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# ON THE HOMOGENIZATION OF THE RENEWAL EQUATION WITH HETEROGENEOUS EXTERNAL CONSTRAINTS

ÉTIENNE BERNARD AND FRANCESCO SALVARANI

ABSTRACT. We study the homogenization limit of the renewal equation with heterogeneous external constraints by means of the two-scale convergence theory. We prove that the homogenized limit satisfies an equation involving non-local terms, which are the consequence of the oscillations in the birth and death terms. We have moreover shown that the numerical approximation of the homogenized equation via the two-scale limit gives an alternative way for the numerical study of the solution of the limiting problem.

## 1. INTRODUCTION

Several mathematical models aim to describe cell dynamics, i.e. a process by which a parent cell divides into two or more daughter cells and then, at the end of its life cycle, it dies (see, for example, [3, 5, 6, 18]).

The first PDE model describing such a phenomenon is the well-known McKendrick model [13], which has first been introduced in the context of epidemiology, and then it has been used for modelling cell cultures evolution by von Foerster [26].

This equation has been widely mathematically studied and has been used as starting point for more elaborate models in biology and epidemiology, we refer to [17, 25] for an overview on the main mathematical properties of the model and examples of development.

The independent variables in the McKendrick model are time and age. Hence, it does not take into account possible spatial heterogeneity of the various terms appearing in it. However, there are several situations in which the cell evolution is influenced by the local properties of the host medium (or substrate).

In the literature, many studies have been concerned with determining the factors influencing growth, development, fission and death of bacteria. Some main factors are well known, such as the pH of the substrate, its temperature, the availability of chemical nutrients. It is indeed well known that these features have a major impact on cell metabolism (among the vast bibliography on the subject, see, for example, [9, 24, 14, 16, 19, 21, 29]). This fact has led to a mathematical literature taking external constraints into account in the model (for a recent survey, see [28]). For linear models taking into account the spatial constraint, the existence and the asymptotic behavior of the solution are well-understood. Yet, the spatial constraint can be heterogeneous on a microscopic scale, see for instance [20] in the case of biofilms, which could lead to costly numerical simulations.

In this article we investigate, from the mathematical viewpoint, how local heterogeneity in the properties of the substrate can modify the behaviour of the cell population with respect to the averaged case. In particular, we consider a periodic spatial structure with high spatial frequency.

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It could mean, for example, that we are able to take into account the local temperature variability in the substrate, or variations in the presence of micronutrients. Our approach is based on the two-scale convergence theory, first proposed by Gabriel Nguetseng [15] and then developed by Grégoire Allaire [1].

From a formal point of view, as we will see below, the renewal equation is a transport equation with a non local boundary condition encoding birth conditions as a function of the total population, and the absorption term encoding death phenomena. There are many studies around the homogenization of the transport equation with or without scattering term, see [7] and references therein. There are also works that take into account the oscillations of the vector field (for example [2, 10, 4, 8]). We underline that the non-local boundary condition, which expresses birth conditions, is a further difficulty with respect to the papers cited above. Our analysis shows that the homogenized limit satisfies an equation involving non-local terms. This behaviour is coherent with a phenomenon observed by Luc Tartar in the context of ordinary differential equations with oscillating terms [22] and also observed by Jiann-Sheng Jiang for the two-scale convergence applied to a first order differential equation [12]. In our case, both the oscillations in the equation and in the boundary conditions have an influence on the emergence of non-local effects in the homogenization procedure.

The structure of the paper is the following. We first describe the problem in Section 2. Then, after a concise description of the two-scale convergence theory (Section 3), we study the two-scale limit in Section 4 and provide some numerical simulations of the target equations, based on the two-scale limit, in Section 5.

## 2. THE RENEWAL EQUATION WITH HETEROGENEOUS EXTERNAL CONSTRAINTS

The renewal equation is one of the standard models which describe the vital dynamics of a population of cells, structured by age, with birth and death phenomena (for a deep study on this equation without external constraints, see [17]).

In this article, we suppose that the vital dynamics is influenced by the properties of the substrate. In what follows, we suppose that the space variable belongs to the interval  $X = (0, 1)$  and that the time horizon  $\tau \in \mathbb{R}_+^*$  of the problem is strictly positive and finite.

The age-dependent spatial density is described by a function  $u : \mathbb{R}_+^* \times (0, \tau) \times X \rightarrow \mathbb{R}_+$ , defined a.e.. Here and in what follows,  $a \in \mathbb{R}_+^*$  is the age variable,  $t \in [0, \tau)$  denotes the time variable and  $x \in X$  denotes the space variable.

We suppose that the speeds of the birth and death processes have a local dependence in space (for example, we suppose that the division process is mediated by some properties of a heterogeneous substrate). We introduce the heterogeneous (in space) age-dependent birth rate  $\sigma_b : \mathbb{R}_+^* \times X \rightarrow \mathbb{R}_+^*$  and the heterogeneous age-dependent death rate  $\sigma_d : \mathbb{R}_+^* \times X \rightarrow \mathbb{R}_+^*$ .

Under the previous assumptions, the evolution of the density  $u$  satisfies the following equation:

$$(1) \quad \partial_t u(a, t, x) + \partial_a u(a, t, x) = -\sigma_d(a, x)u(a, t, x), \quad (a, t, x) \in \mathbb{R}_+^* \times (0, \tau) \times X$$

with boundary condition

$$(2) \quad u(0, t, x) = \int_0^{+\infty} \sigma_b(\alpha, x)u(t, \alpha, x) d\alpha, \quad (t, x) \in (0, \tau) \times X$$

and initial condition

$$(3) \quad u(a, 0, x) = u_{\text{in}}, \quad (a, x) \in \mathbb{R}_+^* \times X.$$

The well-posedness of the problem has been studied by several authors, see for instance [27, 28] and references therein. We provide here an alternative proof, suitable for our goals, which guarantees existence and uniqueness of the solution in a bounded time interval. We underline that the regularity properties of the birth and death rates are different from those required in [28]. Let  $\sigma_d$  and  $\sigma_b$  be two non-negative functions of class  $L^\infty(X; L^2(\mathbb{R}_+^*))$ . If the initial condition  $u_{\text{in}} \in L^\infty(X; L^2(\mathbb{R}_+^*))$  and  $u_{\text{in}} \geq 0$  for a.e.  $(a, x) \in \mathbb{R}_+^* \times X$ , then the initial value problem (1)–(2)–(3) has one and only one non-negative strong solution  $u \in L^\infty((0, \tau) \times X; L^2(\mathbb{R}_+^*))$ .

