2}}(t)\mathcal{D}^{\frac{1}{2}}(t) \end{aligned} \quad (2.9)$$

by Moser-type inequalities. Therefore, using Cauchy-Schwarz and Young inequalities, with the bound on $\frac{v}{2a\tau S + \mu}$, one gets $I_4 \leq CE^{\frac{1}{2}}(t)\mathcal{D}(t)$. Combining the above estimates, summing up α from 1 to 2, and integrating the result, we get (2.7) immediately. \square

Combining the lemmas 2.3 and 2.4, using the fact that $v \in (3/4, 5/4)$ a.e., we get

Lemma 2.5. *Fix $a \in [-1, 1]$, $\mu > 0$ and $\tau \in (0, 1)$. Let $(v-1, u, S) \in C^0([0, T], H^2) \cap C^1([0, T], H^1)$ be a local solution given by Theorem 2.1 for some $T > 0$, with $\delta > E(T)$ small. There exists a constant C independent of T , initial data and τ such that for all times $t \in [0, T]$*

$$\|(v-1, u, \sqrt{\tau}S)(t, \cdot)\|_{H^2} + \int_0^t \|S\|_{H^2}^2 ds \leq C(E_0 + E^{\frac{1}{2}}(t)) \int_0^t \mathcal{D}(s) ds. \quad (2.10)$$

It remains to show the dissipative estimates of v and u . We have the following lemma

Lemma 2.6. *Fix $a \in [-1, 1]$, $\mu > 0$ and $\tau \in (0, 1)$. Let $(v-1, u, S) \in C^0([0, T], H^2) \cap C^1([0, T], H^1)$ be a local solution given by Theorem 2.1 for some $T > 0$, with $\delta > E(T)$ small. There exists a constant C independent of T , initial data and τ such that for all times $t \in [0, T]$*

$$\int_0^t \|(v_x, u_x)(s, \cdot)\|_{H^1}^2 ds \leq C(E_0 + E^{\frac{1}{2}}(t)) \int_0^t \mathcal{D}(s) ds. \quad (2.11)$$

Proof. Take ∂_x^β on equation (1.3)₃ with $\beta = 0, 1$, multiply the result by $\partial_x^{\beta+1}u$ and integrate over $(0, t) \times \mathbb{R} \ni (s, x)$, then one obtains

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \frac{2a\tau S + \mu}{v} (\partial_x^{\beta+1}u)^2 dx ds &= \int_0^t \int_{\mathbb{R}} \partial_x^\beta(\tau S_t) \partial_x^{\beta+1}u dx ds \\ &+ \int_0^t \int_{\mathbb{R}} \left(\partial_x^\beta S + \frac{2a\tau S + \mu}{v} (\partial_x^{\beta+1}u) - \partial_x^\beta \left(\frac{2a\tau S + \mu}{v} u_x \right) \right) \partial_x^{\beta+1}u dx ds =: J_1 + J_2. \end{aligned}$$

Using Young inequality and (2.10) for the first term below after integration by part, then equation (1.3)₂, $\|\tau \partial_x^\beta S\|_{L^\infty} \leq \|\tau S\|_{H^2}$, (2.10) and $\|\partial_x^\beta v_x\|_{L^\infty} \leq \|v-1\|_{H^2}$, it holds

$$\begin{aligned} J_1 &= \int_0^t \frac{d}{dt} \int_{\mathbb{R}} \tau (\partial_x^\beta S) \partial_x^{\beta+1}u dx ds - \int_0^t \int_{\mathbb{R}} \tau (\partial_x^\beta S) \partial_x^{\beta+1}u_t dx ds \\ &= \left[\int_{\mathbb{R}} \tau (\partial_x^\beta S) \partial_x^{\beta+1}u dx \right]_0^t - \int_0^t \int_{\mathbb{R}} \tau (\partial_x^\beta S) \partial_x^{\beta+1}(-p(v)_x + S_x) dx ds \\ &\leq C(E_0 + E^{\frac{1}{2}}(t)) \int_0^t \mathcal{D}(s) ds + \int_0^t \int_{\mathbb{R}} \tau |\partial_x^{2\beta} S| (|p'(v)| |v_{xx}| + |p''(v)| |v_x|^2) dx ds + \int_0^t \int_{\mathbb{R}} \tau |\partial_x^{\beta+1} S|^2 \\ &\leq C \left(E_0 + E^{\frac{1}{2}}(t) \int_0^t \mathcal{D}(s) ds + \int_0^t \int_{\mathbb{R}} |\partial_x^{2\beta} S| (|v_x| + |v_{xx}|) dx ds \right) \end{aligned}$$

which yields using Young inequality $2rq \leq \frac{r^2}{\nu^2} + q^2\nu^2$, Cauchy-Schwarz and (2.10)

$$J_1 \leq \left(C + \frac{1}{\nu^2}\right) (E_0 + E^{\frac{1}{2}}(t)) \int_0^t \mathcal{D}(s) ds + \nu^2 \int_0^t \|v_x(s, \cdot)\|_{H^1}^2 ds.$$

