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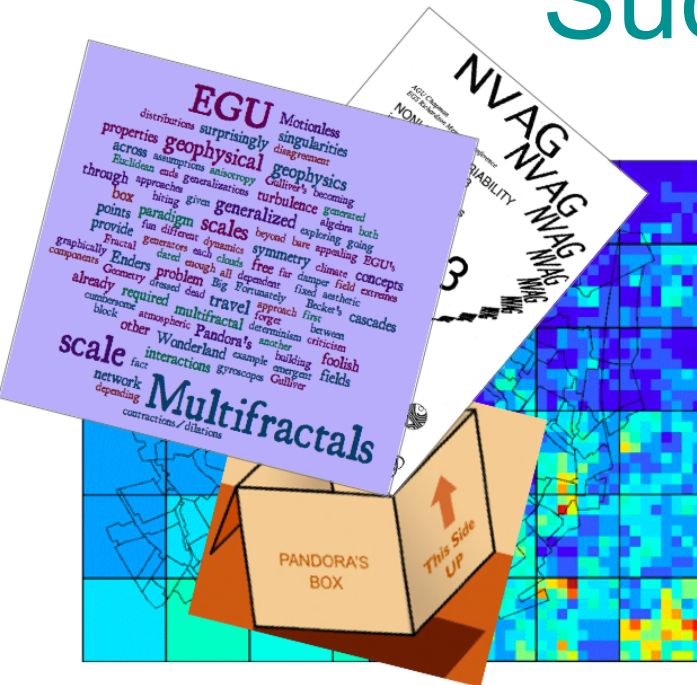
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A Century of Cascade Processes from Turbulence and Intermittency to Climate: Successes, Limitations and Challenges

by Daniel Schertzer and Ioulia Tchiguirinskaia

Hydrology, Meteorology & Complexity (HM&Co)

Ecole des Ponts ParisTech

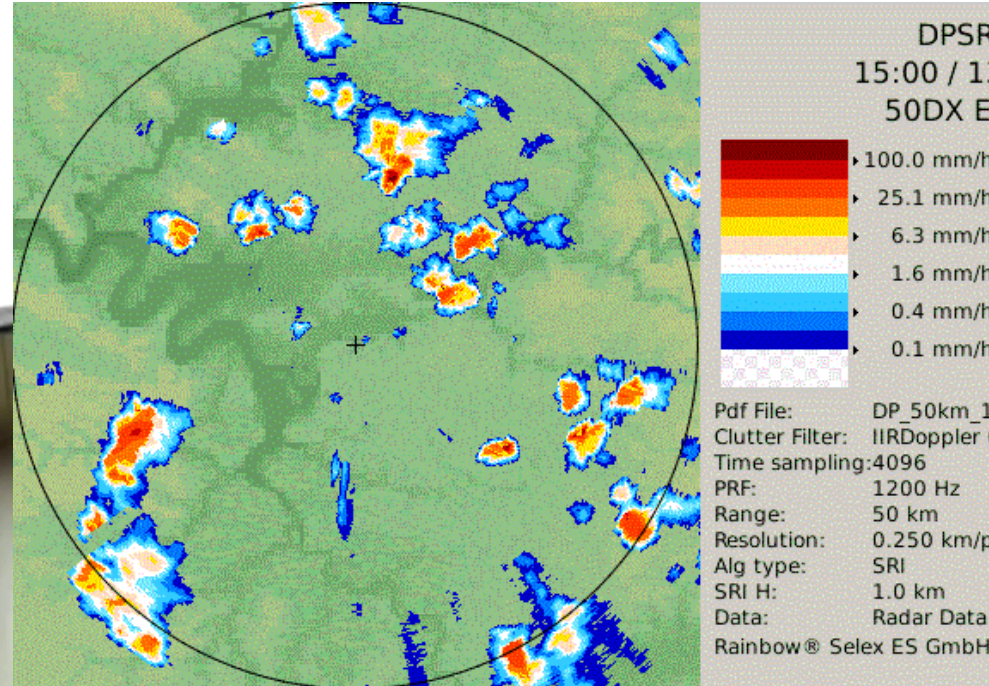


Outstanding Challenges in Nonlinear Dynamics
French-German WE-Heraeus-Seminar
20 - 25 March 2022



Millenium problem of turbulence !

explOatorium®



Art piece 'Windswept' (Ch. Sowers, 2012): 612 freely rotating wind direction indicators to help a large public to understand the complexity of environment near the Earth surface

Polarimetric radar observations of heavy rainfalls over Paris region during 2016 spring (250 m resolution):

- **heaviest rain cells** are much smaller than **moderate ones**
- **complex dynamics** of their aggregation into a large front

IPCC process

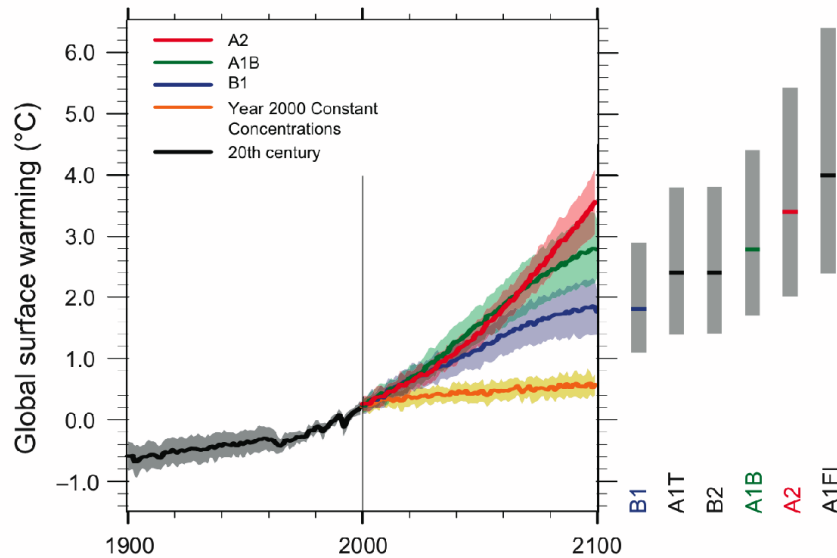


- IPCC reports presumably the most developed attempts of a dialogue between Science and Policy
 - a 2-step filtering
 - Assessment Reports (AR)
 - > Synthesis Report (SYR)
 - > Summary for Policymaker (SPM)
 - evolution of an extremely complex system in few lines

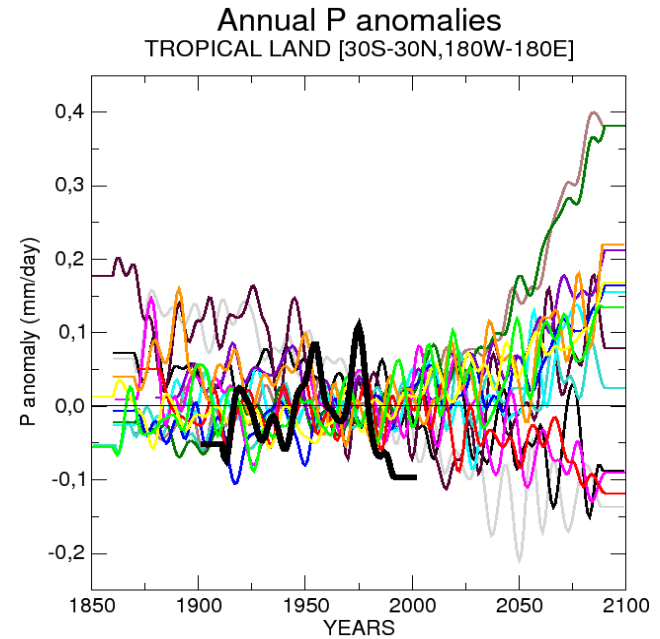
SPM 1. Observed Changes and their Causes

Human influence on the climate system is clear, and recent anthropogenic emissions of greenhouse gases are the highest in history. Recent climate changes have had widespread impacts on human and natural systems. {1}

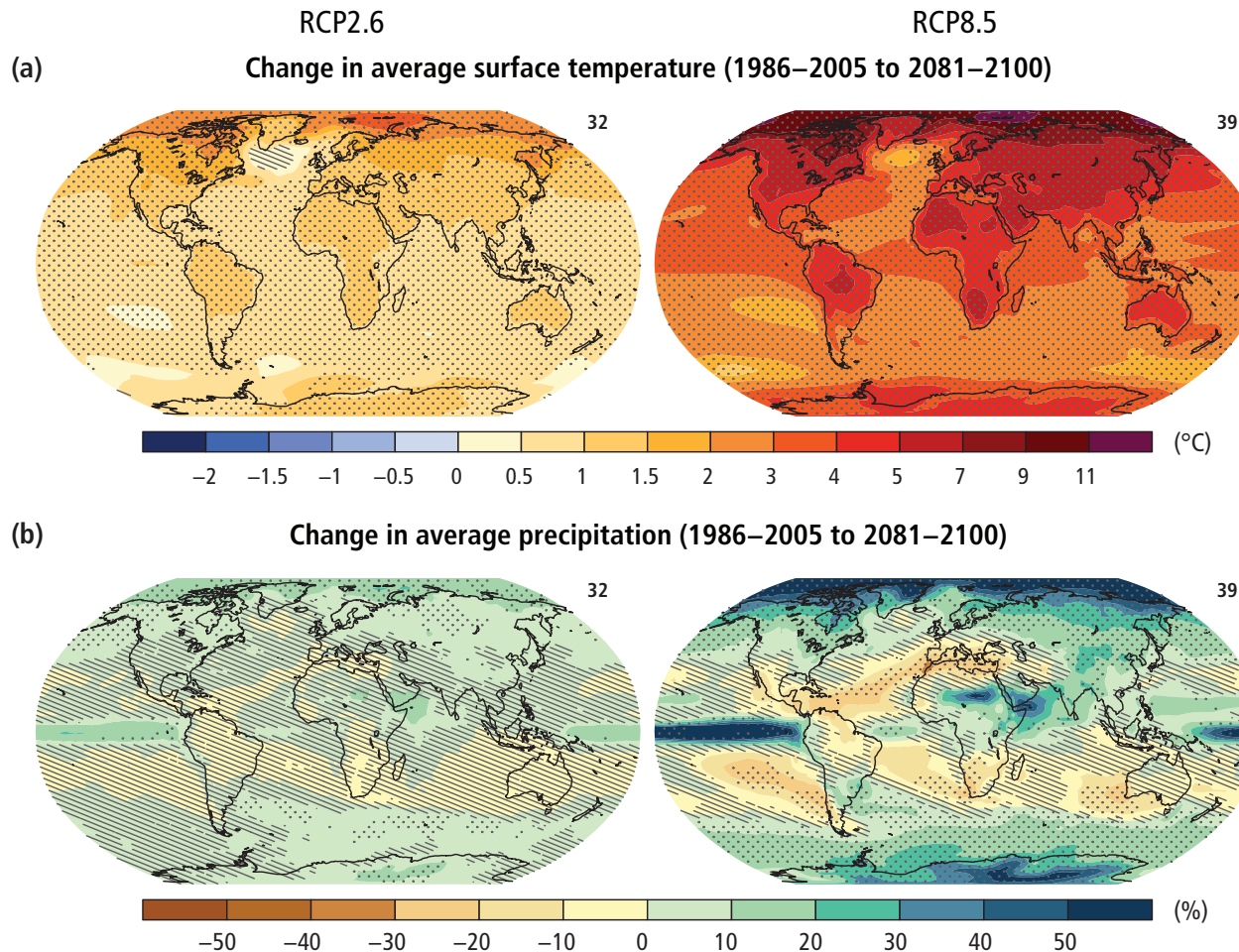
- AR's already filtered products of a vast climate research



Agreement of models on a temperature increase...



but **disagreement** on the evolution of precipitation extremes !



SPM 2.2 Projected changes in the climate system

Surface temperature is projected to rise over the 21st century under all assessed emission scenarios. It is *very likely* that heat waves will occur more often and last longer, and that extreme precipitation events will become more intense and frequent in many regions. The ocean will continue to warm and acidify, and global mean sea level to rise. {2.2}

Figure SPM.7 | Change in average surface temperature (a) and change in average precipitation (b) based on multi-model mean projections for 2081–2100 relative to 1986–2005 under the RCP2.6 (left) and RCP8.5 (right) scenarios.

A century of cascades !

WEATHER PREDICTION
BY
NUMERICAL PROCESS

BY
LEWIS F. RICHARDSON, B.A., F.R.MET.SOC., F.INST.P.

FORMERLY SUPERINTENDENT OF BRIDALEMUR OBSERVATORY
LECTURER ON PHYSICS AT WESTMINSTER TRAINING COLLEGE

We realize thus that: big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity—in the molecular sense.

The Supply of Energy from and to Atmospheric Eddies.

By LEWIS F. RICHARDSON.

(Communicated by Sir Napier Shaw, F.R.S. Received March 9, 1920.)

Introduction.

This paper extends Osborne Reynold's theory of the Criterion of Turbulence, to make it apply to the case in which work is done by the eddies, acting as thermodynamic engines in a gravitating atmosphere. For



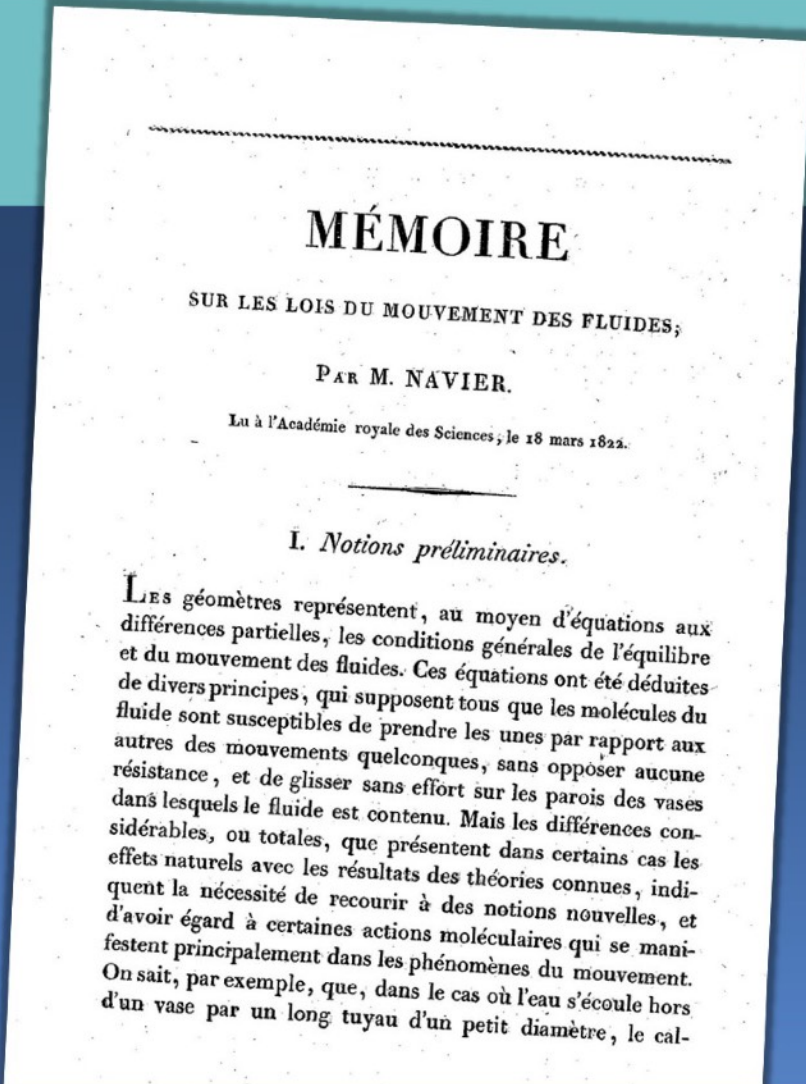
Celebrating 200 years of the Navier memoir: Imagining the microcosm to know the macrocosm

Introduction: S. Mougard (École des Ponts ParisTech)
Speakers: U. Frsich (Observatoire de la Côte d'Azur, Académie
des Sciences), E. Bodenschatz (Max Planck Institute, Göttingen)
Chair: D. Schertzer (École des Ponts ParisTech)

Registration required, but free of charge, with subject:
subscribe_navier_200@liste.enpc.fr

NAVIER 200

18 March 14:00-16:00 (CEST) École des Ponts ParisTech and on line



Navier's memoir

MÉMOIRE

SUR LES LOIS DU MOUVEMENT DES FLUIDES;

PAR M. NAVIER.