*Proof.* We write the initial value problem (1)–(2)–(3) in integral form, by using the method of characteristics. We deduce that, for a.e.  $(a, t, x) \in \mathbb{R}_+^* \times (0, \tau) \times X$ ,

$$(4) \quad \begin{aligned} u(a, t, x) &= \mathbf{1}_{t < a} u_{\text{in}}(a - t, x) \exp\left(-\int_0^t \sigma_d(s + a - t, x) ds\right) \\ &+ \mathbf{1}_{t > a} \left[ \int_0^{+\infty} \sigma_b(\alpha, x) u(\alpha, t - a, x) d\alpha \right] \exp\left(-\int_{t-a}^t \sigma_d(s + a - t, x) ds\right). \end{aligned}$$

If we denote

$$(5) \quad F(u_{\text{in}}, \sigma_d) := \mathbf{1}_{t < a} u_{\text{in}}(a - t, x) \exp\left(-\int_0^t \sigma_d(s + a - t, x) ds\right)$$

and, for any  $h \in L^\infty((0, \tau) \times X; L^2(\mathbb{R}_+^*))$ ,

$$Th := \mathbf{1}_{t > a} \left[ \int_0^{+\infty} \sigma_b(\alpha, x) h(\alpha, t - a, x) d\alpha \right] \exp\left(-\int_{t-a}^t \sigma_d(s + a - t, x) ds\right),$$

Equation (4) can be seen as a fixed-point problem:

$$(6) \quad u = F(u_{\text{in}}, \sigma_d) + Tu.$$

In what follows, we will show that

$$(7) \quad u := \sum_{n=0}^{+\infty} T^n F(u_{\text{in}}, \sigma_d)$$

is well-defined and is the unique solution of Equation (4) in  $L^\infty((0, \tau) \times X; L^2(\mathbb{R}_+^*))$ , which is embedded in  $L^2(\mathbb{R}_+^* \times (0, \tau) \times X)$  because the set  $(0, \tau) \times X$  has finite Lebesgue measure in  $\mathbb{R}^2$ .

The space  $L^\infty((0, \tau) \times X; L^2(\mathbb{R}_+^*))$  is a Banach space with norm

$$\|g\|_{L^\infty((0, \tau) \times X; L^2(\mathbb{R}_+^*))} = \text{esssup}_{(t, x) \in (0, \tau) \times X} \left( \int_0^{+\infty} |g(a, t, x)|^2 da \right)^{1/2}.$$

It is clear that  $T : L^\infty((0, \tau) \times X; L^2(\mathbb{R}_+^*)) \rightarrow L^\infty((0, \tau) \times X; L^2(\mathbb{R}_+^*))$  and that it is linear.

It is moreover bounded because

$$\begin{aligned}
\|Th\|_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))}^2 &= \operatorname{ess\,sup}_{(t,x)\in(0,\tau)\times X} \left( \int_0^{+\infty} |Th(a,t,x)|^2 \, da \right) \\
&= \operatorname{ess\,sup}_{(t,x)\in(0,\tau)\times X} \int_0^{+\infty} \left( \mathbf{1}_{t>a} \left[ \int_{\mathbb{R}_+^*} \sigma_b(\alpha,x) h(\alpha,t-a,x) \, d\alpha \right] \right. \\
&\quad \left. \exp \left( - \int_{t-a}^t \sigma_d(s+a-t,x) \, ds \right) \right)^2 \, da \\
&\leq \|\sigma_b\|_{L^\infty(X;L^2(\mathbb{R}_+^*))}^2 \operatorname{ess\,sup}_{(t,x)\in(0,\tau)\times X} \int_0^t \int_{\mathbb{R}_+^*} h^2(\alpha,t-a,x) \, d\alpha \, da \\
&= \|\sigma_b\|_{L^\infty(X;L^2(\mathbb{R}_+^*))}^2 \|h\|_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))}^2 \tau,
\end{aligned}$$

that is

$$\|T\|_{\mathcal{L}(L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*)))} \leq \|\sigma_b\|_{L^\infty(X;L^2(\mathbb{R}_+^*))} \sqrt{\tau}.$$

We can generalize the previous computations by studying the bound on the iterated operator  $T^n$  with respect to the norm  $\|\cdot\|_{\mathcal{L}(L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*)))}$ . We deduce that, for any  $n \in \mathbb{N}^*$  and for any  $h \in L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))$ , when  $t \in (0,\tau)$ ,

$$\begin{aligned}
&\operatorname{ess\,sup}_{x \in X} \int_0^{+\infty} |T^n h(a,t,x)|^2 \, da \\
&\leq \|\sigma_b\|_{L^\infty(X;L^2(\mathbb{R}_+^*))}^2 \operatorname{ess\,sup}_{x \in X} \int_0^t \|T^{n-1} h(\cdot, t-\cdot, x)\|_{L^2(\mathbb{R}_+^*)}^2 \, da \\
&= \|\sigma_b\|_{L^\infty(X;L^2(\mathbb{R}_+^*))}^2 \operatorname{ess\,sup}_{x \in X} \int_0^t \|T^{n-1} h(\cdot, t_1, x)\|_{L^2(\mathbb{R}_+^*)}^2 \, dt_1 \\
&\leq \|\sigma_b\|_{L^\infty(X;L^2(\mathbb{R}_+^*))}^{2n} \operatorname{ess\,sup}_{x \in X} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \|h(\cdot, t_n, x)\|_{L^2(\mathbb{R}_+^*)}^2 \, dt_n \\
&\leq \frac{1}{n!} \left( \|\sigma_b\|_{L^\infty(X;L^2(\mathbb{R}_+^*))}^2 \right)^n \|h\|_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))}^2 \tau^n.
\end{aligned}$$

By passing to the supremum for  $t \in (0,\tau)$  in both sides of the previous inequality and noticing that the right-hand side is independent on  $t$ , we have that

$$\|T^n\|_{\mathcal{L}(L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*)))} \leq \frac{1}{\sqrt{n!}} \left( \|\sigma_b\|_{L^\infty(X;L^2(\mathbb{R}_+^*))} \sqrt{\tau} \right)^n$$

for all  $n \in \mathbb{N}^*$ .