As for J_2 , using Young inequality $2rq \leq \frac{r^2}{\nu^2} + q^2\nu^2$, Cauchy-Schwarz and (2.10), we have

$$\begin{aligned} J_2 &= \int_0^t \int_{\mathbb{R}} S u_x \, dx ds \\ &\leq \frac{C}{\nu^2} (E_0 + E^{\frac{1}{2}}(t)) \int_0^t \mathcal{D}(s) ds + \nu^2 \int_0^t \|u_x(s, \cdot)\|_{L^2}^2 ds \end{aligned}$$

when $\beta = 0$, and when $\beta = 1$ using moreover $\|(v_x, \sqrt{\tau} S_x)(t, \cdot)\|_{L^\infty} \leq E(t)^{\frac{1}{2}}$

$$\begin{aligned} J_2 &= \int_0^t \int_{\mathbb{R}} \left(\frac{2a\tau S_x v - (2a\tau S + \mu)v_x}{\nu^2} u_x + S_x \right) u_{xx} \, dx ds \\ &\leq (C + \frac{1}{\nu^2})(E_0 + E^{\frac{1}{2}}(t)) \int_0^t \mathcal{D}(s) ds + \nu^2 \int_0^t \|u_{xx}(s, \cdot)\|_{L^2}^2 ds. \end{aligned}$$

Summing up the above estimates for β from 0 to 1, recalling $\frac{2a\tau S + \mu}{\nu}$ is bounded below, we have

$$(1 - \nu^2) \int_0^t \|u_x(s, \cdot)\|_{H^1}^2 ds \leq C(E_0 + E^{\frac{1}{2}}(t)) \int_0^t \mathcal{D}(s) ds + \nu^2 \int_0^t \|v_x(s, \cdot)\|_{H^1}^2 ds \quad (2.12)$$

for $\nu \ll 1$. Now, take ∂_x^β on equation (1.3)₂ with $\beta = 0, 1$, multiply the result by $\partial_x^{\beta+1}v$, it yields

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} -p'(v)(\partial_x^{\beta+1}v)^2 dx ds &= \int_0^t \int_{\mathbb{R}} (\partial_x^\beta u_t \cdot \partial_x^{\beta+1}v - \partial_x^{\beta+1}S \cdot \partial_x^{\beta+1}v + \partial_x^\beta(p'(v))v_x \partial_x^{\beta+1}v) dx ds \\ &=: K_1 + K_2 + K_3. \end{aligned}$$

Using Young inequality and (2.10) for the first term below after integration by part, then (1.3)₁ and another integration by part, we have

$$\begin{aligned} K_1 &= \int_0^t \frac{d}{dt} \int_{\mathbb{R}} \partial_x^\beta u \partial_x^{\beta+1}v dx ds - \int_0^t \int_{\mathbb{R}} \partial_x^\beta u \cdot \partial_x^{\beta+1}v_t dx ds \\ &\leq C(E_0 + E^{\frac{1}{2}}(t)) \int_0^t \mathcal{D}(s) ds + \int_0^t \int_{\mathbb{R}} (\partial_x^{\beta+1}u)^2 dx ds \end{aligned}$$

while Young inequality $2rq \leq \frac{r^2}{\nu^2} + q^2\nu^2$ and (2.10) yield

$$K_2 \leq \frac{C}{\nu^2} (E_0 + E^{\frac{1}{2}}(t)) \int_0^t \mathcal{D}(s) ds + \nu^2 \int_0^t \|\partial_x^{\beta+1}v(s, \cdot)\|_{L^2}^2 ds.$$

Last, $K_3 = 0$ if $\beta = 0$, while using Cauchy-Schwarz inequality with $\|p'(v)v_x\|_{L^\infty} \leq CE^{\frac{1}{2}}$ if $\beta = 1$

$$K_3 = \int_0^t \int_{\mathbb{R}} p''(v)v_x^2 v_{xx} dx ds \leq CE^{\frac{1}{2}}(t) \int_0^t \mathcal{D}(s) ds$$

which yields, after combination with the above estimates for K_1 and K_2 summed up for $\beta \in \{0, 1\}$

$$\int_0^t \|v_x(s, \cdot)\|_{H^1}^2 ds \leq C(E_0 + E^{\frac{1}{2}}(t)) \int_0^t \mathcal{D}(s) ds + \int_0^t \|u_x\|_{H^1}^2 dt. \quad (2.13)$$

Combining (2.12) and (2.13) with ν^2 small enough successively yields

$$\int_0^t \|v_x(s, \cdot)\|_{H^1}^2 ds \leq C(E_0 + E^{\frac{1}{2}}(t)) \int_0^t \mathcal{D}(s) ds$$

and

$$\int_0^t \|u_x(s, \cdot)\|_{H^1}^2 ds \leq C(E_0 + E^{\frac{1}{2}}(t)) \int_0^t \mathcal{D}(s) ds$$

which is exactly (2.11): the proof of Lemma 2.6 is finished. \square

Combining lemmas 2.3-2.6, the proof of Proposition 2.2 is finished.