Lu à l'Académie royale des Sciences, le 18 mars 1822.

$$\frac{4 \cdot F(\rho)}{\rho^2} \left(\begin{array}{l} \alpha'^2 \left\{ \begin{array}{l} (u \sin.^2 r - v \sin. r \cos. r) \delta u \\ (u \sin. r \cos. r + v \cos.^2 r) \delta v \end{array} \right\} \\ \beta'^2 \left\{ \begin{array}{l} (u \cos.^2 r \sin.^2 s + v \sin. r \cos. r \sin.^2 s + w \cos. r \sin. s \cos. s) \delta u \\ (u \sin. r \cos. r \sin.^2 s + v \sin.^2 r \sin.^2 s + w \sin. r \sin. s \cos. s) \delta v \\ (u \cos. r \sin. s \cos. s + v \sin. r \sin. s \cos. s + w \cos.^2 s) \delta w \end{array} \right\} \\ \gamma'^2 \left\{ \begin{array}{l} (u \cos.^2 r \cos.^2 s + v \sin. r \cos. r \cos.^2 s - w \cos. r \sin. s \cos. s) \delta u \\ (u \sin. r \cos. r \cos.^2 s + v \sin.^2 r \cos.^2 s - w \sin. r \sin. s \cos. s) \delta v \\ (-u \cos. r \sin. s \cos. s - v \sin. r \sin. s \cos. s + w \sin.^2 s) \delta w \end{array} \right\} \end{array} \right)$$

- 52 pages (389-
- on average 1-2 equations per page
 - not numbered
 - no vector/tensor notation
 - some fluctuations in notations
 - a unique figure
- no bibliography
 - but several online references
- published in 1823

From Navier (1822)... to Stokes (1843)



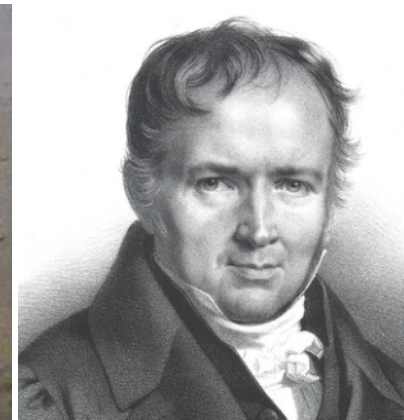
Claude Navier



Augustin-Louis
Cauchy



Adhémar Jean Claude
Barré de Saint-Venant



Simeon Denis
Poisson



Sir George Stokes

► Hot debates at Ecole des Ponts and outside...

Key points of Navier's derivation

- a paradigm shift:
 - viscosity often invoked, but first time explicit
 - moreover in a (3D) partial differential equation
 - obtained with a given level of rigour
- microcosm to macrocosm:
 - based on a few principles by Navier
 - not a unique path, see lattice gas dynamics
 - even numerous microcosms for a given macrocosm?
- localisation of the relative velocity
 - rather ad-hoc
 - but indispensable to avoid divergences
- why Ecole des Ponts was at the forefront of this research?
 - mixture of maths and engineering expertise?
 - but practitioners vs. “savants” ?

► To be pursued at Les Houches in 2023 !!

From scaling analysis to cascade processes

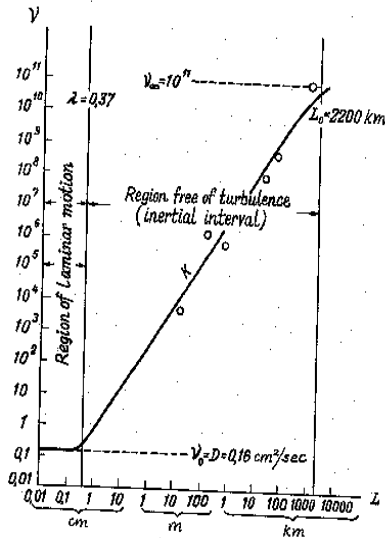


Fig. 1 The vertical diffusion coefficient $\nu(L)$ as a function of the turbulence scale L . Empirical points after Richardson.¹⁹

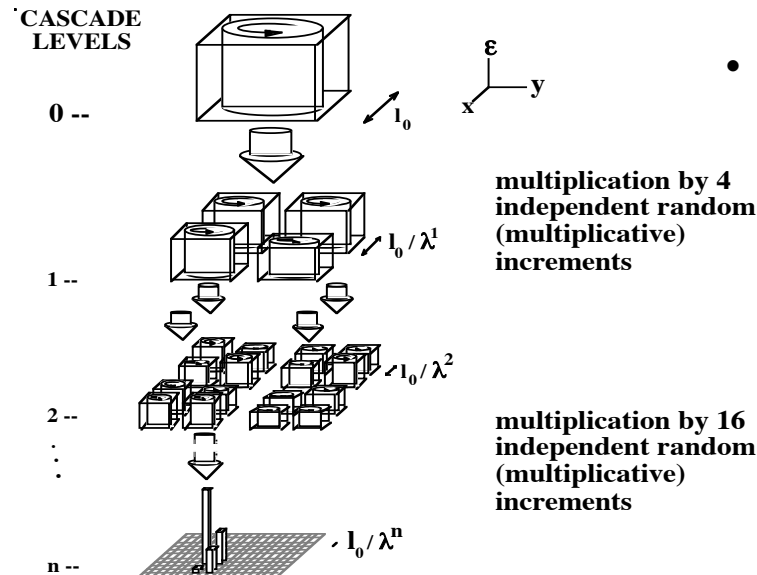
• Scaling analysis

- Passive scalar dispersion: Richardson (1926)
- Structure function: Kolmogorov (1941)
- Energy spectrum: Obukhov (1941)
- Higher order structure functions: Kolmogorov (1962), Obukhov (1962)
- Renyi dimensions: Grassberger and Procaccia (1983), Hentschel, and Procaccia (1983)
- Legendre transform to dimensions: Parisi and Frisch (1985)
- Fractal measures: Halsey et al. (1987)
-

• Cascade processes

- β -model: Novikov and Stewart (1964), Frisch et al. (1978)
- Log normal model: Yaglom (1966)
- Limit log-normal: Mandelbrot (1974)
- α -model: S+L (1984, 1985), Log-Poisson model: Dubrulle (1994)
- Multiplicative chaos: Kahane (1985)
- Universal multifractals/Levy multiplicative chaos: S+L (1987a&b, 1997), Fan (1987)
- Log-Poisson model: Dubrulle (1994), S&al (1995)
- Scaling Gyroscopes Cascade Tchiguirinskaia (1998)
-

Richardson, 1926



Schertzer & Lovejoy, 1989b



École des Ponts
ParisTech

Isotropic Cascades

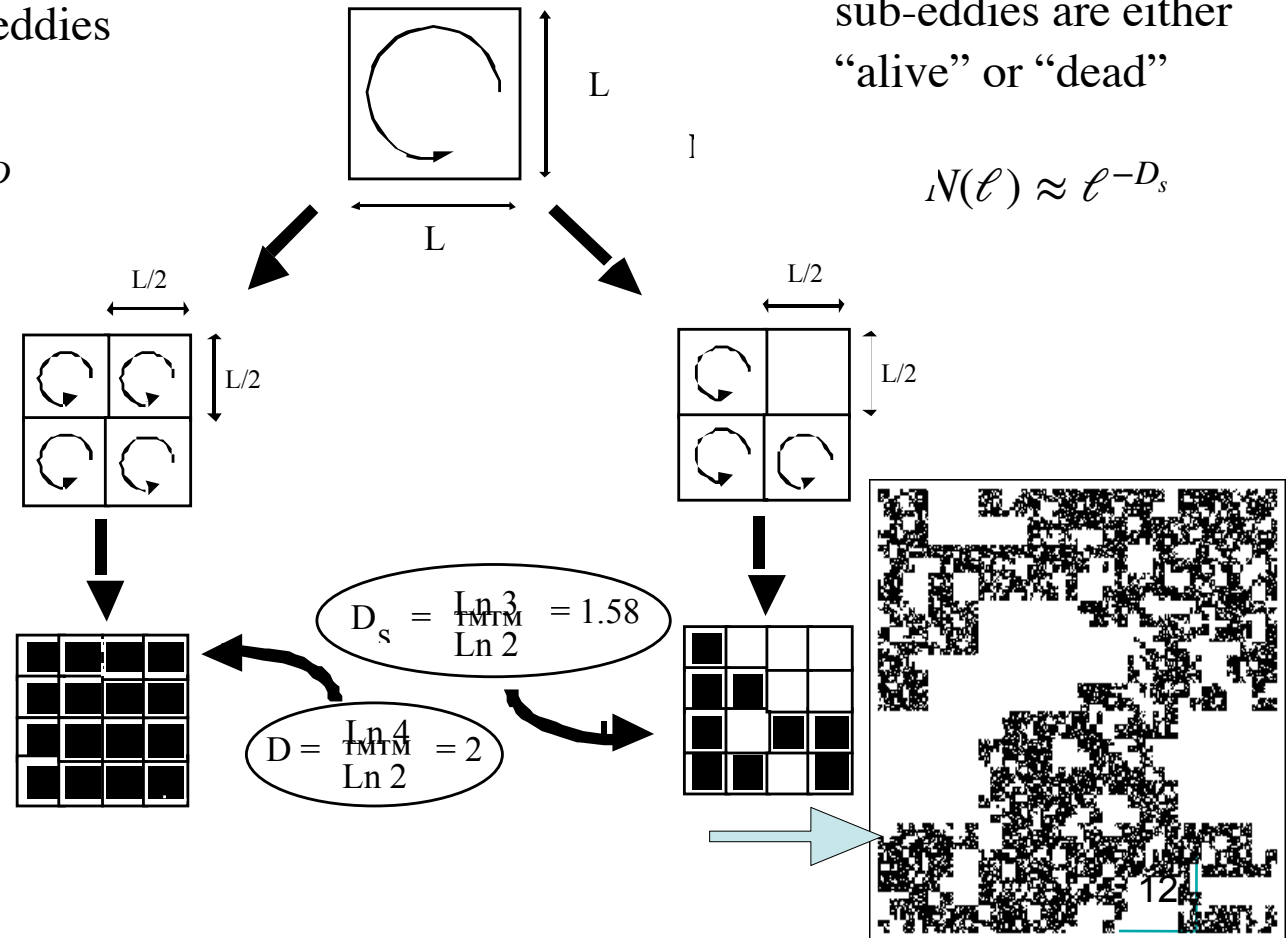
Non-intermittent
("homogeneous")
cascade: all eddies
same energy

$$N(\ell) \approx \ell^{-D}$$

β -model of
intermittency:

sub-eddies are either
"alive" or "dead"

$$N(\ell) \approx \ell^{-D_s}$$



Scale symmetry and equations

Whereas elementary mathematical properties of Navier-Stokes solutions are still unknown (existence, uniqueness: a Millennium problem):

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{grad}(\underline{u}) = \underline{f} - \frac{1}{\rho} \underline{grad}(p) + \nu \Delta \underline{u}$$

one can point out a **scale symmetry** (*): $\underline{x} \mapsto \underline{x}/\lambda$
 $t \mapsto t/\lambda^{1-\gamma}; \underline{u} \mapsto \underline{u}/\lambda^\gamma;$

$$\underline{f} \mapsto \underline{f}/\lambda^{2\gamma-1}; \nu \mapsto \nu/\lambda^{1+\gamma}; \rho \mapsto \rho/\lambda^{\gamma'}$$

Kolmogorov's scaling (K41) obtained with: $\epsilon(\ell) \sim \frac{u^3}{\ell} \sim \bar{\epsilon} \Rightarrow \gamma_K = -1/3$

More general case: multiple singularities γ 's:

$$\Pr(\gamma' > \gamma) \approx \lambda^{-c(\gamma)}$$

(*) from self-similarity (Sedov, 1961), symmetry (Parisi +Frisch, 1985), to generalised Galilean invariance (S+al, 2010)

Varenna summer school (1983)

- “Turbulence and Predictability in Geophysical Fluid Dynamics” organised by M. Ghil, R. Benzi et G. Parisi
 - a primary version of the multifractal formalism of Parisi and Frisch (1985). It concludes by: **“Still the multifractal model appears to be somewhat more restrictive than Mandelbrot’s weighted-curdling model which does include the lognormal case”**.

- S+L short formal presentation (the conference proceedings (1985) refers to S+L (1984)):
 - a small perturbation of the β -model is no longer limited to a unique dimension (α -model)
 - divergence of higher order moments: generic in cascade models
 - the later can introduce spurious scaling, an analytical approximation depending on a unique scaling exponent H and the critical order α was proposed:

$$\xi(p) = pH + \theta(p - \alpha)(1 - p/\alpha)$$

- it fits the experimental points from Anselmet et al. (1983), see fig. 1 with $H = 1/3, \alpha = 5, 5.5, 6$

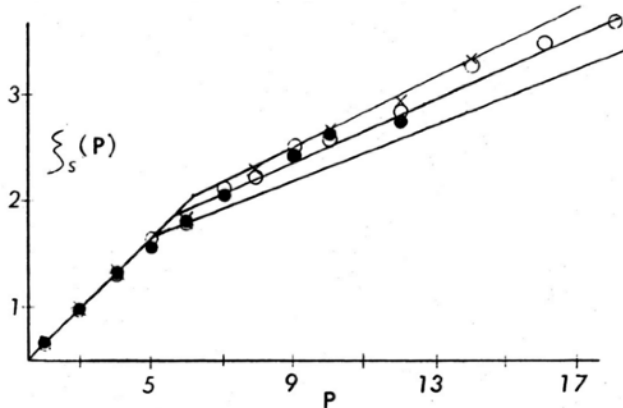
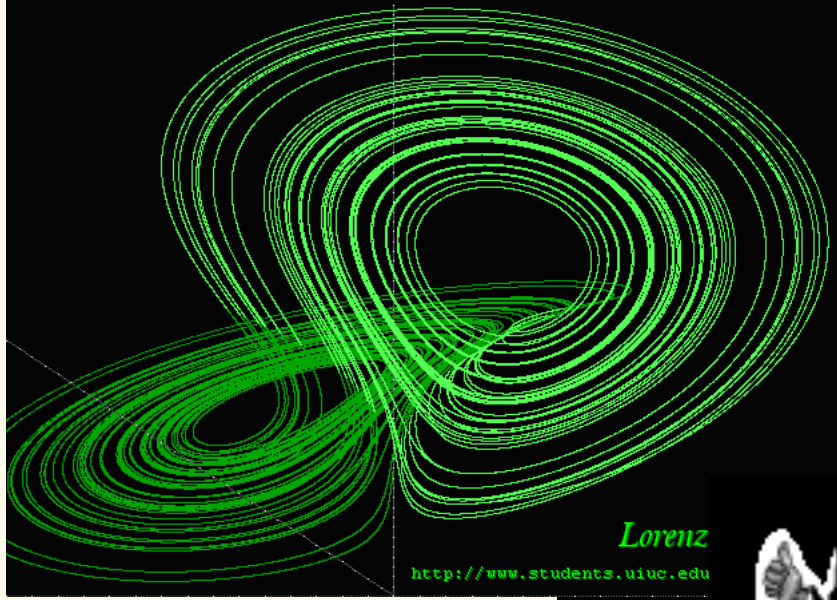


Fig.1 from S+L (1984)



Lorenz
<http://www.students.uiuc.edu>

More you average
wind spectra, more
you obtain Kolmogorov's
spectrum...