Because of (5), we have that

$$(8) \quad \|F(u_{\text{in}}, \sigma_d)\|_{L^\infty((0,\tau)\times X;L^2(\mathbb{R}_+^*))} \leq \|u_{\text{in}}\|_{L^\infty(X;L^2(\mathbb{R}_+^*))}.$$

Consequently,

$$\begin{aligned} & \sum_{n=0}^{+\infty} \|T^n F(u_{\text{in}}, \sigma_d)\|_{L^\infty((0,\tau) \times X; L^2(\mathbb{R}_+^*))} \\ & \leq \|F(u_{\text{in}}, \sigma_d)\|_{L^\infty((0,\tau) \times X; L^2(\mathbb{R}_+^*))} \sum_{n=0}^{+\infty} \frac{1}{\sqrt{n!}} \left( \|\sigma_b\|_{L^\infty(X; L^2(\mathbb{R}_+^*))} \sqrt{\tau} \right)^n \\ & \leq \|u_{\text{in}}\|_{L^\infty(X; L^2(\mathbb{R}_+^*))} \sum_{n=0}^{+\infty} \frac{1}{\sqrt{n!}} \left( \|\sigma_b\|_{L^\infty(X; L^2(\mathbb{R}_+^*))} \sqrt{\tau} \right)^n. \end{aligned}$$

The right-hand side of the previous inequality is a numerical series which converges thanks to the ratio test. Therefore,  $u$  defined as in (7) exists and is norm-bounded. Moreover,  $u$  solves the integral problem (6) because

$$\begin{aligned} u &= \sum_{n=0}^{+\infty} T^n F(u_{\text{in}}, \sigma_d) = F(u_{\text{in}}, \sigma_d) + \sum_{n=1}^{+\infty} T^n F(u_{\text{in}}, \sigma_d) \\ &= F(u_{\text{in}}, \sigma_d) + T \sum_{n=0}^{+\infty} T^n F(u_{\text{in}}, \sigma_d) = F(u_{\text{in}}, \sigma_d) + Tu. \end{aligned}$$

The solution is moreover unique. Indeed, suppose that there exist two distinct (in a.e. sense) solutions  $u_1$  and  $u_2$  of (6). Then, by finite induction, their difference  $u_1 - u_2$  is such that, for any  $n \in \mathbb{N}^*$

$$u_1 - u_2 = T(u_1 - u_2) = T^2(u_1 - u_2) = \dots = T^n(u_1 - u_2).$$

By passing to the norm

$$\begin{aligned} & \|u_1 - u_2\|_{L^\infty((0,\tau) \times X; L^2(\mathbb{R}_+^*))} \\ &= \|T^n\|_{\mathcal{L}(L^\infty((0,\tau) \times X; L^2(\mathbb{R}_+^*)))} \|u_1 - u_2\|_{L^\infty((0,\tau) \times X; L^2(\mathbb{R}_+^*))} \\ &\leq \frac{1}{\sqrt{n!}} \left( \|\sigma_b\|_{L^\infty(X; L^2(\mathbb{R}_+^*))} \sqrt{\tau} \right)^n \|u_1 - u_2\|_{L^\infty((0,\tau) \times X; L^2(\mathbb{R}_+^*))} \rightarrow 0 \end{aligned}$$

as  $n \rightarrow +\infty$ . But  $u_1$  and  $u_2$  are distinct by hypothesis. Hence we have a contradiction and so the solution is unique.  $\square$

### 3. BASIC CONCEPTS OF TWO-SCALE CONVERGENCE

The concept of two-scale convergence has been introduced by Gabriel Nguetseng [15] and developed by Grégoire Allaire [1]. Its definition is the following: Let  $X$  be a domain of  $\mathbb{R}^d$  and  $Y = (0, 1)^d$ . Denote with  $C_{\text{per}}(Y)$  the space of continuous functions on  $Y$  which are  $Y$ -periodic. A family of functions  $z^\varepsilon(x) \subset L^2(X)$  two-scale converges to a limit  $z^0(x, y) \in L^2(X \times Y)$  if, for any test function  $\psi(x, y) \in L^2(X; C_{\text{per}}(Y))$ , we have

$$\lim_{\varepsilon \rightarrow 0} \int_X z^\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_X \int_Y z^0(x, y) \psi(x, y) dx dy.$$

The following compactness result is crucial for using the two-scale convergence theory. Let  $X$  be a domain of  $\mathbb{R}^d$  and  $Y = (0, 1)^d$ . Let  $z^\varepsilon(x) \subset L^2(X)$  be a uniformly bounded family of functions such that

$$\|z^\varepsilon\|_{L^2(X)} \leq C$$

where the constant  $C$  is independent of  $\varepsilon$ . Then, there exists a subsequence extracted from  $z^\varepsilon$  (still denoted  $z^\varepsilon$ ) such that  $z^\varepsilon$  two-scale converges to some limit  $z^0(x, y) \in L^2(X \times Y)$ .

Another important property of the two-scale limit is given by the following proposition. Let  $X$  be a domain of  $\mathbb{R}^d$  and  $Y = (0, 1)^d$ . Let  $z^\varepsilon \subset L^2(X)$  a family of functions which two-scale converges to a limit  $z^0 \in L^2(X \times Y)$ . Then  $z^\varepsilon(x)$  converges to

$$\langle z \rangle(x) = \int_Y z^0(x, y) \, dy$$

weakly in  $L^2(X)$ , that is

$$\lim_{\varepsilon \rightarrow 0} \int_X z^\varepsilon(x) \varphi(x) \, dx = \int_X \varphi(x) \int_Y z^0(x, y) \, dy \, dx \quad \text{for all } \varphi \in L^2(X).$$

Hence, the two-scale convergence, which is given in terms of test functions (see Definition 3), is a form of weak convergence which implies the standard weak convergence in  $L^2$ . However, the following result gives a sufficient condition for improving this weak-type convergence. Let  $X$  be a domain of  $\mathbb{R}^d$  and  $Y = (0, 1)^d$ . Let  $z^\varepsilon(x)$  be a family such that it two-scale converges to  $z^0(x, y)$ . Then

$$\lim_{\varepsilon \rightarrow 0} \|z^\varepsilon\|_{L^2(X)} \geq \|z^0\|_{L^2(X \times Y)} \geq \|\bar{z}\|_{L^2(X)}$$

where  $\bar{z}(x)$  is the weak  $L^2$ -limit of the family  $z^\varepsilon(x)$ . Moreover, if

$$(9) \quad \lim_{\varepsilon \rightarrow 0} \|z^\varepsilon\|_{L^2(X)} = \|z^0\|_{L^2(X \times Y)}$$

and if the two-scale limit  $z^0(x, y) \in L^2(X; C_{\text{per}}(Y))$ , then the following strong two-scale convergence holds

$$\lim_{\varepsilon \rightarrow 0} \left\| z^\varepsilon(\cdot) - z^0\left(\cdot, \frac{\cdot}{\varepsilon}\right) \right\|_{L^2(X)} = 0.$$

#### 4. TWO-SCALE HOMOGENIZATION OF THE RENEWAL EQUATION

We reformulate here the initial-value problem (1)–(2)–(3) by introducing a scale parameter  $\varepsilon \in \mathbb{R}_+^*$ . The parameter  $\varepsilon > 0$  represents the heterogeneity length scale of the problem. The smaller the value of the parameter  $\varepsilon$ , the smaller the period of the spatial oscillations for the initial conditions, for the birth rate and for the death rate.