3. PROOF OF MAIN THEOREMS

In this section, we first prove Theorem 1.1 by the usual bootstrap (equiv. continuation) method and we next prove Theorem 1.2 by compactness arguments.

Proof of Theorem 1.1: First, δ being the constant defined in (2.3), choose $\epsilon \in (0, \delta)$ small enough such that

$$C\epsilon^{\frac{1}{2}} < \frac{1}{2},$$

where C is the constant in (2.4). Then there exists $\epsilon_0 \in (0, \epsilon)$ such that, for some $T > 0$, whatever the initial value satisfying $E(0) \equiv E_0 \leq \epsilon_0$, the problem (1.3)–(1.4) has a unique local solution $(v - 1, u, S) \in C^0([0, T], H^2) \cap C^1([0, T], H^1)$ satisfying

$$E(T) \leq \epsilon$$

and, according to Proposition 2.2, for all $t \in [0, T]$

$$E(t) + \frac{1}{2} \int_0^t \mathcal{D}(s) ds \leq CE_0. \quad (3.1)$$

Now, having fixed ϵ , if one furthermore requires

$$C\epsilon_0 < \frac{1}{2}\epsilon$$

where C is the constant in (2.4) then, according to Proposition 2.2, it holds in fact for all $t \in [0, T]$

$$E(t) \leq \frac{\epsilon}{2}$$

which implies that one can continue *infinitely in time* the local solutions with initial condition satisfying $E(0) \equiv E_0 \leq \epsilon_0$. The latter global-in-time solutions are unique and satisfy for all $t \geq 0$

$$E(t) + \int_0^t \mathcal{D}(s) ds \leq 2CE_0 \quad (3.2)$$

which finishes the proof of Theorem 1.1.

Proof of Theorem 1.2: Let (v^τ, u^τ, S^τ) be global solutions obtained in Theorem 1.1, so

$$\sup_{0 \leq t < \infty} \|(v^\tau - 1, u^\tau, \sqrt{\tau}S^\tau)(t, \cdot)\|_{H^2}^2 + \int_0^\infty (\|(v_x^\tau, u_x^\tau)(t, \cdot)\|_{H^1}^2 + \|S^\tau(t, \cdot)\|_{H^2}^2) dt \leq C_0 E_0, \quad (3.3)$$

holds with C_0 a constant independent of τ . Thus, there exists $(v^0, u^0) \in L^\infty((0, \infty), H^2)$ and $S^0 \in L^2((0, \infty), H^2)$ such that

$$\begin{aligned} (v^\tau, u^\tau) &\rightharpoonup (v^0, u^0) \quad \text{weakly-}^* \text{ in } L^\infty((0, \infty), H^2), \\ S^\tau &\rightharpoonup S^0 \quad \text{weakly in } L^2((0, \infty), H^2). \end{aligned}$$

Using (3.3) and $v \in (\frac{3}{4}, \frac{5}{4})$, note also for any $T > 0$ that $\partial_t v^\tau = u_x^\tau$ and $\partial_t u^\tau = S_x^\tau - p'(v^\tau)v_x^\tau$ are bounded in $L^2((0, T), H^1)$ uniformly whatever $\tau \in (0, 1)$, which implies $(v^0, u^0) \in C([0, T], H^1)$ and the fact that (v^τ, u^τ) are in fact relatively compact in $C([0, T], H_{loc}^{2-\delta_0})$ for any $\delta_0 \in (0, 1)$ using e.g. Simon-Lions-Aubin theorem [21] with the compact embedding $H_{loc}^2 \subset\subset H_{loc}^{2-\delta_0}$ so it holds

$$(v^\tau, u^\tau) \rightarrow (v^0, u^0) \quad \text{strongly in } C([0, T], H_{loc}^{2-\delta_0})$$

as $\tau \rightarrow 0$ and up to subsequences.

One can now let $\tau \rightarrow 0$ in (1.3)₁ as an identity in L^2 for all $t \in (0, T)$, but in (1.3)₂ only as an identity in $L^2(0, T; L^2)$ while it remains to identify S_x^0 . This can be done letting $\tau \rightarrow 0$ in (1.3)₃ as an identity in $\mathcal{D}'((0, \infty), H^1)$. The uniform boundedness of $\sqrt{\tau}S^\tau$ yields $\tau S^\tau \rightarrow 0$ in

$L^\infty((0, \infty), H^2)$ as $\tau \rightarrow 0$, then $\tau \partial_t S^\tau \rightarrow 0$ in $\mathcal{D}'((0, \infty), H^2)$ as $\tau \rightarrow 0$. Recalling v is uniformly bounded one finally obtains in $L^2(0, T; H^1)$

$$S^0 = \frac{\mu u_x^0}{v^0} \quad (3.4)$$

which finishes the proof on noting $\frac{\mu u_x^0}{v^0} \in L^\infty(0, T; H^1)$ for any $T > 0$.

Acknowledgement: Yuxi Hu's Research is supported by the Fundamental Research Funds for the Central Universities (No. 2023ZKPYLX01).

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