Ed's statement to cool
down a hot debate
(Varenna, 1983)



2008 AGU
Lorenz lecture



Van der Hoven wind spectrum (1957)

Richardson cascade is split into macro, meso, micro oscillations...

quasi-2D

quasi-3D

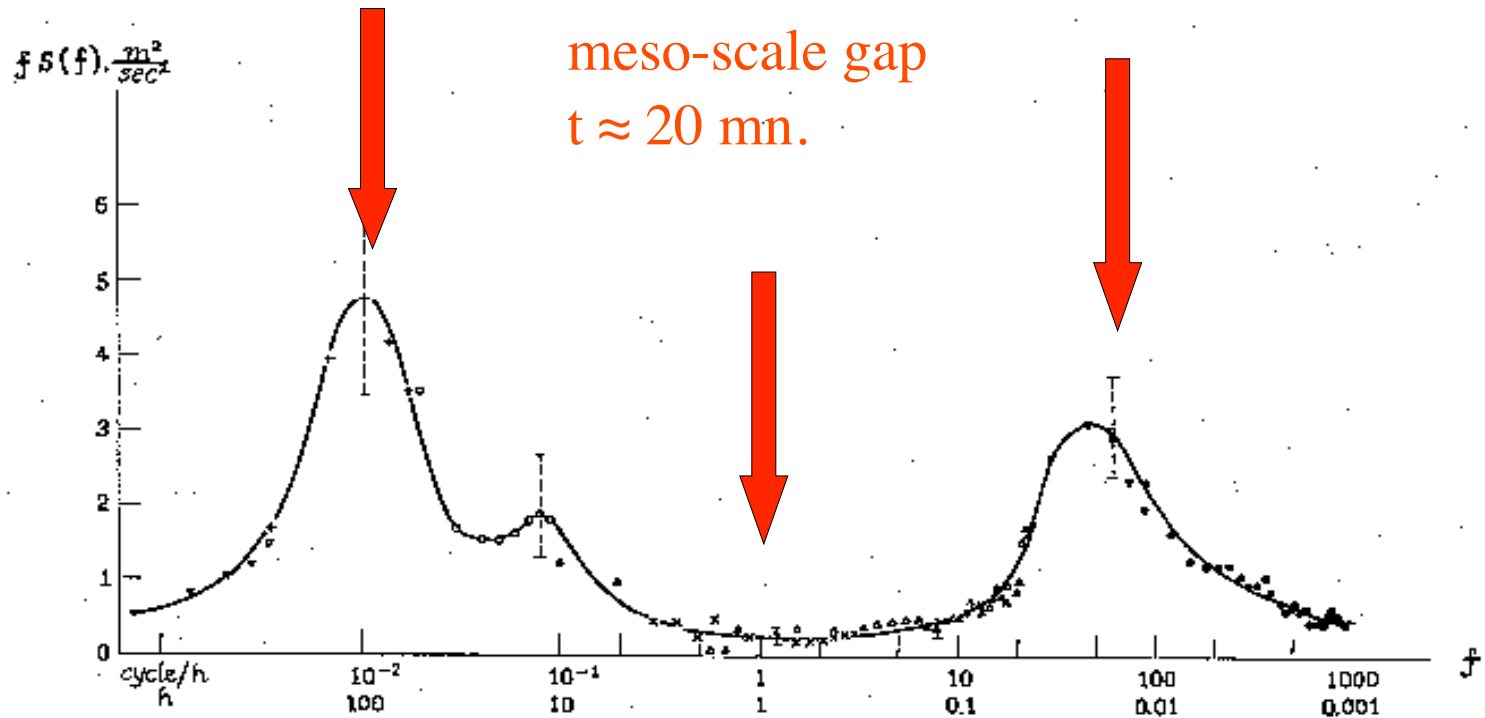
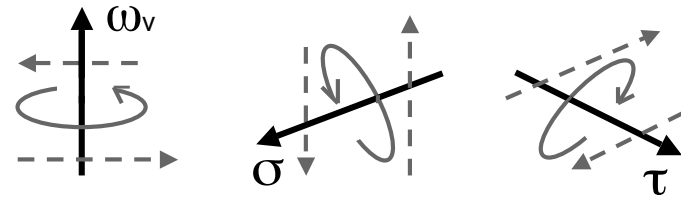
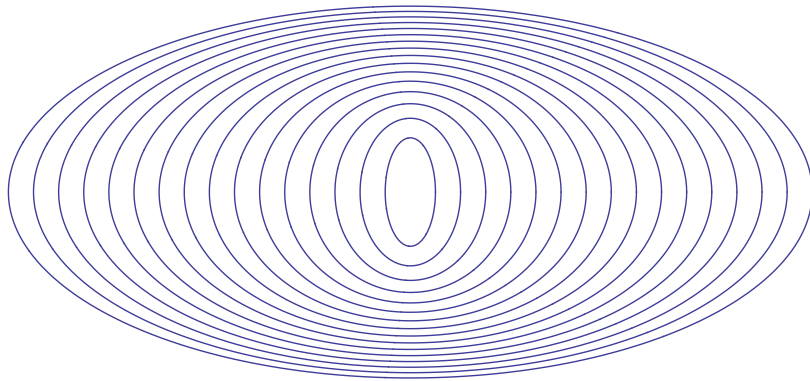


Fig. 3 Spectrum of the horizontal wind velocity. After Van der Hoven.²⁶ Some experimental points are shown on the graph; see reference 26.

2+ H_z -dimensional vorticity equation ($0 < H_z < 1$)

Scaling stratified /convective atmosphere:



$$D\vec{\sigma}/Dt = (\vec{\sigma} \cdot \vec{\nabla}_h)\vec{u}_h$$

$$D\vec{\tau}/Dt = (\vec{\tau} \cdot \vec{\nabla}_h + \boxed{\vec{\omega}_v \cdot \vec{\nabla}_v})\vec{u}_h$$

$$D\vec{\omega}_v/Dt = (\vec{\tau} \cdot \vec{\nabla}_h + \vec{\omega}_v \cdot \vec{\nabla}_v)\vec{u}_v$$

Strong interactions between *local generalized* scales,
= *strongly non local* (Euclidean) scales !

- a difficulty for direct numerical simulations ?
- easy for stochastic simulations !

S&aI, APC, 2012

Cascades and statistical physics

Transformation of a measure σ with the help of a “density” ε into another measure Π :

$$d\Pi = \varepsilon d\sigma$$

Generalisation with a non trivial limit ε of densities ε_λ of increasing resolution: $\lambda = L/\ell \rightarrow \infty$

$$\varepsilon_\lambda = e^{\Gamma_\lambda}; \quad \mathbb{E}e^{q\Gamma_\lambda} = Z_\lambda(q) = e^{K_\lambda(q)} \approx \lambda^{K(q)} \longleftrightarrow \mathbb{P}(\varepsilon_\lambda > \lambda^\gamma) \approx \lambda^{-c(\gamma)}$$

Mellin
transform



Legendre
transform

$$K(q) \longleftrightarrow c(\gamma)$$

main “trick”: $\log \lambda$ -divergence of the generator \Rightarrow that of the cumulant generating function

Γ : generator \approx hamiltonian

q : statistical order \approx inverse of temperature

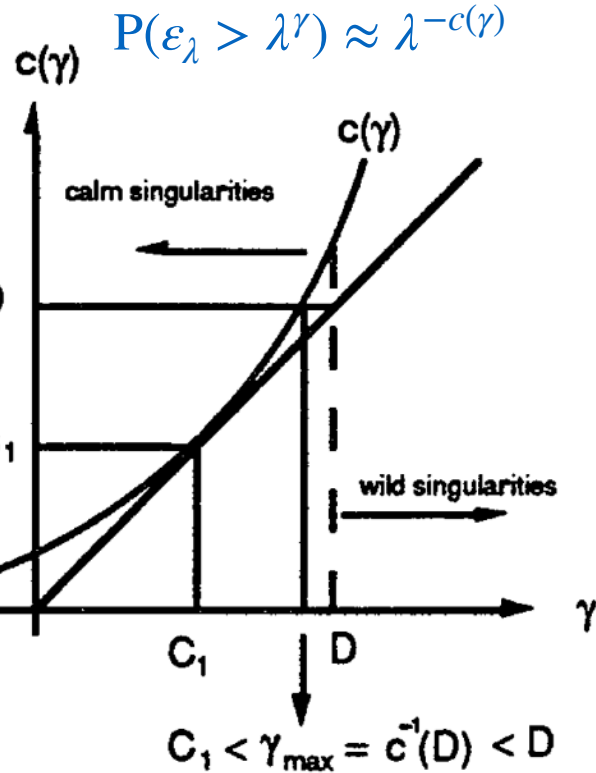
Z : 1st characteristic or moment generating function \approx partition function

K : 2nd characteristic or cumulant generating function \approx Mathieu Potential = Gibbs free entropy

c : codimension or Kramer function \approx entropy

γ : singularity or Hölder exponent \approx energy

Codimension vs. Dimension formalisms



$$\dim(A) + \text{codim}(A) = D$$

- codimensions easier for stochastic processes (S+L 85, 87, & 88, M88, 92 (Kramer functions)) **proba = ratio of 2 numbers!**

- conservation vs. degenerescence independent of the domain dimension

upper $\dim(\sigma) < C_1 \Rightarrow \varepsilon\sigma = 0$ (degenerescence)

lower $\dim(\sigma) > C_1 \Rightarrow E\varepsilon\sigma = \sigma$ (conservative)

$\Rightarrow E\varepsilon$ is a projector

- relations between deterministic dimensions and stochastic codimensions for **a given D** domain dimension:

$$\alpha_D + \gamma = D = f(\alpha_D) + c(\gamma)$$

$$D(q) + C(q) = D; \tau_D(q) = (q - 1)D(q); K(q) = (q - 1)C(q)$$

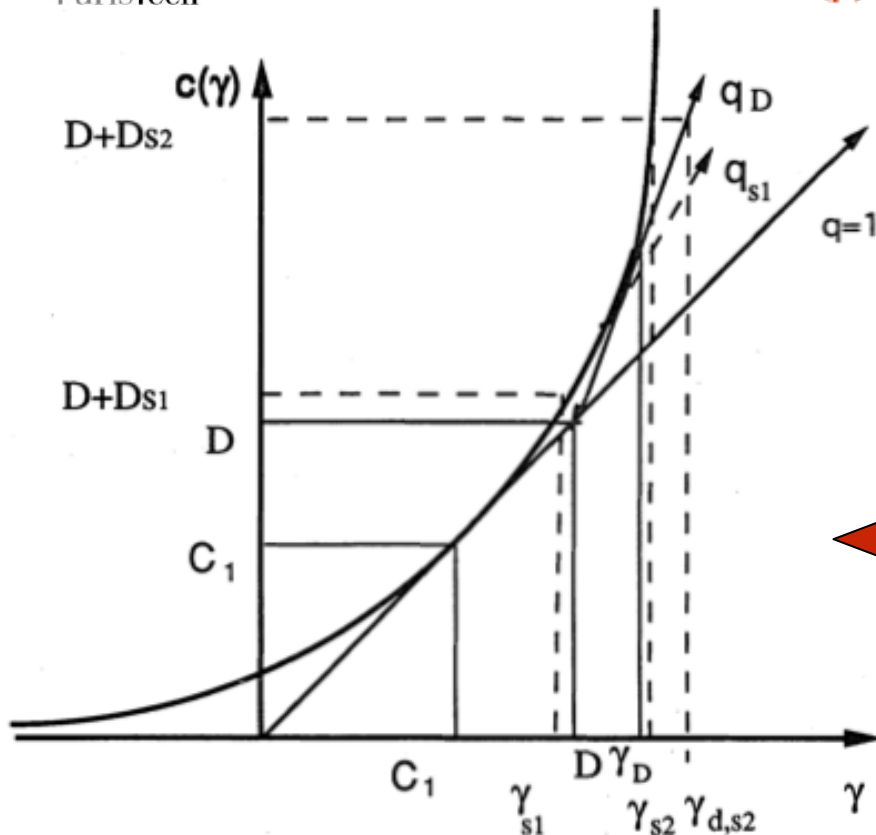
Multifractal phase transitions

$$P(\varepsilon_\lambda > \lambda^r) \approx \lambda^{-c(r)}$$

$$K(q) = \sup_\gamma [q\gamma - c(\gamma)]$$

$$c(\gamma) = \sup_q [q\gamma - K(q)]$$

$$\langle \varepsilon_\lambda^q \rangle \approx \lambda^{K(q)}$$



Legendre
transform

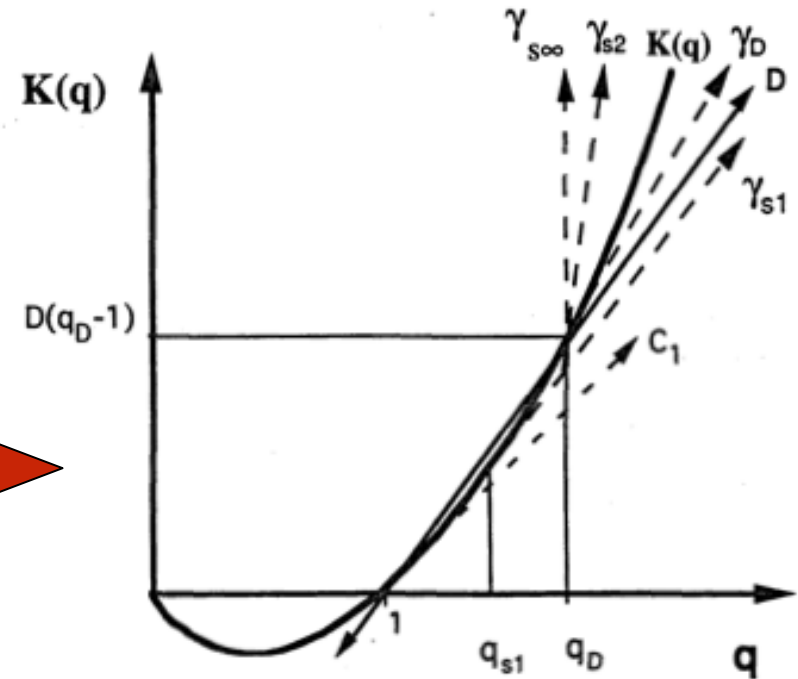
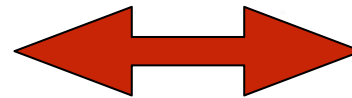


Fig. 5.5: Schematic diagram of $c(\gamma)$, $c_d(\gamma)$ indicating two sampling dimensions D_{s1} , D_{s2} and their corresponding $\gamma_{s1} < \gamma_D < \gamma_{s2} < \gamma_{d,s2}$; the critical tangent (slope q_D) contains the point (D, D) .

Fig. 5.6: Schematic diagram of $K(q)$, with straight lines of slopes $\gamma_{s1} < \gamma_D < \gamma_{d,s2} < \gamma_{d,s\infty}$ (∞) indicating the behavior for increasing sample size N_s ($N_{s\infty} = \infty$). The line of slope D defining q_D is also shown.

Universality

- Strong statistical universality: stable Lévy variables

$$\forall n \in \mathbb{N}, \exists a(n), b(n) \in \mathbb{R} : \sum_{i=1}^n X_i \stackrel{d}{=} a(n)X + b(n)$$

$\exists \alpha \in (0, 2] : a(n) = n^{1/\alpha}; \alpha < 2, \forall s \gg 1 : P(|X| > s) \approx s^{-\alpha}$ (hyperbolic/Pareto tail)
 $\alpha = 2 : \text{Gauss}$

A stable Levy X is attractive for any Y_i having same type of tail:

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n Y_i - b(n)}{a(n)} \stackrel{d}{=} X$$

► Poisson sum instead!