**4.1. The initial-boundary value problem.** Our goal is to study the two-scale limit, as  $\varepsilon \rightarrow 0^+$ , of the following problem:

$$(10) \quad \partial_t u^\varepsilon(a, t, x) + \partial_a u^\varepsilon(a, t, x) = -\sigma_d^\varepsilon(a, x) u^\varepsilon(a, t, x), \quad (a, t, x) \in \mathbb{R}_+^* \times (0, \tau) \times X$$

with boundary conditions

$$(11) \quad u^\varepsilon(0, t, x) = \int_0^{+\infty} \sigma_b^\varepsilon(\alpha, x) u^\varepsilon(\alpha, t, x) \, d\alpha, \quad (t, x) \in (0, \tau) \times X$$

and initial conditions

$$(12) \quad u^\varepsilon(a, 0x) = u_{\text{in}}^\varepsilon(a, x), \quad (a, x) \in \mathbb{R}_+^* \times X.$$

Let

$$\sigma_b^* = \sigma_b^*(a, x, y), \quad \sigma_d^* = \sigma_d^*(a, x, y) \quad \text{and} \quad u_{\text{in}}^* = u_{\text{in}}^*(a, x, y).$$

The coefficients and data in (10) are of the form

$$\sigma_d^\varepsilon(a, x) := \sigma_d^* \left( a, x, \frac{x}{\varepsilon} \right), \quad \sigma_b^\varepsilon(a, x) := \sigma_b^* \left( a, x, \frac{x}{\varepsilon} \right)$$

and

$$u_{\text{in}}^\varepsilon(a, x) := u_{\text{in}}^* \left( a, x, \frac{x}{\varepsilon} \right),$$

with  $u_{\text{in}}^* \in L^2(\mathbb{R}_+^*; C(X; C_{\text{per}}(Y)))$ , where  $Y = (0, 1)$  and  $C_{\text{per}}(Y)$  denotes the space of continuous functions on  $Y$  which are  $Y$ -periodic.

We suppose that  $\sigma_d^*$  and  $\sigma_b^*$  are non-negative, locally periodic with respect to the last independent variable and that

$$(13) \quad \begin{aligned} \sigma_d^* \text{ and } \sigma_b^* &\in L^2(\mathbb{R}_+^*; C(X; C_{\text{per}}(Y))), \\ \sigma_d^\varepsilon \text{ and } \sigma_b^\varepsilon &\in L^\infty(X; L^2(\mathbb{R}_+^*)). \end{aligned}$$

Moreover, we assume that there exists a strictly positive constant  $\bar{\sigma}$  such that

$$(14) \quad \|\sigma_b^\varepsilon\|_{L^\infty(X; L^2(\mathbb{R}_+^*))} \leq \bar{\sigma} \quad \text{and} \quad \|\sigma_d^\varepsilon\|_{L^\infty(X; L^2(\mathbb{R}_+^*))} \leq \bar{\sigma}$$

Let  $\varphi \in L^2(\mathbb{R}_+^*; C(X; C_{\text{per}}(Y)))$ . The hypotheses on  $\sigma_d^*$  and  $\sigma_b^*$  guarantee that  $\sigma_d^* \varphi \in L^2(\mathbb{R}_+^*; C(X; C_{\text{per}}(Y)))$  and  $\sigma_b^* \varphi \in L^2(\mathbb{R}_+^*; C(X; C_{\text{per}}(Y)))$ .

**4.2. The homogenization procedure.** In the asymptotics as  $\varepsilon \rightarrow 0^+$ , we derive the corresponding homogenized equation. In this setting, the variable  $x$  is a parameter. Note that the regularity of the birth rate  $\sigma_b^*$  and of the death rate  $\sigma_d^*$ , as well as the composition with continuous functions like the exponential, makes them suitable as test functions in the two-scale convergence (see Definition 3).

Clearly, Theorem 2 guarantees the existence and the uniqueness of the solution of (10)-(11)-(12) for all  $\varepsilon > 0$ . Because of the boundedness of  $(0, \tau) \times X$ , we can deduce that  $u^\varepsilon \in L^2(\mathbb{R}_+^* \times (0, \tau) \times X)$  for all  $\varepsilon > 0$ .

Consider now the family  $(u^\varepsilon)_{\varepsilon > 0}$  of solutions to the initial-boundary value problem (10)-(11)-(12) and study the limit of the family as  $\varepsilon \rightarrow 0^+$ . By analogy with a remark by Tartar [23, 22], we can expect, for this system, the existence of memory effects induced by the two-scale homogenization procedure.

Denote with  $L_{\text{per}}^2(Y)$  the set of  $L^2$  functions on  $Y$  which are periodic in  $Y$ . For any  $g \in L^\infty(Y)$ , we introduce, as in [11], the linear operator

$$\mathcal{L}_g h := gh - \langle gh \rangle \quad \forall h \in L_{\text{per}}^2(Y),$$

We underline that the operator  $\mathcal{L}_g$  is bounded in  $L_{\text{per}}^2(Y)$  because

$$\|\mathcal{L}_g h\|_{L_{\text{per}}^2(Y)}^2 = \int_Y |g(y)h(y) - \langle gh \rangle|^2 dy = \int_Y |g(y)h(y)|^2 dy - \langle gh \rangle^2$$

and, by applying the Cauchy-Schwarz inequality,

$$|\langle gh \rangle| = \left| \int_Y g(y)h(y) dy \right| \leq \left( \int_Y |g(y)h(y)|^2 dy \right)^{1/2}.$$

We are now ready to prove our homogenization result for the evolution (10)-(11)-(12) in the framework of the two-scale convergence theory.



Let  $u^\varepsilon(t, x)$  be the solution of the evolution problem (10)-(11)-(12), with an  $\varepsilon$ -dependent initial condition  $u_{\text{in}}^\varepsilon \in L^2(\mathbb{R}_+^*; C(X; C_{\text{per}}(Y)))$ , and let  $u_{\text{in}}^0 \in L^2(\mathbb{R}_+^* \times X \times Y)$  be its two-scale limit.

Suppose moreover that the birth and death rates satisfy the hypotheses of Subsection 4.1. This implies that  $\sigma_b^\varepsilon$  and  $\sigma_d^\varepsilon$  two-scale converge to their two-scale limits  $\sigma_b^0$  and  $\sigma_d^0$  in  $L^2(\mathbb{R}_+^* \times X \times Y)$ .

Then

$$u^\varepsilon \rightharpoonup u_{\text{hom}} \quad \text{weakly in } L^2(\mathbb{R}_+^* \times (0, \tau) \times X)$$

and  $u_{\text{hom}}(t, x)$  solves the following integro-differential equation

$$(15) \quad \begin{cases} \partial_t u_{\text{hom}}(a, t, x) + \partial_a u_{\text{hom}}(a, t, x) = \\ \quad - \langle \sigma_d^0 \rangle(a, x) u_{\text{hom}}(a, t, x) - \langle \sigma_d^0 \sum_{n=0}^{+\infty} \mathcal{S}^n \mathcal{Q}(u_{\text{in}}^0, \sigma_d^0) \rangle(a, t, x) \\ u_{\text{hom}}(0, t, x) = \int_0^{+\infty} [\langle \sigma_b^0 \rangle(\alpha, x) u_{\text{hom}}(\alpha, t, x) + \langle \sigma_b^0 \sum_{n=0}^{+\infty} \mathcal{S}^n \mathcal{Q}(u_{\text{in}}^0, \sigma_d^0) \rangle(\alpha, t, x)] d\alpha, \\ u_{\text{hom}}(a, 0, x) = \langle u_{\text{in}}^0 \rangle(a, x), \end{cases}$$

where

$$\begin{aligned} \mathcal{S}h &:= \mathbf{1}_{t>a} e^{-\int_{t-a}^t \sigma_d^0(a+\theta-t, x, y) d\theta} \int_0^{+\infty} [\sigma_b^0(\alpha, x, y) h(\alpha, t-a, x, y) \\ &\quad - \int_Y \sigma_b^0(\alpha, x, y) h(\alpha, t-a, x, y) dy] d\alpha \\ &+ \int_{(t-a)_+}^t e^{-\int_s^t \sigma_d^0(a+\theta-t, x, y) d\theta} \int_Y \sigma_d^0(a, x, y) h(a+s-t, s, x, y) dy ds \end{aligned}$$

for any  $h \in L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)$  and

$$\mathcal{Q}(u_{\text{in}}^0, \sigma_d^0) := \mathbf{1}_{t<a} e^{-\int_0^t \sigma_d^0(a+\theta-t, x, y) d\theta} \left[ u_{\text{in}}^0(a-t, x, y) - \int_Y u_{\text{in}}^0(a-t, x, y) dy \right].$$

Note that the two-scale homogenized equation exhibits two memory terms, both in the equation and in the boundary conditions at  $a = 0$ . Moreover,  $u_{\text{in}}^\varepsilon$  two-scale converges to  $\langle u_{\text{in}}^0 \rangle$  because of the regularity hypothesis on the family  $(u_{\text{in}}^\varepsilon)$ .

*Proof.* The first step of the proof consists in studying the uniqueness of the solution of (15).

We know, by hypothesis, that the two-scale limit  $u_{\text{in}}^0 \in L^2(\mathbb{R}_+^*; C(X; C_{\text{per}}(Y)))$ . Notice that  $\langle u_{\text{in}}^0 \rangle \in L^\infty(X; L^2(\mathbb{R}_+^*))$  since  $L^2(\mathbb{R}_+^*; L^\infty(X)) \hookrightarrow L^\infty(X; L^2(\mathbb{R}_+^*))$ . By linearity of Equation (15), it suffices to show that any solution of equation (15) with 0 as initial condition is necessarily zero. Let  $v_{\text{hom}}$  be such a solution, then it satisfies:

$$\partial_t v_{\text{hom}}(a, t, x) + \partial_a v_{\text{hom}}(a, t, x) = - \langle \sigma_d^0 \rangle(a, x) v_{\text{hom}}(a, t, x),$$

for  $(a, t, x) \in \mathbb{R}_+^* \times (0, \tau) \times X$ , with boundary condition:

$$(16) \quad v_{\text{hom}}(0, t, x) = \int_0^{+\infty} \langle \sigma_b^0 \rangle(\alpha, x) v_{\text{hom}}(t, \alpha, x) d\alpha, \quad (t, x) \in (0, \tau) \times X$$

and initial condition

$$(17) \quad v_{\text{hom}}(a, 0, x) = 0, \quad (a, x) \in \mathbb{R}_+^* \times X.$$

By applying Theorem 2, with  $\langle \sigma_b^0 \rangle$  and  $\langle \sigma_d^0 \rangle$  instead of  $\sigma_b$  and  $\sigma_d$ , implies then the desired conclusion.

The existence proof is based on the integral form of the evolution problem (10)-(11)-(12):

$$u^\varepsilon(a, t, x) = \mathbb{1}_{t < a} u_{\text{in}}^* \left( a - t, x, \frac{x}{\varepsilon} \right) \exp \left( - \int_0^t \sigma_d^* \left( s + a - t, x, \frac{x}{\varepsilon} \right) ds \right) \\ + \mathbb{1}_{t > a} \left[ \int_{\star}^{+\infty} \sigma_b^* \left( \alpha, x, \frac{x}{\varepsilon} \right) u^\varepsilon(\alpha, t - a, x) d\alpha \right] \exp \left( - \int_{t-a}^t \sigma_d^* \left( s + a - t, x, \frac{x}{\varepsilon} \right) ds \right)$$

By applying the same strategy of Theorem 2, we deduce that, for all  $\varepsilon > 0$ ,

$$(18) \quad \|u^\varepsilon(t, \cdot, \cdot)\|_{L^\infty(X; L^2(\mathbb{R}_+^*))} \leq \|u_{\text{in}}^\varepsilon(t, \cdot)\|_{L^\infty(X; L^2(\mathbb{R}_+^*))} \sum_{n=0}^{+\infty} \frac{1}{\sqrt{n!}} \left( \|\sigma_b^\varepsilon\|_{L^\infty(X; L^2(\mathbb{R}_+^*))} \sqrt{\tau} \right)^n \\ \leq \|u_{\text{in}}^\varepsilon(t, \cdot)\|_{L^\infty(X; L^2(\mathbb{R}_+^*))} \sum_{n=0}^{+\infty} \frac{1}{\sqrt{n!}} (\bar{\sigma} \sqrt{\tau})^n =: C < +\infty,$$

thanks to the bounds (14). We have hence deduced an  $\varepsilon$ -uniform bound for all members of the family  $(u^\varepsilon)_{\varepsilon > 0}$ . Therefore, by Theorem 3, there exists a subsequence, still denoted  $u^\varepsilon$ , which two-scale converges to a function  $u^0 \in L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)$ , i.e.:

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}_+^* \times (0, \tau) \times X} u^\varepsilon(a, t, x) \psi \left( a, t, x, \frac{x}{\varepsilon} \right) dadtdx \\ = \int_{\mathbb{R}_+^* \times (0, \tau) \times X \times Y} u^0(a, t, x, y) \psi(a, t, x, y) dadtdxdy,$$

for any test-function  $\psi$  satisfying the regularity hypotheses of Definition 3.