Log-Levy e^{qX} : its moments $E(e^{qX})$ are finite for any $q > 0$, iff it has **only a negative Pareto tail**, i.e. iff X is an extremely asymmetric/skewed Lévy stable

Universal multifractals



K(q)/C1 versus q

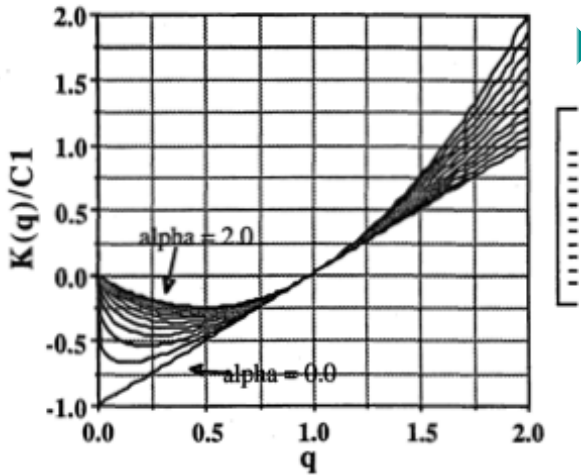


Fig. 1.41 Universal K(q) vs q, for different $\alpha=0$ to 2 by increment $\Delta\alpha=.2$.

c(gamma)/C1 versus gamma/C1

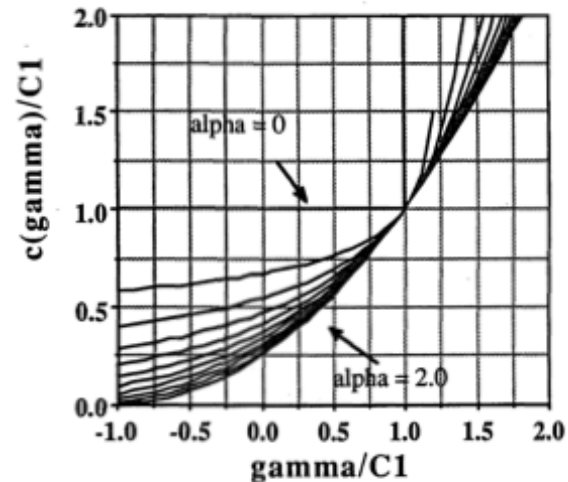


Fig. 1.42 Universal c(γ) vs γ , for different $\alpha=0$ to 2 by increment $\Delta\alpha=.2$

▶ stable and attractive multifractals under multiplication

• 3 basic parameters:

- H , deviation from conservation, $H=0$ for conservative fields:

$$\langle \varepsilon_\lambda \rangle \approx \lambda^{-H}, H = -K(1)$$

- c_1 , mean intermittency, codimension of the singularity of the mean field, $C_1=0$ for homogeneous field:

$$c_1 = dK/dq \Big|_{q=1}$$

- α , multifractality index, variability of the intermittency, $\alpha=0$ for uni/monofractals:

$$\alpha c_1 = d^2K/dq^2 \Big|_{q=1}$$

Spectral analysis and closures (L69)

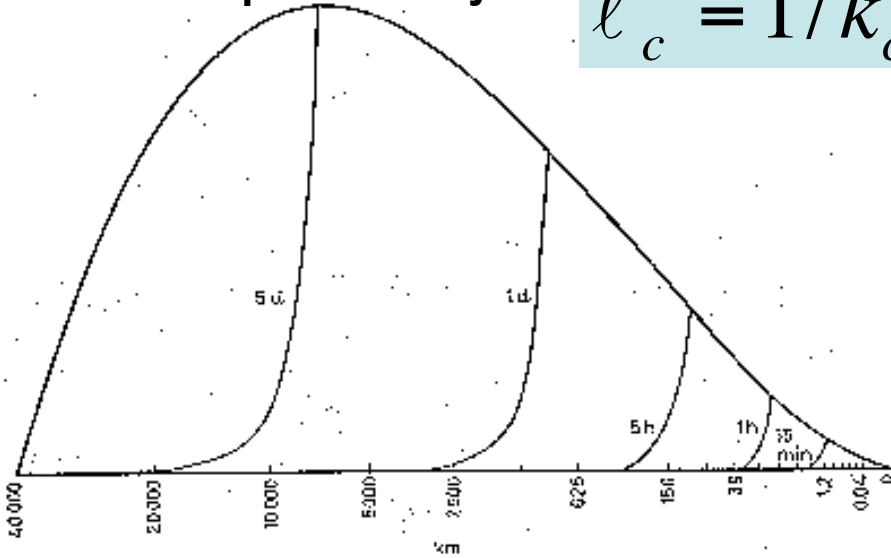
Scale of unpredictability

$$\ell_c = 1/k_c \approx t^{3/2}$$

Lorenz (1969)

Leith and Kraichnan(1972)

Metais and Lesieur (1986)



Flux from correlated e^c
to decorrelated energy e^Δ

$$e^c(\underline{x}, t) = \underline{u}^2(\underline{x}, t) \cdot \underline{u}^1(\underline{x}, t)$$

$$e^\Delta(\underline{x}, t) = \frac{1}{2} \left(\underline{u}^2(\underline{x}, t) - \underline{u}^1(\underline{x}, t) \right)^2$$

Similar results based

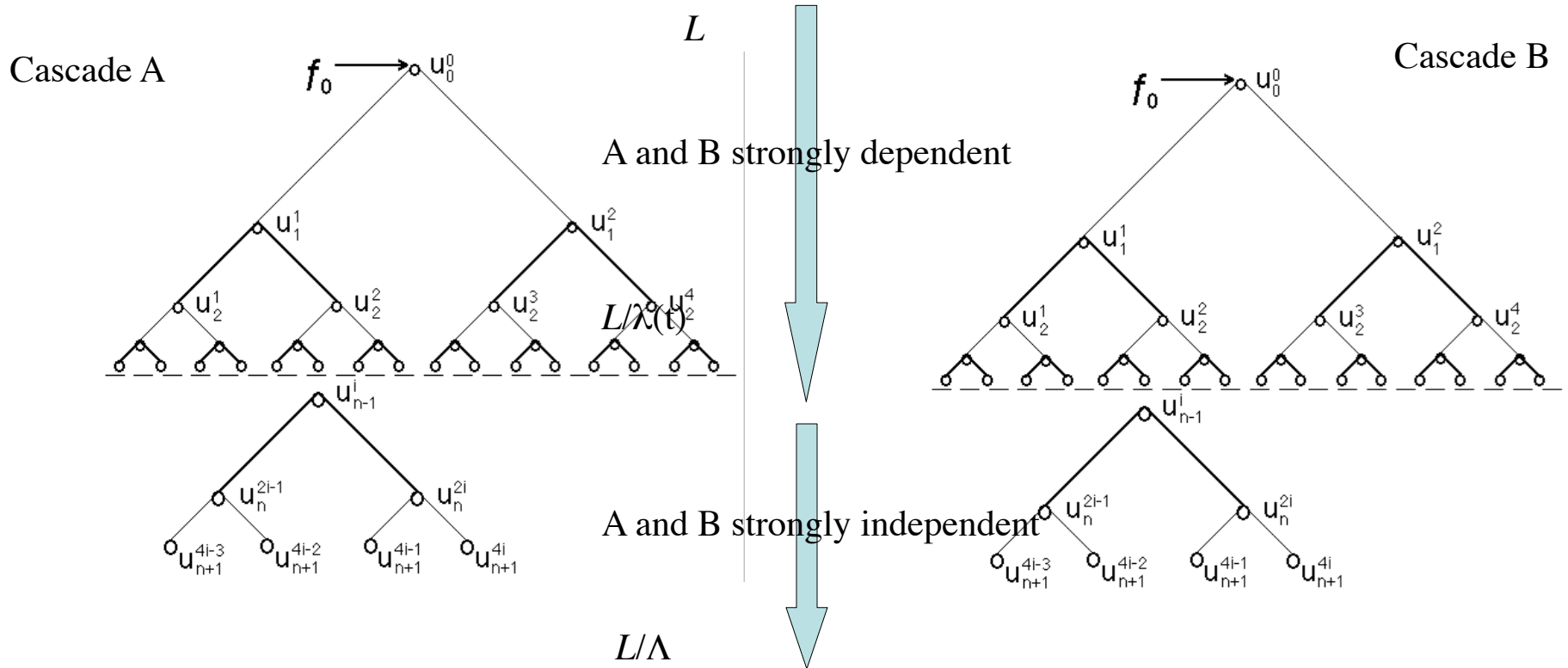
on turbulence phenomenology: $\ell_c^{2/3} = \bar{\varepsilon}^{-1/2} t^{3/2}$; $\bar{\varepsilon} = 10^{-3} \text{ m}^2 \text{ s}^{-3}$, $\eta \approx 10^{-3} \text{ m}$;

**Divergent Lyapunov
exponent:**

$$\mu_\ell \approx u_\ell / \ell \approx \bar{\varepsilon}^{1/3} \ell^{-2/3} \Rightarrow \lim_{\ell \rightarrow 0} (\mu_\ell) = \infty !$$

Multifractal predictability

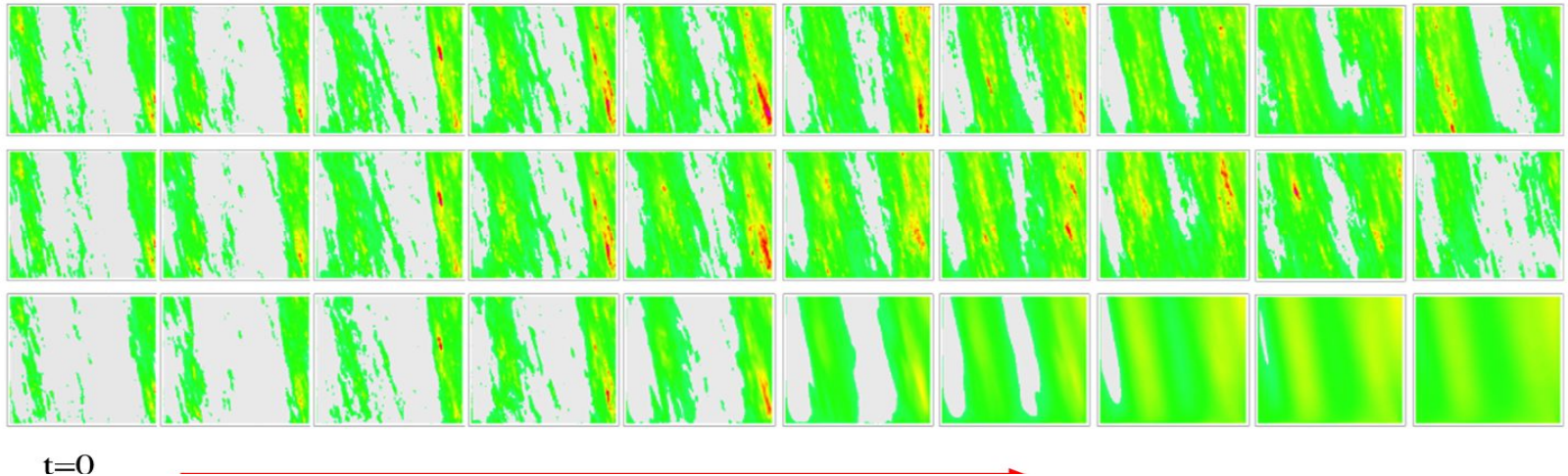
Crude idea: relaxation of (common) past structures \implies flux of the past
 (new) independent structures \implies flux of the future



Multifractal predictability

Rain simulation ($\alpha=1.5$, $C_1=0.2$, $H=0.1$ on log scale. Realizations A, B are identical until $t=0$, then they diverge.

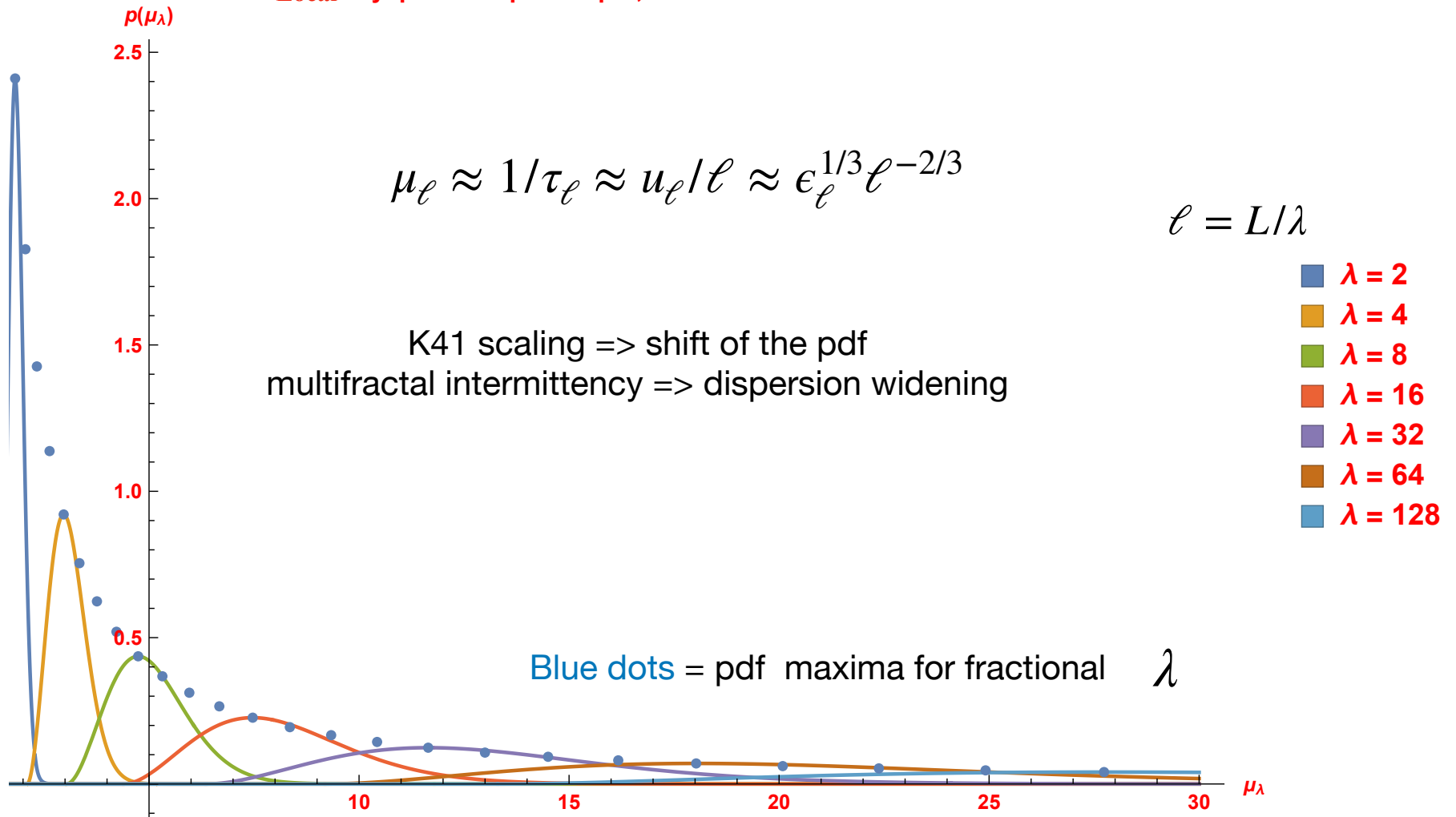
Top: Realization A. Middle: Realization B. Bottom, forecast



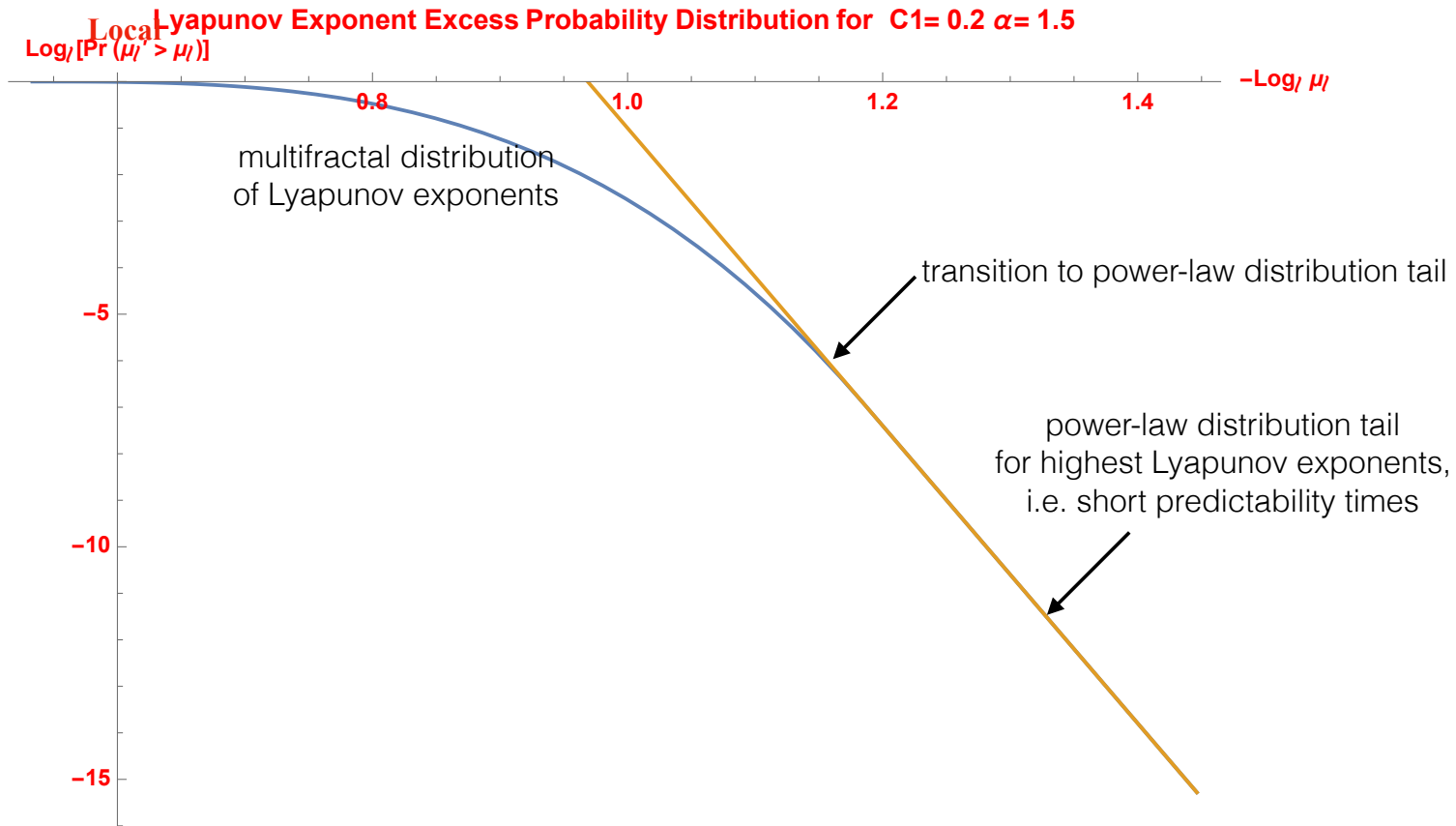
- **Power law divergence** between the realizations A and B,
=> **irrelevance** of the finite dimensional ‘LE + MET’ scenario !
- **Drastic loss of variability of forecast C** with deterministic sub-grid modeling (based on the conservation of the flux) => ‘**baby theorem**’: **stochastic sub-grid modeling** does much better than deterministic one!

Multifractal eddy turn-over times

Local Lyapunov Exponent pdf, $C1= 0.2 \alpha= 1.5$



Multifractal eddy turn-over times



$$\text{Log}_\ell[\text{Pr}(\mu'_\ell > \mu_\ell)] \text{ vs. } -\text{Log}_\ell[\mu_\ell]$$

where: μ_ℓ is the Lyapunov exponent at scale ℓ

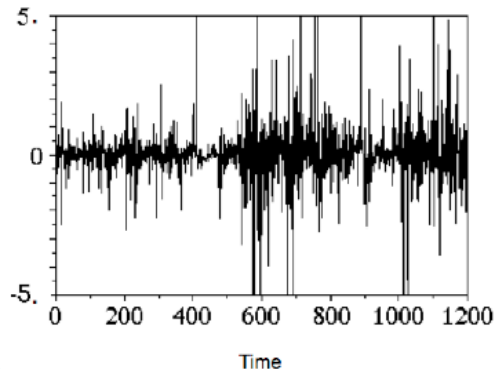
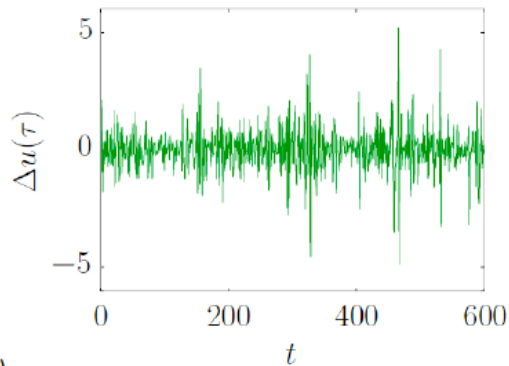
$$-\text{Log}_\ell[\mu_\ell] > 0 \Leftrightarrow \lim_{\ell \rightarrow 0} (\mu_\ell) = \infty$$

huge fluctuations are introduced by the distribution power-law tail.

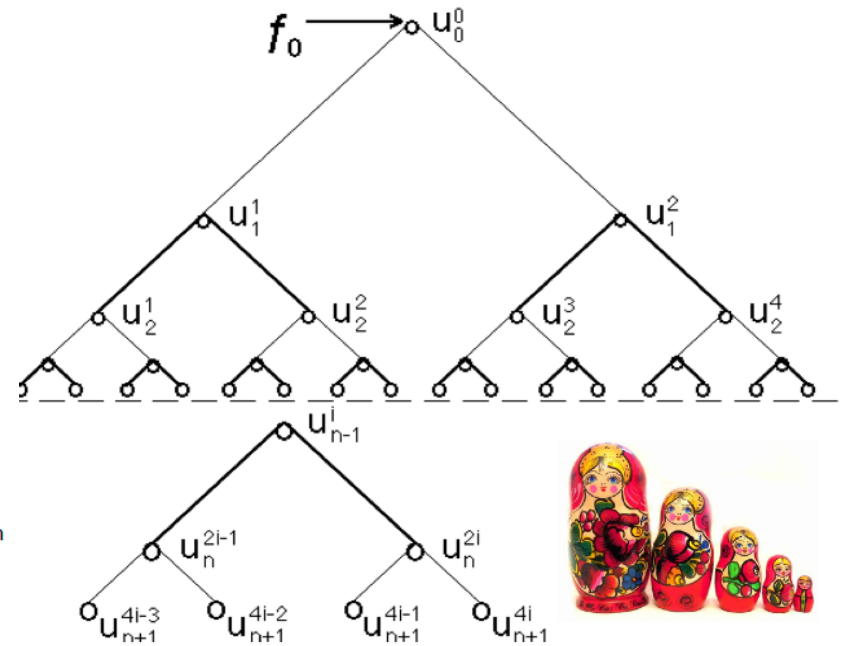
Scaling Gyroscope Cascade

$$\left(\frac{d}{dt} + \nu k_n^2\right) \hat{u}_n^i = i \{ k_{n+1} [|\hat{u}_{n+1}^{2i-1}|^2 - |\hat{u}_{n+1}^{2i}|^2] + (-1)^i k_n \hat{u}_n^i * \hat{u}_{n-1}^{a(i)} \}$$

$a(i)$ is an ancestor.



(a) Figure 2: Comparison of fluctuations: (a) atmospheric turbulence at 100m (Fitton, 2013) and (b) SGC simulation for $n=6$ (Chigirinskaya and Schertzer, 1996), both display somehow similar strong intermittency.



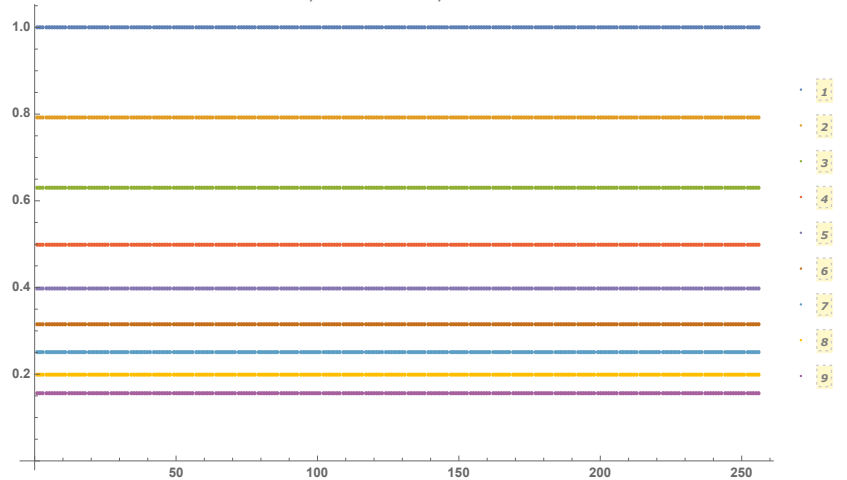
Local flux of energy:

$$\epsilon_n^i = - \sum_{r=0}^n k_{n-r+1} \left[\left| \hat{u}_{n-r+1}^{2a^r(i)-1} \right|^2 - \left| \hat{u}_{n-r+1}^{2a^r(i)} \right|^2 \right] \text{Im}(\hat{u}_n^{a^r(i)}) + (-1)^{a^r(i)+1} k_{n-r} \left| \hat{u}_n^{a^{r+1}(i)} \right|^2 \text{Im}(\hat{u}_{n-1}^{a^{r+1}(i)})$$

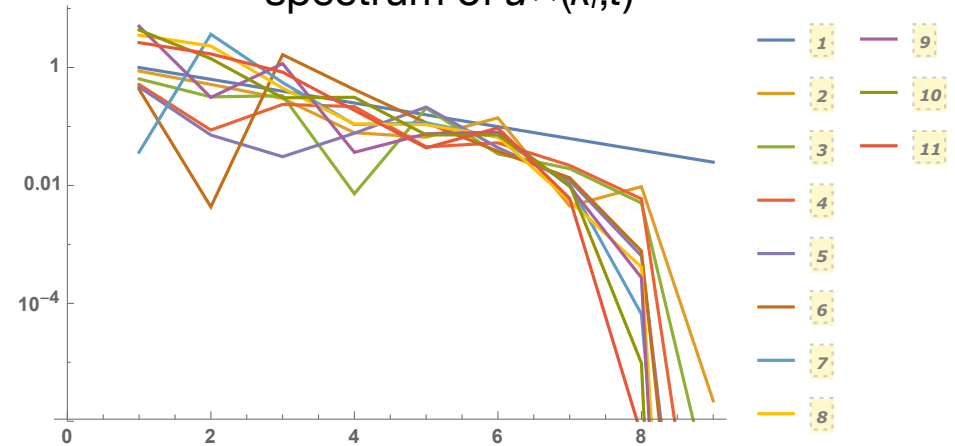
SGC multifractal predictability

initial energy distribution per eddy

itP = 1, itP*nNoOut = 50, t = 0.121535

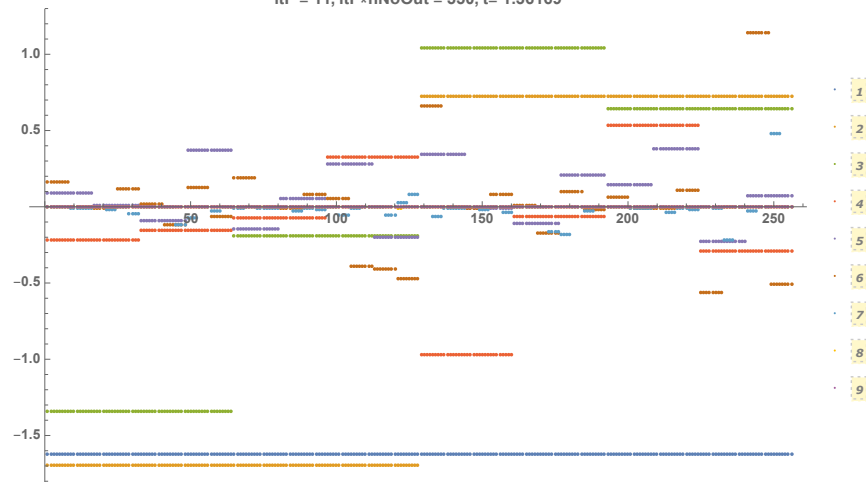


time evolution of the energy spectrum of $u^{(1)}(k_i, t)$

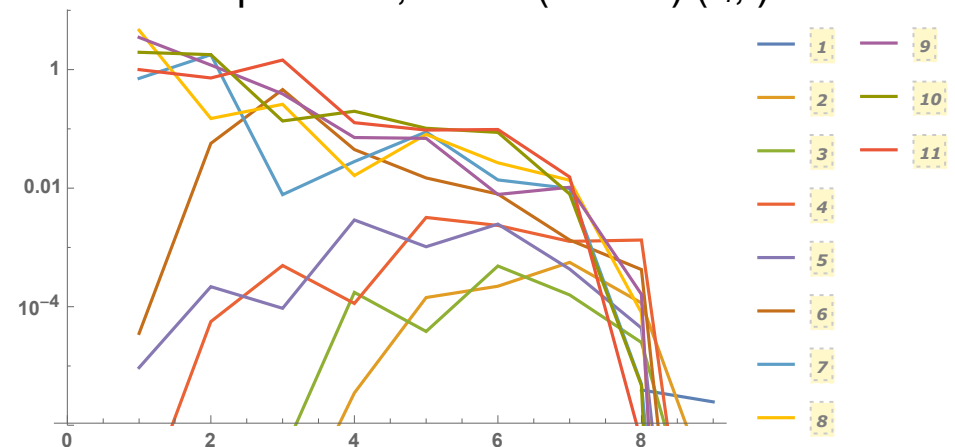


energy distribution per eddy $t \approx 1.4$

itP = 11, itP*nNoOut = 550, t = 1.36169



time evolution of the decorrelation spectrum, i.e. of $(u^{(1)} - u^{(2)})(k_i, t)$



SGC cascade as a dynamical system

SGC Jacobian matrix, scale levels: $m=0,3$ location index $j=1,2^m$

Out[863]/M

$$\begin{pmatrix}
 0 & 2 k_1 u_{1,1} & -2 k_1 u_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -k_1 u_{1,1} & -k_1 u_{1,1} & 0 & 2 k_2 u_{2,1} & -2 k_2 u_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 k_1 u_{1,2} & 0 & k_1 u_{1,1} & 0 & 0 & 2 k_2 u_{2,3} & -2 k_2 u_{2,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & -k_2 u_{2,1} & 0 & -k_2 u_{2,1} & 0 & 0 & 0 & 2 k_3 u_{3,1} & -2 k_3 u_{3,2} & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & k_2 u_{2,2} & 0 & 0 & k_2 u_{2,1} & 0 & 0 & 0 & 0 & 2 k_3 u_{3,3} & -2 k_3 u_{3,4} & 0 & 0 & 0 & 0 \\
 0 & 0 & -k_2 u_{2,3} & 0 & 0 & -k_2 u_{2,2} & 0 & 0 & 0 & 0 & 0 & 2 k_3 u_{3,5} & -2 k_3 u_{3,6} & 0 & 0 \\
 0 & 0 & k_2 u_{2,4} & 0 & 0 & 0 & k_2 u_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 2 k_3 u_{3,7} & -2 k_3 u_{3,8} \\
 0 & 0 & 0 & -k_3 u_{3,1} & 0 & 0 & 0 & -k_3 u_{3,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & k_3 u_{3,2} & 0 & 0 & 0 & 0 & k_3 u_{3,1} & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & -k_3 u_{3,3} & 0 & 0 & 0 & 0 & -k_3 u_{3,2} & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & k_3 u_{3,4} & 0 & 0 & 0 & 0 & 0 & k_3 u_{3,2} & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & -k_3 u_{3,5} & 0 & 0 & 0 & 0 & 0 & -k_3 u_{3,3} & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & k_3 u_{3,6} & 0 & 0 & 0 & 0 & 0 & 0 & k_3 u_{3,3} & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & -k_3 u_{3,7} & 0 & 0 & 0 & 0 & 0 & 0 & -k_3 u_{3,4} & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & k_3 u_{3,8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_3 u_{3,4}
 \end{pmatrix}$$

m and j have quite different roles => need to innovate w.r.t. to classical MET

- no longer the norm of the separation of 2 “points”, therefore of the Jacobian **matrix**
- but a volume error, therefore the Jacobian **determinant J_E** over this volume:

$$\left| \wedge_{i=1}^d \delta X_i(t) \right| / \left| \wedge_{i=1}^d \delta X_i(o) \right| \approx e^{\mu_E t}$$

$$\mu_E = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \left(\int_0^t \left| \det[J_E(t')] \right| dt' \right)$$

SGC cascade as a dynamical system

SGC Jacobian matrix, scale levels: $m=0,3$ location index $j=1,2^m$

Out[863]/Matrix

$$\begin{pmatrix}
 - & 2 k_1 u_{1,1} & -2 k_1 u_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -k_1 u_{1,1} & -k_1 u_{1,1} & 0 & 2 k_2 u_{2,1} & -2 k_2 u_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 k_1 u_{1,2} & 0 & k_1 u_{1,1} & 0 & 0 & 2 k_2 u_{2,3} & -2 k_2 u_{2,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & -k_2 u_{2,1} & 0 & -k_2 u_{2,1} & 0 & 0 & 0 & 2 k_3 u_{3,1} & -2 k_3 u_{3,2} & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & k_2 u_{2,2} & 0 & 0 & k_2 u_{2,1} & 0 & 0 & 0 & 0 & 2 k_3 u_{3,3} & -2 k_3 u_{3,4} & 0 & 0 & 0 & 0 \\
 0 & 0 & -k_2 u_{2,3} & 0 & 0 & -k_2 u_{2,2} & 0 & 0 & 0 & 0 & 0 & 2 k_3 u_{3,5} & -2 k_3 u_{3,6} & 0 & 0 \\
 0 & 0 & k_2 u_{2,4} & 0 & 0 & 0 & k_2 u_{2,2} & 0 & 0 & 0 & 0 & 0 & 0 & 2 k_3 u_{3,7} & -2 k_3 u_{3,8} \\
 0 & 0 & 0 & -k_3 u_{3,1} & 0 & 0 & 0 & -k_3 u_{3,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & k_3 u_{3,2} & 0 & 0 & 0 & 0 & k_3 u_{3,1} & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & -k_3 u_{3,3} & 0 & 0 & 0 & 0 & -k_3 u_{3,2} & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & k_3 u_{3,4} & 0 & 0 & 0 & 0 & 0 & k_3 u_{3,2} & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & -k_3 u_{3,5} & 0 & 0 & 0 & 0 & 0 & -k_3 u_{3,3} & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & k_3 u_{3,6} & 0 & 0 & 0 & 0 & 0 & 0 & k_3 u_{3,3} & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & -k_3 u_{3,7} & 0 & 0 & 0 & 0 & 0 & 0 & -k_3 u_{3,4} & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & k_3 u_{3,8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_3 u_{3,4}
 \end{pmatrix}$$

- only a diagonal + 2 pseudo-diagonals that are anti-symmetric up to a factor -2
- synchronisation of sibling eddies $\Rightarrow J_E$ reduces to products of diagonal term
 - recurrence relation obtained by 3 strategies, in particular with the help of the Schur complement:

$$J_{0,n+1} = \begin{pmatrix} J_{0,n} & B \\ C & D \end{pmatrix} \simeq \begin{pmatrix} J_{0,n} - BD^{-1}C & B \\ 0 & D \end{pmatrix}$$

$$\Rightarrow \det(J_{0,n+1}) = \det(D) \det(J_{0,n} - BD^{-1}C)$$

Scaling & multifractality of GRIP data

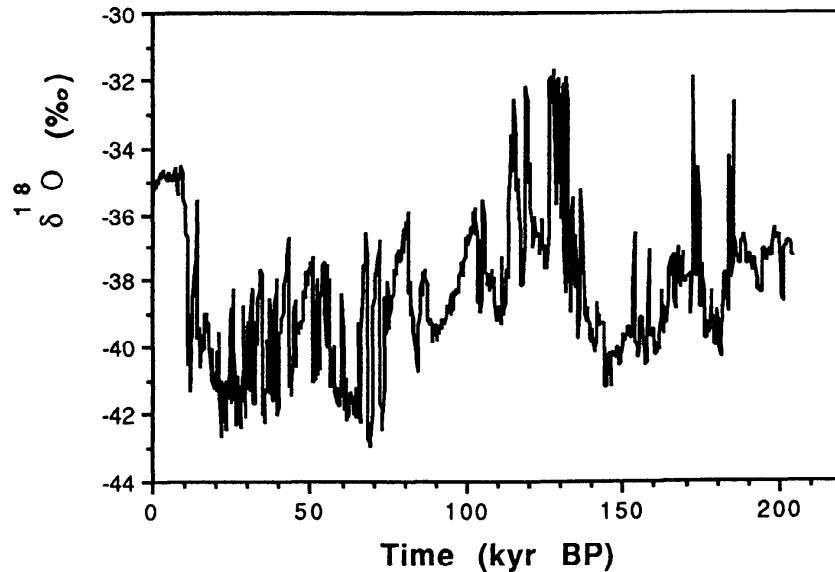


Figure 1. High resolution (200 yr average) $\delta^{18}O$ record coming from the recent GRIP Greenland ice core. Sharp fluctuations occurring on small time scales are clearly visible.