We can hence deduce the following equality, in the sense of the two-scale limit and up to a subsequence :

$$u^0(a, t, x, y) = \mathbb{1}_{t < a} u_{\text{in}}^0(a - t, x, y) \exp \left( - \int_0^t \sigma_d^0(s + a - t, x, y) ds \right) \\ + \mathbb{1}_{t > a} \left[ \int_0^{+\infty} \sigma_b^0(\alpha, x, y) u^0(\alpha, t - a, x, y) d\alpha \right] \exp \left( - \int_{t-a}^t \sigma_d^0(s + a - t, x, y) ds \right).$$

Consequently, the limit  $u^0$  solves the two-scale evolution equation

$$(19) \quad \partial_t u^0(a, t, x, y) + \partial_a u^0(a, t, x, y) = -\sigma_d^0(a, x, y) u^0(a, t, x, y)$$

for any  $(a, t, x, y) \in \mathbb{R}_+^* \times (0, \tau) \times X \times Y$  with boundary conditions :

$$(20) \quad u^0(0, t, x, y) = \int_0^{+\infty} \sigma_b^0(\alpha, x, y) u^0(\alpha, t, x, y) d\alpha, \quad (t, x, y) \in (0, \tau) \times X \times Y$$

and initial conditions

$$(21) \quad u^0(a, 0, x, y) = u_{\text{in}}^0(a, x, y), \quad (a, x, y) \in \mathbb{R}_+^* \times X \times Y.$$

By Proposition 3, we deduce that the sequence  $u^\varepsilon$  converges weakly in  $L^2(\mathbb{R}^+ \times (0, \tau) \times X \times Y)$  to

$$u_{\text{hom}}(a, t, x) := \langle u^0 \rangle(a, t, x).$$

We conclude our proof by deducing the equation satisfied by  $u_{\text{hom}}$ . We decompose the two-scale limit into a homogeneous part, denoted  $u_{\text{hom}}$ , and a remainder  $r$ , with zero mean over the periodic cell, i.e.

$$(22) \quad u^0(a, t, x, y) = u_{\text{hom}}(a, t, x) + r(a, t, x, y) \quad \text{and} \quad \langle r \rangle = 0.$$

We then replace (22) into Equation (19), which governs the space-time evolution of the two-scale limit  $u^0$ . We obtain

$$(23) \quad \begin{aligned} \partial_t u_{\text{hom}}(a, t, x) + \partial_a u_{\text{hom}}(a, t, x) + \partial_t r(a, t, x, y) + \partial_a r(a, t, x, y) \\ = -\sigma_d^0(a, x, y) u_{\text{hom}}(a, t, x) - \sigma_d^0(a, x, y) r(a, t, x, y) \end{aligned}$$

for  $(a, t, x, y) \in \mathbb{R}_+^* \times (0, \tau) \times X \times Y$ . Equations (20) and (21) become respectively

$$(24) \quad u_{\text{hom}}(0, t, x) + r(0, t, x, y) = \int_0^{+\infty} \sigma_b^0(\alpha, x, y) [u_{\text{hom}}(\alpha, t, x) + r(\alpha, t, x, y)] d\alpha$$

and

$$(25) \quad u_{\text{hom}}(a, 0, x) + r(a, 0, x, y) = u_{\text{in}}^0(a, x, y).$$

We integrate Equation (23) over the periodicity cell  $Y$ , thus obtaining

$$(26) \quad \partial_t u_{\text{hom}}(a, t, x) + \partial_a u_{\text{hom}}(a, t, x) = -\langle \sigma_d^0 \rangle(a, x) u_{\text{hom}}(a, t, x) - \langle \sigma_d^0 r \rangle(a, t, x).$$

On the other hand, if we integrate Equations (20) and (21) over the periodicity cell  $Y$ , we deduce

$$(27) \quad u_{\text{hom}}(0, t, x) = \int_0^{+\infty} [\langle \sigma_b^0 \rangle(\alpha, x) u_{\text{hom}}(\alpha, t, x) + \langle \sigma_b^0 r \rangle(\alpha, t, x)] d\alpha,$$

and

$$(28) \quad u_{\text{hom}}(a, 0, x) = \langle u_{\text{in}}^0 \rangle(a, x).$$

By inserting (26), (27) and (28) respectively in (23), (24) and (25), we get the initial-boundary value problem for the remainder term:

$$(29) \quad \partial_t r(a, t, x, y) + \partial_a r(a, t, x, y) = -[\sigma_d^0(a, x, y) r(a, t, x, y) - \langle \sigma_d^0 r \rangle(a, t, x)].$$

The initial and the boundary conditions become respectively

$$(30) \quad \begin{aligned} r(0, t, x, y) &= \int_0^{+\infty} [\sigma_b^0(\alpha, x, y) r(\alpha, t, x, y) - \langle \sigma_b^0 r \rangle(\alpha, t, x)] d\alpha \\ &= \int_0^{+\infty} \mathcal{L}_{\sigma_b^0} r(\alpha, t, x, y) d\alpha \end{aligned}$$

and initial conditions

$$(31) \quad r(a, 0, x, y) = u_{\text{in}}^0(a, x, y) - \langle u_{\text{in}}^0 \rangle(a, x) = \mathcal{L}_1 u_{\text{in}}^0(a, x, y).$$

We have thus deduced the following coupled initial-boundary problems for the unknowns  $r$  and  $u_{\text{hom}}$ :

$$(32) \quad \begin{cases} \partial_t r(a, t, x, y) + \partial_a r(a, t, x, y) = \int_Y \sigma_d^0(a, x, y) r(a, t, x, y) dy - \sigma_d^0(a, x, y) r(a, t, x, y) \\ r(0, t, x, y) = \int_0^{+\infty} [\sigma_b^0(\alpha, x, y) r(\alpha, t, x, y) - \int_Y \sigma_b^0(\alpha, x, y) r(\alpha, t, x, y) dy] d\alpha \\ r(a, 0, x, y) = u_{\text{in}}^0(a, x, y) - \langle u_{\text{in}}^0 \rangle(a, x) \end{cases}$$

and

$$(33) \quad \begin{cases} \partial_t u_{\text{hom}}(a, t, x) + \partial_a u_{\text{hom}}(a, t, x) = -\langle \sigma_d^0 \rangle(a, x) u_{\text{hom}}(a, t, x) - \langle \sigma_d^0 r \rangle(a, t, x) \\ u_{\text{hom}}(t, 0, x) = \int_0^{+\infty} [\langle \sigma_b^0 \rangle(\alpha, x) u_{\text{hom}}(\alpha, t, x) + \langle \sigma_b^0 r \rangle(\alpha, t, x)] d\alpha, \\ u_{\text{hom}}(0, a, x) = \langle u_{\text{in}}^0 \rangle(a, x). \end{cases}$$