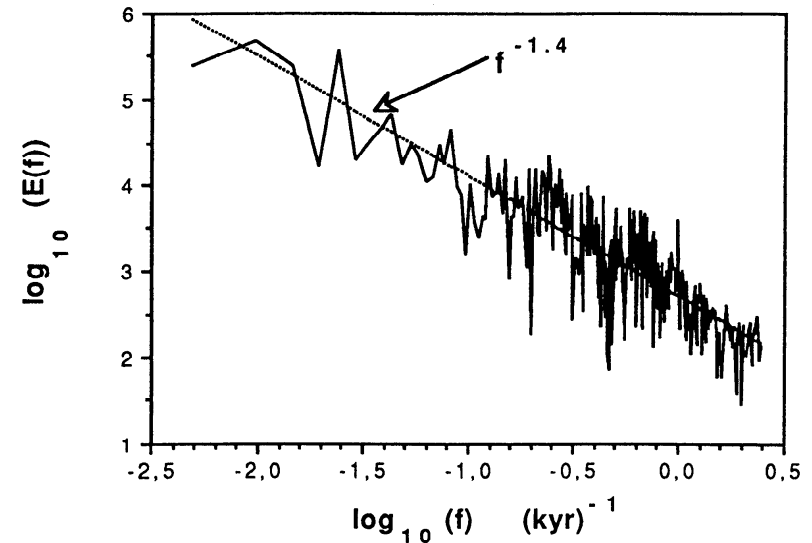


Figure 2. The power spectrum of the $\delta^{18}O$ data, shown in a log-log plot. The global straight line $f^{-1.4}$ is an indication of scaling. There is no evidence of periodic variations of frequencies of $(20 \text{ kyr})^{-1}$ or $(40 \text{ kyr})^{-1}$.

$$E(f) \approx f^{-\beta}; \beta \approx 1.4$$

Scaling & multifractality of GRIP data

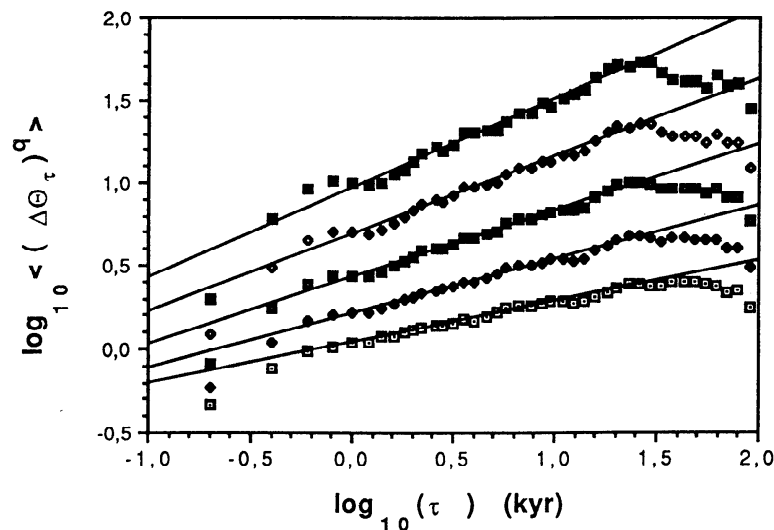


Figure 3. The structure functions $\langle (\Delta\theta_\tau)^q \rangle$ vs. τ in a log-log plot for $q=1, 1.5, 2, 2.5,$ and 3 . Linear trends are clearly visible for all order of moments, for a range of scales going from ≈ 400 yr to ≈ 40 kyr. The straight lines indicate the best linear regression over this range of scales for each value of q . This gives in particular: $H = \zeta(1) = 0.24 \pm 0.02$, and $\zeta(2) = 0.40 \pm 0.03$.

$$\left\langle \left(\Delta\theta \left(\frac{T}{\lambda} \right)^q \right) \right\rangle \approx \lambda^{-\zeta(q)} \left\langle (\Delta\theta(T))^q \right\rangle$$

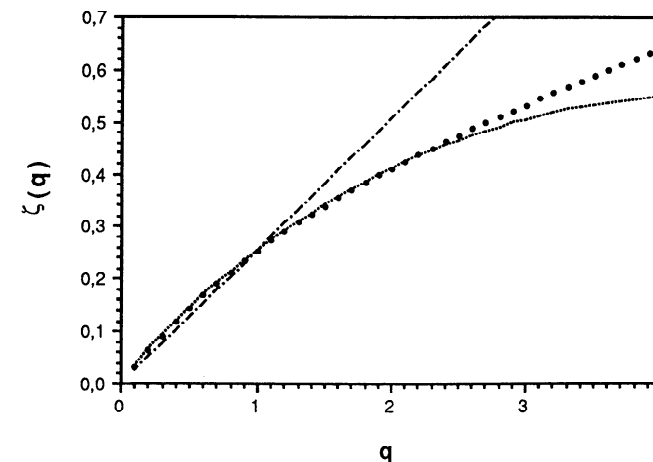
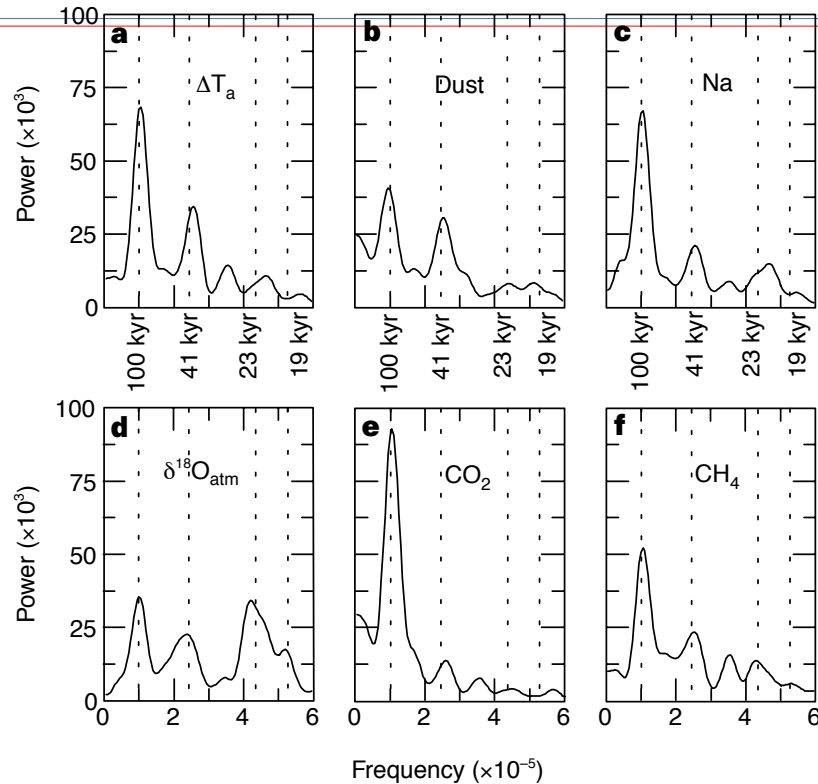


Figure 4. The scaling exponent structure function $\zeta(q)$ empirical curve (dots), compared to the monofractal curve $\zeta(q) = qH$ (dashed and dotted line), and to the universal multifractal function obtained with $\alpha = 1.6$ and $C_1 = 0.05$ in equation (2) (dotted curve). The empirical curve is nonlinear, indicating multifractality. For moments of order $q > 1$, the departure from monofractal behaviour is strong. The universal multifractal fit is excellent until moment order $q_{\max} \approx 2.5 \pm 0.3$, corresponding to a multifractal phase transition occurring because of sample limitations.

Vostok data



“integrated spectra”: $\omega E(\omega)$ vs. $\text{Log}(\omega)$

well identified but broad peaks/spikes

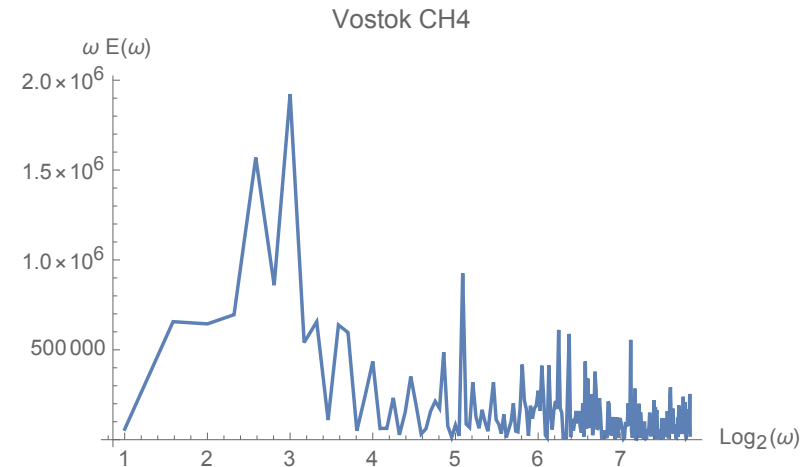


Figure 4 Spectral properties of the Vostok time series. Frequency distribution (in cycles yr^{-1}) of the normalized variance power spectrum (arbitrary units). Spectral analysis was done using the Blackman-Tukey method (calculations were performed with the Analyseries software⁴⁷): **a**, isotopic temperature; **b**, dust; **c**, sodium; **d**, $\delta^{18}\text{O}_{\text{atm}}$; **e**, CO_2 ; and **f**, CH_4 . Vertical lines correspond to periodicities of 100, 41, 23 and 19 kyr.

Vostok data: spectral scaling

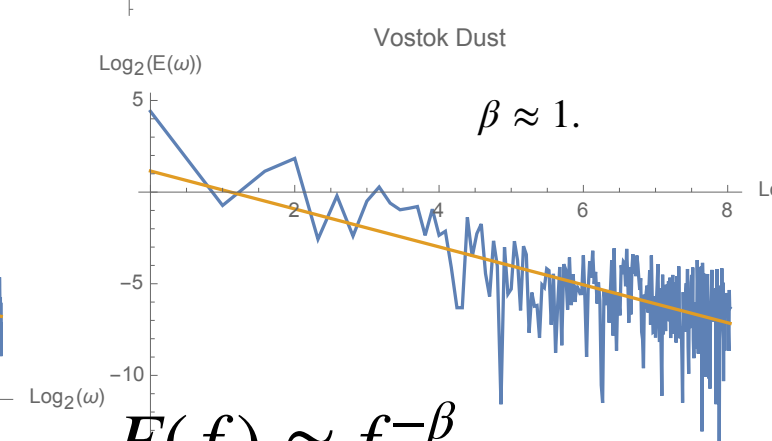
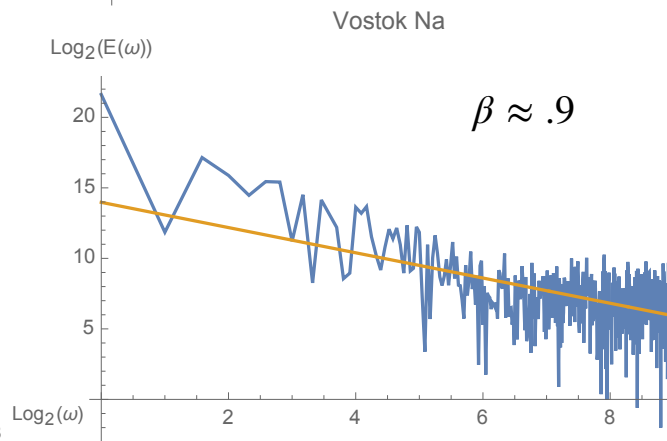
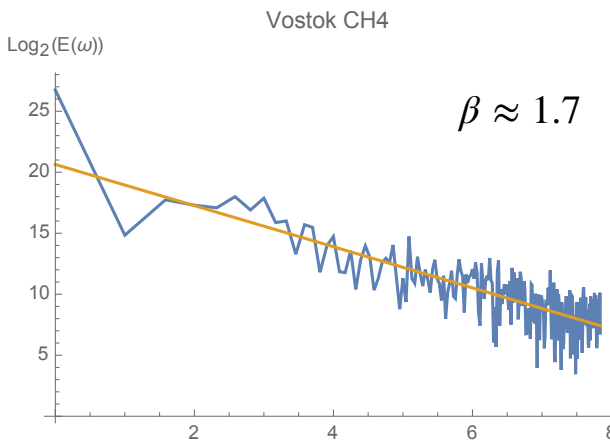
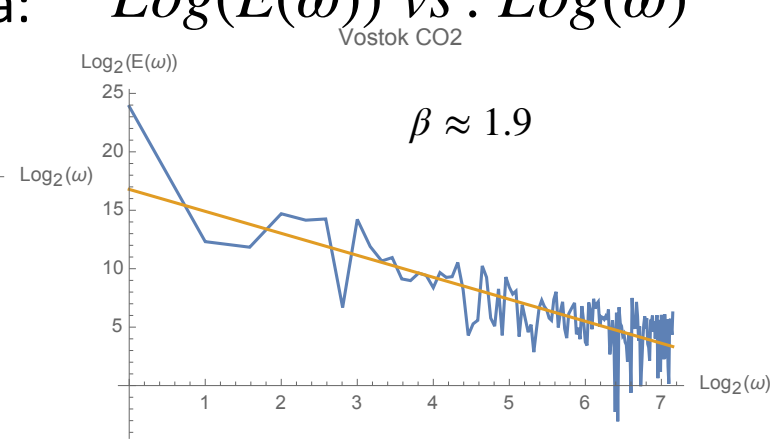
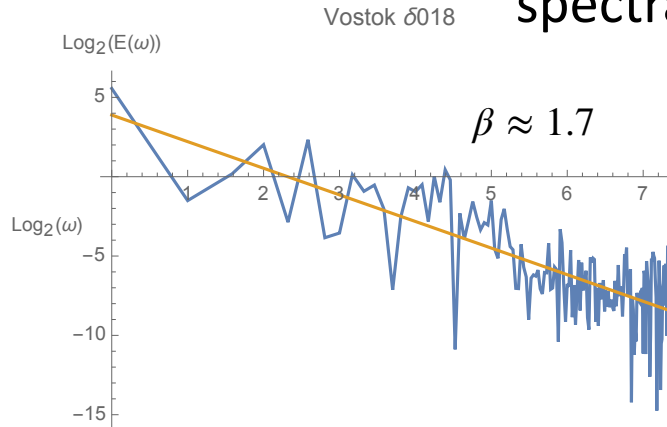
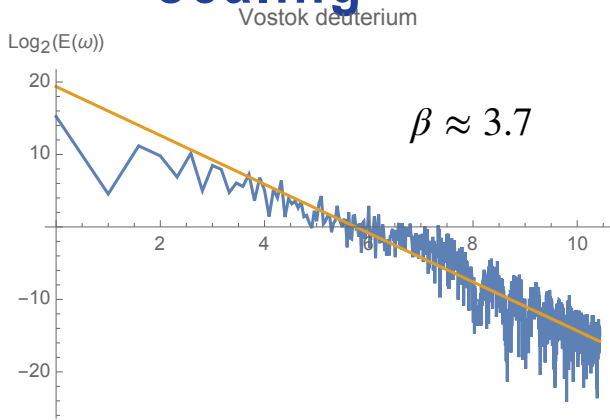
СТО ГОДИНА
МИЛАНКОВИЋЕВЕ ТЕОРИЈЕ
О КЛИМАТСКИМ
ПРОМЕНАМА