Note that no term involving  $u_{\text{hom}}$  appears in problem (32). We write it in integral form, thus obtaining

$$\begin{aligned} r(a, t, x, y) &= \mathbf{1}_{t < a} e^{-\int_0^t \sigma_d^0(a+\theta-t, x, y) d\theta} \left[ u_{\text{in}}^0(a-t, x, y) - \int_Y u_{\text{in}}^0(a-t, x, y) dy \right] \\ &+ \mathbf{1}_{t > a} e^{-\int_{t-a}^t \sigma_d^0(a+\theta-t, x, y) d\theta} \int_0^{+\infty} \left[ \sigma_b^0(\alpha, x, y) r(\alpha, t-a, x, y) \right. \\ &\quad \left. - \int_Y \sigma_b^0(\alpha, x, y) r(\alpha, t-a, x, y) dy \right] d\alpha \\ &+ \int_{(t-a)_+}^t e^{-\int_s^t \sigma_d^0(a+\theta-t, x, y) d\theta} \int_Y \sigma_d^0(a, x, y) r(a+s-t, s, x, y) dy ds. \end{aligned}$$

We then introduce the quantities

$$(34) \quad \mathcal{Q}(u_{\text{in}}^0, \sigma_d^0) := \mathbf{1}_{t < a} e^{-\int_0^t \sigma_d^0(a+\theta-t, x, y) d\theta} \left[ u_{\text{in}}^0(a-t, x, y) - \int_Y u_{\text{in}}^0(a-t, x, y) dy \right]$$

and, for any  $h \in L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)$ ,

$$\begin{aligned} \mathcal{S}h &:= \mathbf{1}_{t > a} e^{-\int_{t-a}^t \sigma_d^0(a+\theta-t, x, y) d\theta} \int_0^{+\infty} \left[ \sigma_b^0(\alpha, x, y) h(\alpha, t-a, x, y) \right. \\ &\quad \left. - \int_Y \sigma_b^0(\alpha, x, y) h(\alpha, t-a, x, y) dy \right] d\alpha \\ &+ \int_{(t-a)_+}^t e^{-\int_s^t \sigma_d^0(a+\theta-t, x, y) d\theta} \int_Y \sigma_d^0(a, x, y) h(a+s-t, s, x, y) dy ds. \end{aligned}$$

By means of an argument similar to the proof of Theorem 2, we look for solutions of the fixed-point problem

$$(35) \quad r = \mathcal{Q}(u_{\text{in}}^0, \sigma_d^0) + \mathcal{S}r.$$

We hence introduce the following *ansatz* on the structure of the solution:

$$(36) \quad r := \sum_{n=0}^{+\infty} \mathcal{S}^n \mathcal{Q}(u_{\text{in}}^0, \sigma_d^0),$$

and show that it gives the unique solution of (35) in  $L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)$ .

We first remark that the linear operator  $\mathcal{S}$  is well defined on  $L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)$  and that its image belongs to  $L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)$ .

Thanks to the triangular inequality and the standard Cauchy-Schwarz inequality, we indeed have that, for  $h \in L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)$ ,

$$\begin{aligned} &\|\mathcal{S}h\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 \\ &\leq \left( 2\|\sigma_b^0\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 + \|\sigma_d^0\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 \tau \right) \|h\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2. \end{aligned}$$

This allows to deduce that

$$\|\mathcal{S}\|_{\mathcal{L}(L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y))} \leq \left( 2\|\sigma_b^0\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 + \|\sigma_d^0\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 \right)^{1/2},$$

i.e.  $\mathcal{S}$  is bounded in  $L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)$  and hence continuous.

By following the same strategy, we obtain an estimate of the  $L^2$ -norm of the iterated operator  $\mathcal{S}^n$ . For any  $n \in \mathbb{N}_*$  and for any  $h \in L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)$ , we have that

$$\begin{aligned} & \|\mathcal{S}^n h\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 \\ & \leq \left( 2\|\sigma_b^0\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 + \|\sigma_d^0\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 \right) \int_0^t \|\mathcal{S}^{n-1} h(\cdot, t-a, x, y)\|_{L^2(\mathbb{R}_+^*)}^2 dt_1 \\ & \leq \frac{1}{n!} \left( 2\|\sigma_b^0\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 + \|\sigma_d^0\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 \right)^n \|h\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 \end{aligned}$$

i.e.

$$\begin{aligned} & \|\mathcal{S}^n\|_{\mathcal{L}(L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y))} \\ & \leq \frac{\tau^{n/2}}{\sqrt{n!}} \left( 2\|\sigma_b^0\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 + \|\sigma_d^0\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 \right)^{n/2} \end{aligned}$$

for all  $n \in \mathbb{N}^*$ .

Equation (34) implies that

$$\|\mathcal{Q}(u_{\text{in}}^0, \sigma_d^0)\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 \leq 4\|u_{\text{in}}^0\|_{L^2(\mathbb{R}_+^* \times X \times Y)}^2 \tau.$$

Consequently,

$$\begin{aligned} & \sum_{n=0}^{+\infty} \|\mathcal{S}^n \mathcal{Q}(u_{\text{in}}^0, \sigma_d^0)\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 \leq 2\|u_{\text{in}}^0\|_{L^2(\mathbb{R}_+^* \times X \times Y)}^2 \tau^{1/2} \times \\ & \sum_{n=0}^{+\infty} \tau^{n/2} \sqrt{n!} \left( 2\|\sigma_b^0\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 + \|\sigma_d^0\|_{L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)}^2 \right)^{n/2} < +\infty \end{aligned}$$

because of the convergence of the numerical series on the right-hand side. Therefore,  $r$  exists and its norm is bounded. The same argument used in the existence and uniqueness proof for the solution of (6) shows that  $r$  is the unique solution of the integral formulation of (32). Hence, from (33), we deduce that  $u_{\text{hom}}$  solves Equation (15), i.e.