1920 2021 20

100 YEARS of
MILANKOVIC'S THEORY of
CLIMATE CHANGE



spectra: $\text{Log}(E(\omega))$ vs. $\text{Log}(\omega)$

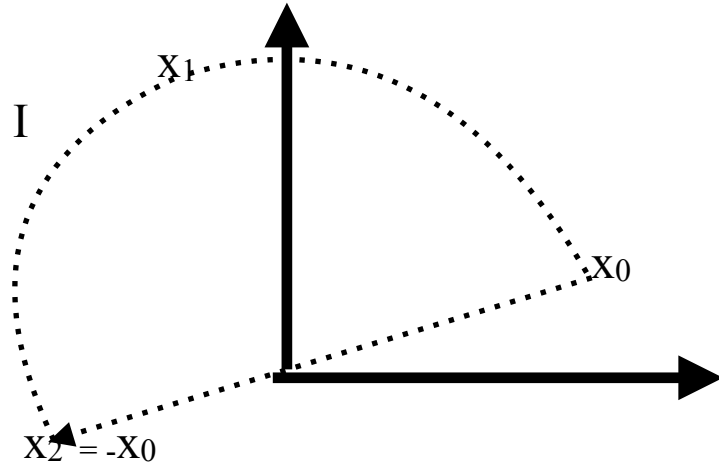


$$E(f) \approx f^{-\beta}$$

Strong nonlinearity: you need “integrated spectra” to see the astronomical spikes

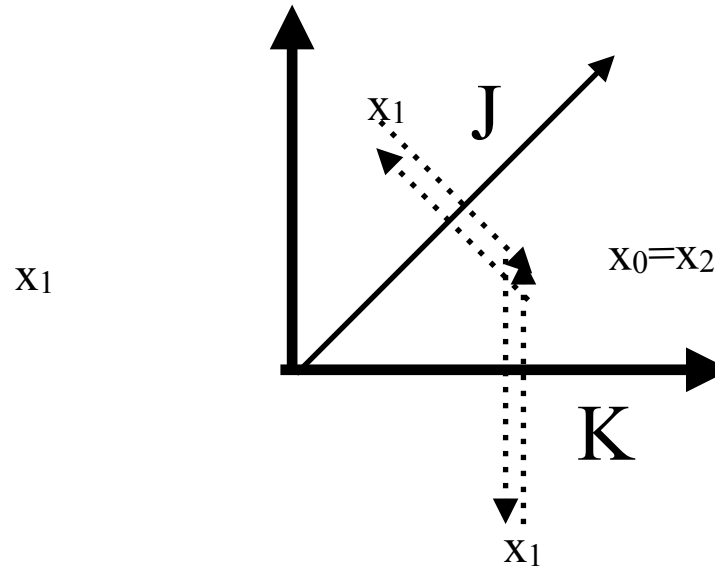
Symmetries and unity roots

$I = \text{rot}(\pi/2)$



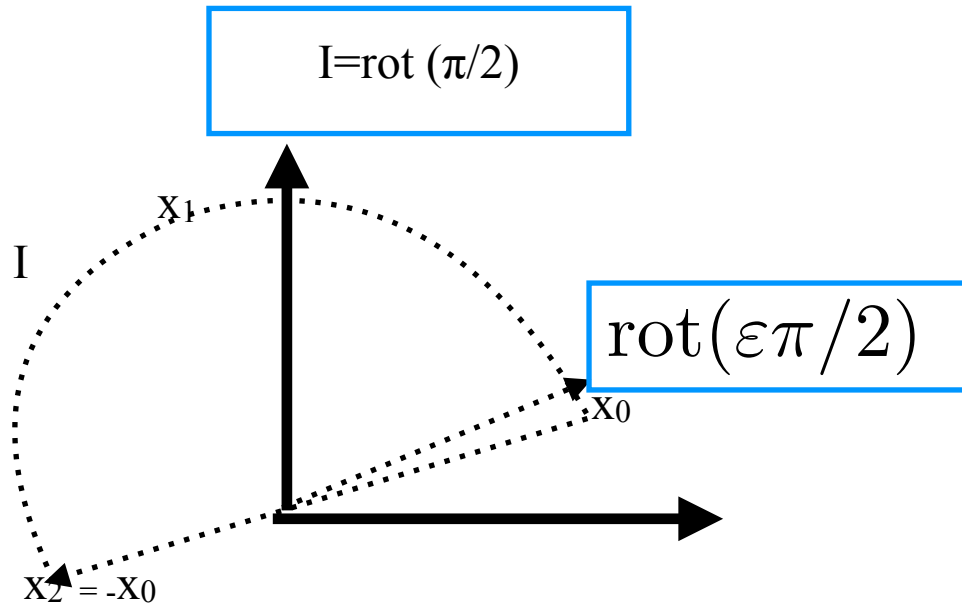
$$I^2 = -1$$

$J, K = \text{mirror symmetries}$

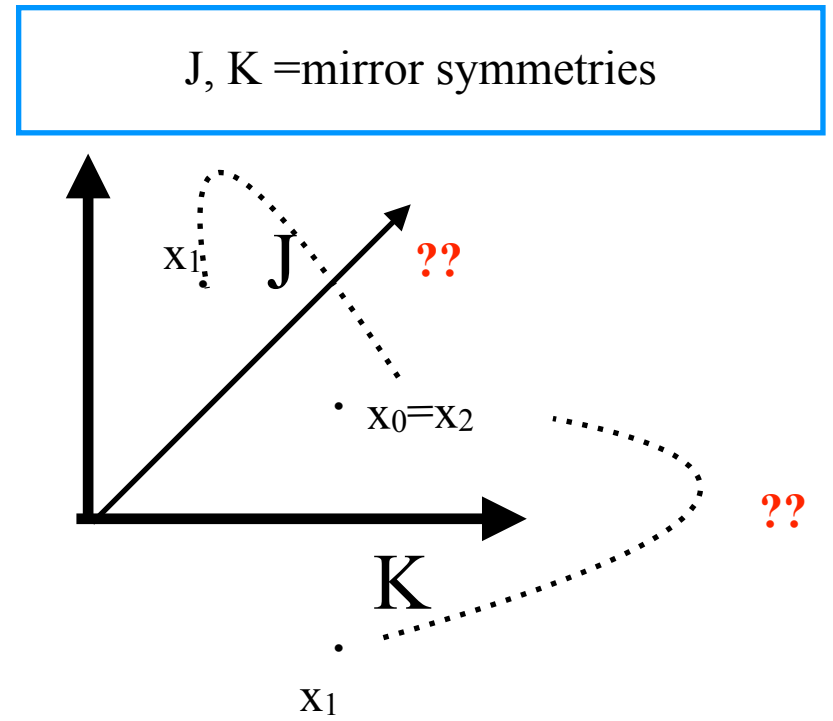


$$K^2 = J^2 = 1$$

Symmetries and unity roots

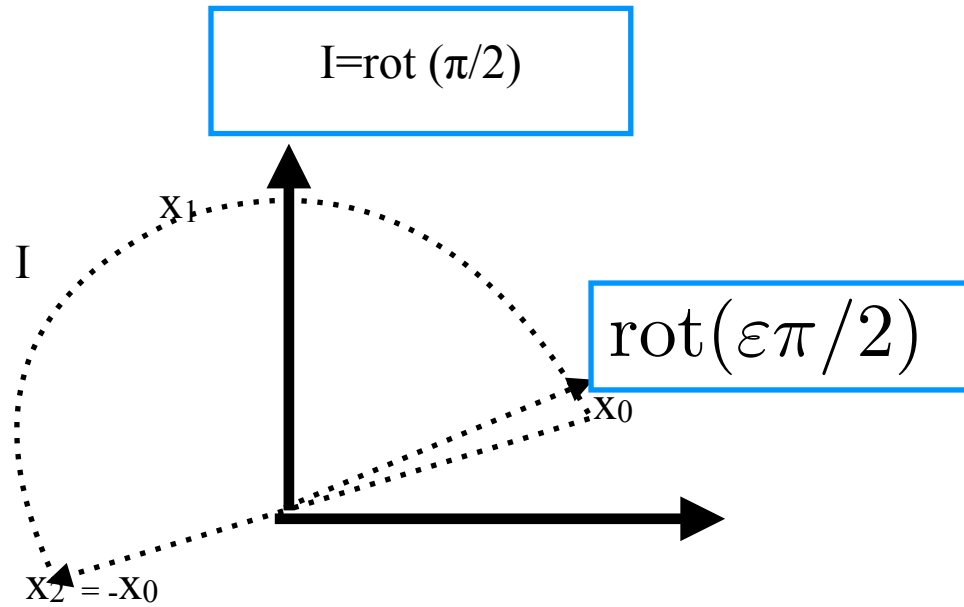


$I^2 = -1$

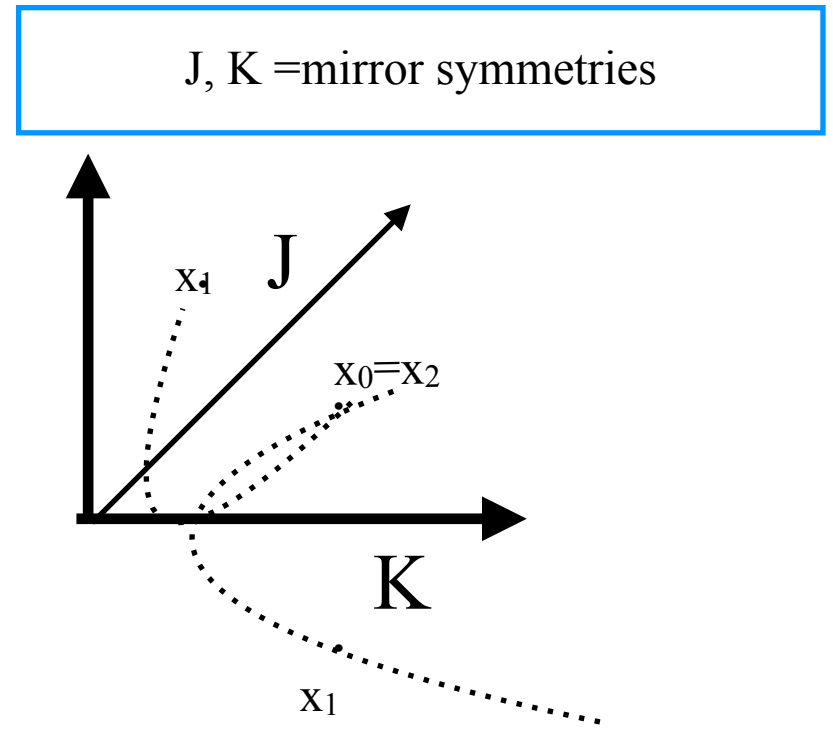


$K^2 = J^2 = 1$

Symmetries and unity roots



$$I^2 = -1$$



$$K^2 = J^2 = 1$$

Spherical geometry \longrightarrow **Hyperbolic geometry**

Combining symmetries

2D linear Lie algebra $\mathfrak{H}' = \mathfrak{so}(2, \mathbb{R})$:

$$G = d\mathbf{1} + e\mathbf{I} + f\mathbf{J} + c\mathbf{K};$$

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$2I = [J, K], \quad 2J = [I, K], \quad 2K = [J, I]$$

anti-commutators:

$$\{I, J\} = \{J, K\} = \{K, I\} = 0$$

$$I^2 = -J^2 = -K^2 = IJK = -1$$

(pseudo or split quaternions)



“quaternion equation” (Hamilton, 16/10/1843)

$$I_2^2 = J_2^2 = K_2^2 = I_2 J_2 K_2 = -1$$

Combining symmetries

2D linear Lie algebra $\mathfrak{H}' = \mathfrak{I}(2, \mathbb{R})$:

$$G = d\mathbf{1} + e\mathbf{I} + f\mathbf{J} + c\mathbf{K};$$

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$2I = [J, K], \quad 2J = [I, K], \quad 2K = [J, I]$$

anti-commutators:

$$\{I, J\} = \{J, K\} = \{K, I\} = 0$$

$$I^2 = -J^2 = -K^2 = IJK = -1$$



$$I_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; J_2 = \begin{bmatrix} 0 & -K \\ K & 0 \end{bmatrix}; K_2 = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

“quaternion equation” (Hamilton, 16/10/1843)

$$I_2^2 = J_2^2 = K_2^2 = I_2 J_2 K_2 = -1$$

Algebra of cascade generators

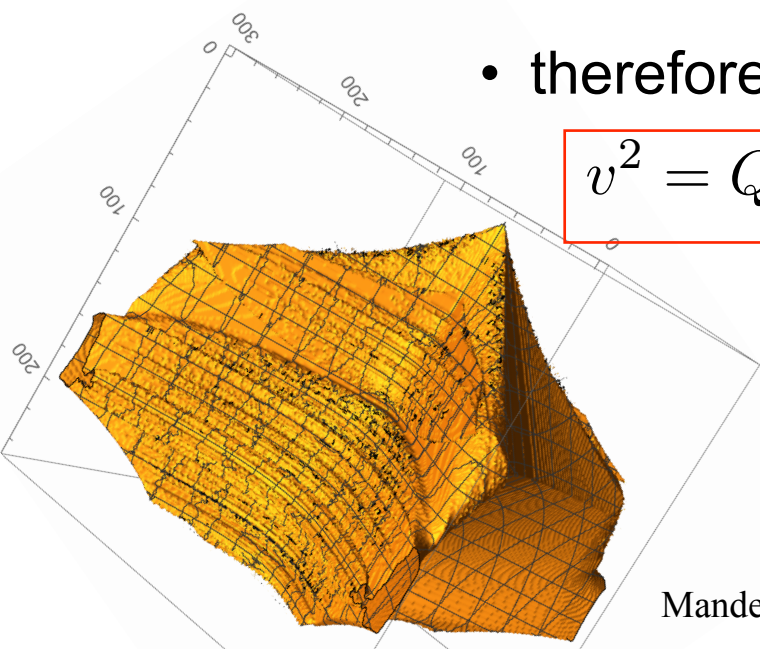
- Clifford algebra, dimension = 2^n
 - real numbers R ($n=0$), complex numbers C ($n=1$), quaternions H ($n=2$) other hyper-complex numbers, external algebras and many more!
- $Cl_{p,q}$: generated by operators $\{e\}$ that anti-commute and square to plus or minus the identity:

$$e^i e^j = -e^j e^i \quad (i \neq j) \quad (e^i)^2 = \pm 1$$

- therefore a quadratic form Q of signature $(p, q, p+q=n)$:

$$v^2 = Q(v)1 \quad Q(v) = v_1^2 + v_2^2 \dots + v_p^2 - v_{p+1}^2 - v_{p+2}^2 \dots - v_{p+q}^2$$

ex.: $R = Cl_{0,0}$; $C = Cl_{0,1}$; $H = Cl_{0,2}$
 $H' = I(2, R) = Cl_{2,0} = Cl_{1,1}$
 “pseudo-/split- quaternions”



From algebra to group

$\{K, I\}$

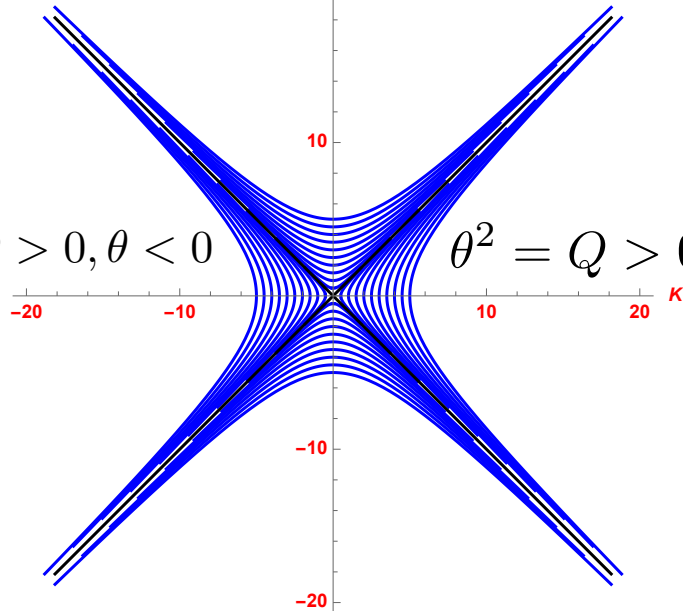
Hyperbolic Geometry

$$\theta^2 = Q < 0, \theta = i\theta', \theta' \in R^+$$

$$\theta^2 = Q > 0, \theta < 0$$

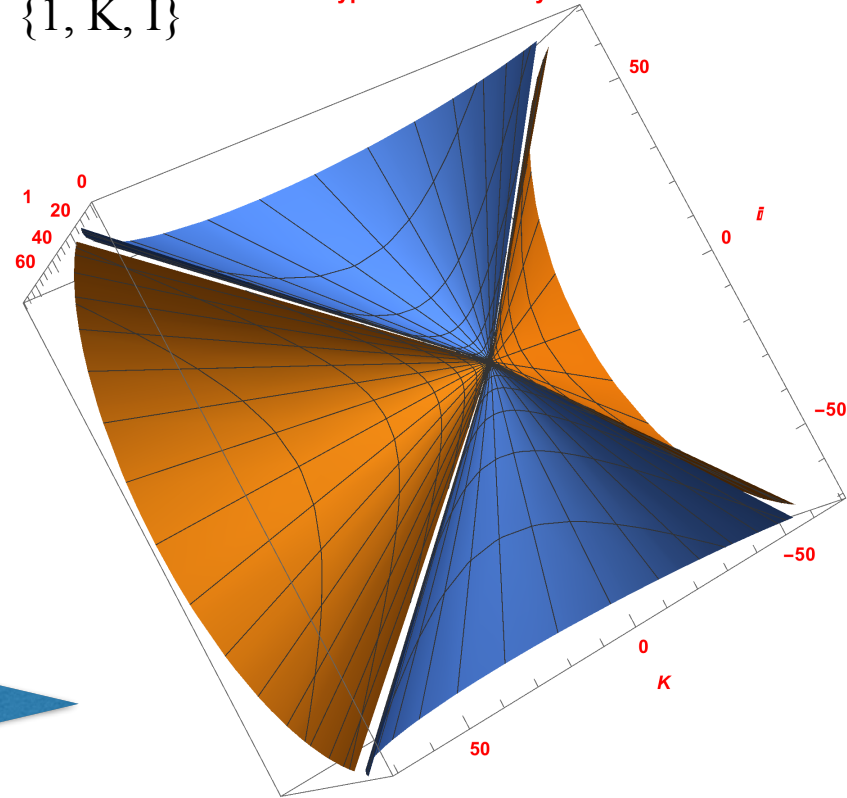
$$\theta^2 = Q > 0, \theta > 0$$

$$\theta^2 = Q < 0, \theta = i\theta', \theta' \in R^-$$



$\{1, K, I\}$

Hyperbolic Geometry



Exp



Generalised Moivre-Euler formula: $(e^{u\theta})^\alpha = \cosh(\alpha\theta)1 + \sinh(\alpha\theta)u$

infinite number of u , $u^2 = \pm 1!$

Stochastic Clifford?

- Statistical universality: stable Lévy vectors

$$\forall n \in \mathbb{N}, \exists a(n), b(n) \in \mathbb{R} : \sum_{i=1}^n X_i \stackrel{d}{=} a(n)X + b(n)$$

$\exists \alpha \in (0, 2] : a(n) = n^{1/\alpha}; \alpha < 2, \forall s \gg 1 : P(|X| > s) \approx s^{-\alpha}$ (hyperbolic/Pareto tail)
 $\alpha = 2 : \text{Gauss}$

A stable Levy X is attractive for any Y_i having same type of tail:

$$\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n Y_i - b(n)}{a(n)} \stackrel{d}{=} X$$

- classical “quasi- scalar” case: only b is a vector like X_i and Y_i
- ‘real’ vector case: a and α are matrices (S. et al., 2001)

Exponentiation of Lévy-Clifford algebra

- Existence ?

- Q defines a bilinear form $\langle \cdot, \cdot \rangle$

$$\langle X, Y \rangle = \frac{1}{2} (Q(X + Y) - Q(X) - Q(Y))$$

- which defines a Laplace-Clifford transform,

- hence a second characteristic function (cumulant generating function)

$$E \exp(\langle q, \Gamma_\lambda \rangle) = Z_\lambda(q) = \exp(K_\lambda(q))$$

finite over set of cones \mathcal{A}^\downarrow

the opposite cones to that supporting the extremely
assymmetric Lévy stable component \mathcal{A}^\uparrow

Adjoint analysis of operator cascades

- The adjoint representation of any \mathfrak{g} Lie algebra is the canonical linear representation defined by

$$\forall X \in \mathfrak{g}, \text{ad} : X \rightarrow \text{ad}X \in L(\mathfrak{g}) \qquad \forall Y \in \mathfrak{g} : \text{ad}X(Y) = [X, Y]$$

– it defines the **bilinear Killing form**:

$$K(X, Y) = \text{Tr}(\text{ad}X \cdot \text{ad}Y)$$

a new difficulty is that K is degenerate on the radical of \mathfrak{g}

Fractionnaly Integrated Flux model (FIF, vector version)

FIF assumes that both the renomalized propagator G_R and force f_R are known:

$$G_R^{-1} * u = f_R$$

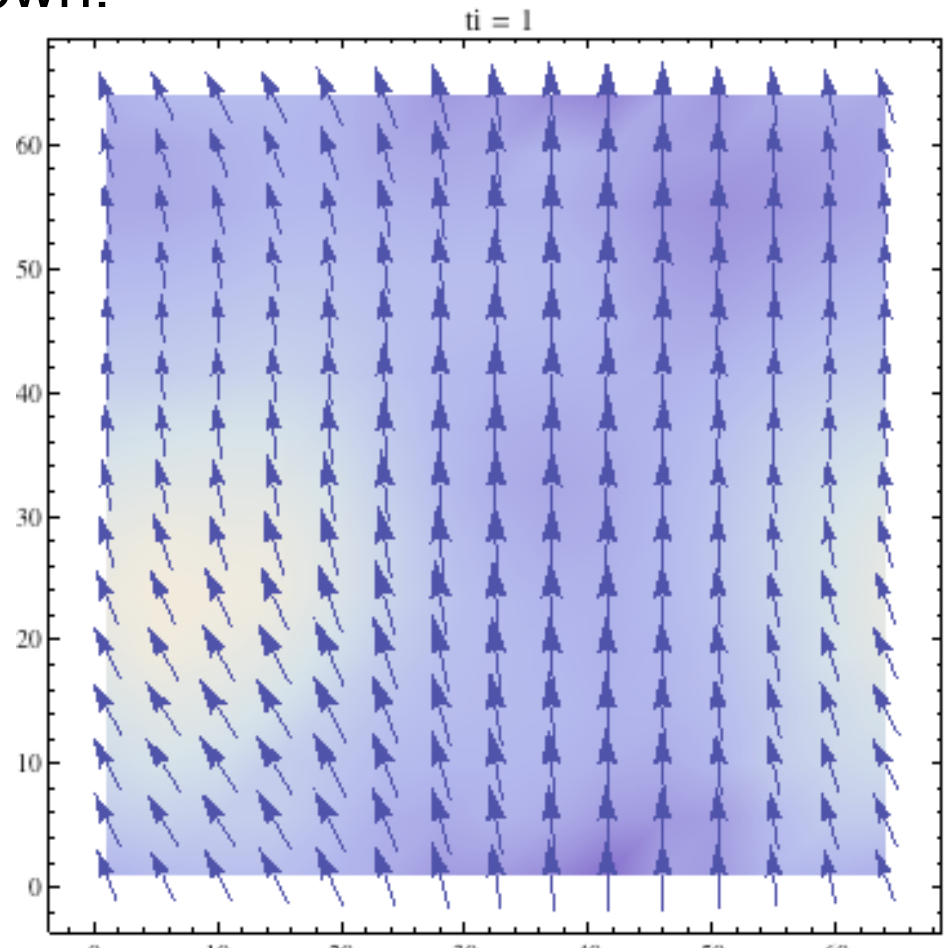
where:

$$f_R = \varepsilon^a$$

G_R^{-1} is a fractionnal differential operator

ε results from a continuous, vector, multiplicative cascade (Lie cascade)

Complex FIF simulation of a 2D cut of wind and its vorticity (color)



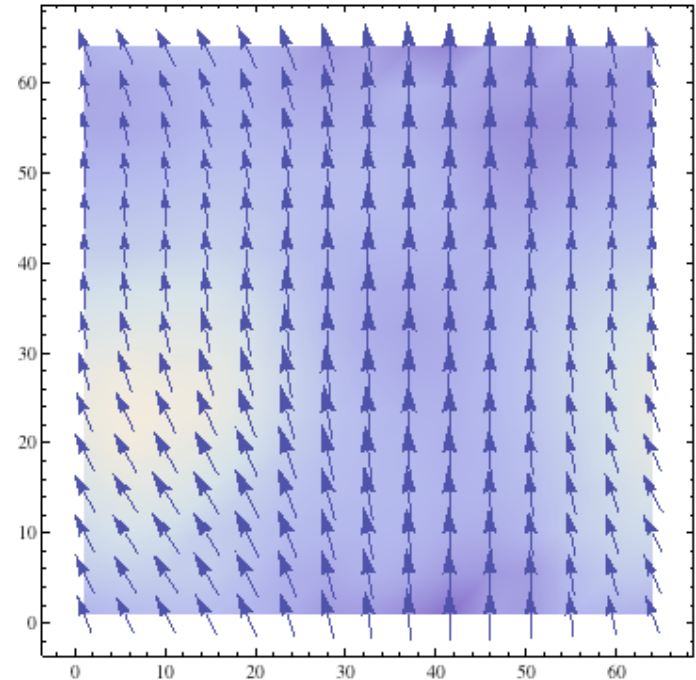
Surface layer complexity!

explOatorium®



Art piece 'Windswept' (Ch. Sowers, 2012): 612 freely rotating wind direction indicators to help a large public to understand the complexity of environment near the Earth surface

WAUDIT Wind resource assessment
audit and standardization



Multifractal FIF simulation (S et al., 2013) of a 2D+1 cut of wind and its vorticity (color). This stochastic model has only a few parameters that are physically meaningful.

Both movies illustrate the challenge of the near surface wind that plays a key role in the heterogeneity of the precipitations... and wind energy!

Fractionnaly Integrated Flux model (FIF, vector version)

FIF assumes that both the renomalized propagator G_R and force f_R are known:

$$G_R^{-1} * u = f_R$$

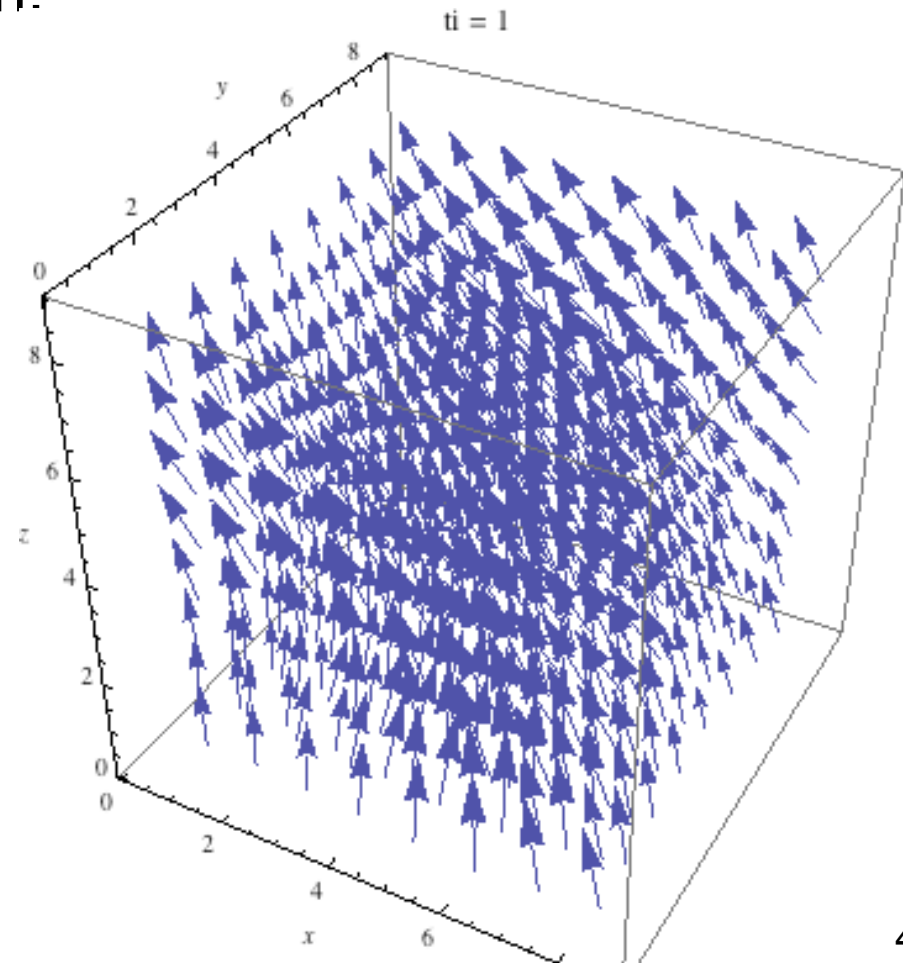
where:

$$f_R = \varepsilon^a$$

G_R^{-1} is a fractionnal differential operator

ε results from a continuous, vector, multiplicative cascade (Lie cascade)

3D FIF wind simulation based on quaternions

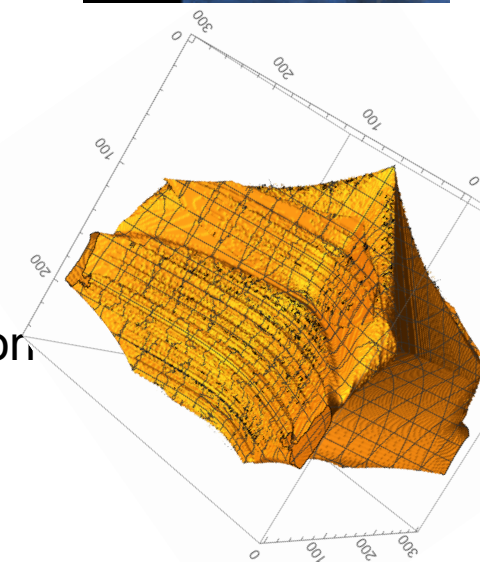
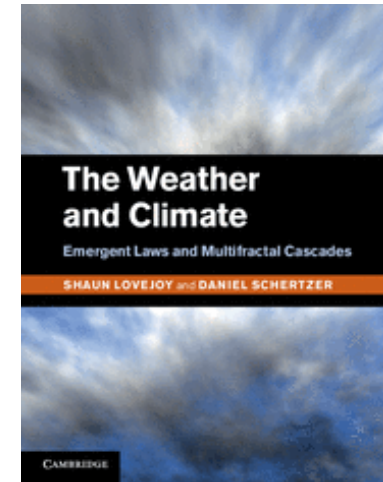


Conclusions

S&T, Earth& Space, 2020
Chaos 2015, S&al. ACP, 2012,
S&L, IJBC, 2011,
Fitton&al., JMI 2013

- Intermittency: a key issue in geophysics and a major breakthrough with multifractals in the 1980's:
 - infinite hierarchy of fractal supports of the field singularities
 - beyond commonalities significant differences of approaches and applications
- No longer limited to scalar valued fields
 - **multifractal operators**: exponentiation from a stochastic Lie algebra of generators onto its Lie group of transformations
 - ex. **Clifford algebra** $Cl_{p,q}$
 - physically meaningful and convenient to understand, analyse & simulate intermittent vector fields, more generally multidimensional systems.

=> from field physics to singularity physics



Conclusions

S&T, Earth& Space, 2020
Chaos 2015, S&al. ACP, 2012,
S&L, IJBC, 2011,
Fitton&al., JMI 2013

Final conclusion: the Nobel Committee for Physics is right to quote the saying reported by Philip Anderson (Phys. Today 41 526 1988):

“A real scientific mystery is worth pursuing to the ends of the Earth for its own sake, independently of any obvious practical importance or intellectual glamour.”

Intermittency is without doubt such a mystery, but not without multifaceted practical importance.

More than illustrated by thousands of communications in EGS/EGU NP since 1988, in particular those of my colleagues of HM&Co and CNRS GDRs Turbulence and Multifractals.