$$\left\{ \begin{array}{l} \partial_t u_{\text{hom}}(a, t, x) + \partial_a u_{\text{hom}}(a, t, x) = \\ \quad - \langle \sigma_d^0 \rangle(a, x) u_{\text{hom}}(a, t, x) - \langle \sigma_d^0 \sum_{n=0}^{+\infty} \mathcal{S}^n \mathcal{Q}(u_{\text{in}}^0, \sigma_d^0) \rangle(a, t, x) \\ u_{\text{hom}}(0, t, x) = \int_0^{+\infty} [\langle \sigma_b^0 \rangle(\alpha, x) u_{\text{hom}}(\alpha, t, x) + \langle \sigma_b^0 \sum_{n=0}^{+\infty} \mathcal{S}^n \mathcal{Q}(u_{\text{in}}^0, \sigma_d^0) \rangle(\alpha, t, x)] d\alpha, \\ u_{\text{hom}}(a, 0, x) = \langle u_{\text{in}}^0 \rangle(a, x), \end{array} \right.$$

where

$$\begin{aligned} Sh := & \mathbf{1}_{t>a} e^{-\int_{t-a}^t \sigma_d^0(a+\theta-t,x,y) d\theta} \int_0^{+\infty} \left[ \sigma_b^0(\alpha, x, y) h(\alpha, t-a, x, y) \right. \\ & \left. - \int_Y \sigma_b^0(\alpha, x, y) h(\alpha, t-a, x, y) dy \right] d\alpha \\ & + \int_{(t-a)_+}^t e^{-\int_s^t \sigma_d^0(a+\theta-t,x,y) d\theta} \int_Y \sigma_d^0(a, x, y) h(a+s-t, s, x, y) dy ds \end{aligned}$$

for any  $h \in L^2(\mathbb{R}_+^* \times (0, \tau) \times X \times Y)$  and

$$\mathcal{Q}(u_{\text{in}}^0, \sigma_d^0) := \mathbf{1}_{t<a} e^{-\int_0^t \sigma_d^0(a+\theta-t,x,y) d\theta} \left[ u_{\text{in}}^0(a-t, x, y) - \int_Y u_{\text{in}}^0(a-t, x, y) dy \right].$$

Thanks to the proof on the uniqueness of the solution of the initial-boundary value problem (15), we deduce that the whole family  $\{u^\varepsilon\}$  converges to  $u_{\text{hom}}$  weakly in  $L^2(\mathbb{R}_+^* \times (0, \tau) \times X)$ . The proof is hence complete.  $\square$

The result of Theorem 4.2 shows that the limit equation (15) has a much more complex structure than the two-scale limit problem (19)-(21). In particular, it contains memory terms. Such memory terms can be complicated to deal with numerically. Therefore, the two-scale limit problem can be used to numerically study the solution of the homogenized equation. The price to be paid is the introduction of an additional variable into the periodic cell, the advantage is that it allows to keep the local in-time character of the equation. In particular, a numerical strategy based on the two-scale limit does not require the entire time evolution of the solution to be handled at each time step.

## 5. NUMERICAL RESULTS

In this section, we provide some numerical simulations, based on the two-scale description, which illustrate the behavior of the homogenized equation, and compare them with the non-homogenized equation with oscillating birth and death rate. We moreover analyze the numerical solutions of the non-homogenized problem with averaged birth and death rates (and with the same initial condition).

We base our numerical study on the theoretical framework described in the previous Section: we have worked with the two-scale problem (19)-(21) and then we have averaged on the periodicity cell the numerical solution and obtained a numerical approximation of the homogenized limit  $u_{\text{hom}}$ .

The numerical approximation of the solution of (19)-(21) has been obtained by exact integration on the characteristic curves of the two-scale transport equation.

The initial condition (21) for all numerical simulations has been the following:

$$(37) \quad u^0(a, 0, x, y) = u_{\text{in}}^0(a, x, y) = \sin^2(2\pi x)(1-2a)\mathbf{1}_{a \in [0,1]}.$$

The numerical method is defined on an uniform spatial grid  $x_0 = 0, x_1, \dots, x_{N_x} = 1$ , composed of  $N_x = 101$  points and with uniform mesh  $\Delta x = (N_x - 1)^{-1} = 0.01$ , a time grid, composed of  $N_t = 201$  points  $t_0 = 0, t_1, \dots, t_{N_t} = \tau = 2$ , with uniform mesh  $\Delta t = 2(N_t - 1)^{-1} = 0.01$ , and an age grid, composed of  $N_a = 801$  points and with uniform mesh  $\Delta a = 2(N_a - 1)^{-1}$ ,  $a_0 = 0, a_1, \dots, a_{N_a} = a_{\text{max}} = 2\tau = 4$ . In all our simulations,  $\Delta a = \Delta t$ .

We have moreover used the standard trapezoidal quadrature rule for the boundary conditions at  $a = 0$ . Because of the choice on the support of the initial conditions, on  $\tau$  and on  $a_{\text{max}}$ , we have no numerical losses in the computation of the integral boundary conditions (20) at  $a = 0$ .

We have run two numerical experiments. In the first experiment, we imposed the following birth rate in the boundary condition (20), for all  $(a, t, x, y) \in [0, 4] \times [0, 2] \times [0, 1] \times [0, 1]$ :

$$(38) \quad \sigma_b^0(a, y) = 8(2 - \cos(2\pi y))a(1-a)\mathbb{1}_{a(1-a) \geq 0}$$

and the death rate (see Equation (19))

$$(39) \quad \sigma_d^0(a, y) = 8(2 - \cos(2\pi y))(1+a).$$

Both quantities depend on the highly oscillating variable  $y$  and are age-dependent.

The results of the first numerical experiment are plotted in Figure 1. In the image on the top, we visualize the profile of the numerical approximation of the homogenized approximation

$$(t, x) \mapsto \int_{\mathbb{R}_+} u_{\text{hom}}(a, t, x) da = \int_{\mathbb{R}_+ \times (0,1)} u^0(a, t, x, y) dy da$$

and, in the image on the bottom, the solution of the initial-boundary value problem (1)-(3), integrated with respect to the age variable, with the same initial condition and with averaged death and birth rates, i.e.

$$(40) \quad \sigma_b(a, x) = \int_0^1 \sigma_b^0(a, x, y) dy, \quad \sigma_d(a, x) = \int_0^1 \sigma_d^0(a, x, y) dy.$$

It is apparent that both problems lead to very similar results.

In the second numerical experiment, we have kept the same initial condition, the same death rate (39) as in the previous numerical test. We have modified only the birth rate, which is now

$$(41) \quad \sigma_b^0(a, y) = 8(2 - \sin(2\pi y))a(1-a)\mathbb{1}_{a(1-a) \geq 0}.$$

Note that the averages in  $y$  of both birth rates (38) and (41) are equal.

The results of this second numerical experiment are plotted in Figure 2. The behavior of the numerical solution of the homogenized problem is, in this case, very different with respect to the solution of problem (1)-(3) with averaged birth and death rates (40).

This numerical behavior is in full agreement with the theoretical analysis carried out in the previous section, showing that the behaviour of the homogenized problem is very sensitive with respect to the choice of the oscillating birth and death rates.

We conclude this section by showing the emergence of the aforementioned asymptotic behaviors. We show the numerical results of the  $\varepsilon$ -dependent initial-boundary value problem (10)-(12), with  $\varepsilon = 0.02$  and initial condition (37), as in the previous steps.

The death rate in this last pair of numerical experiments is

$$(42) \quad \sigma_d^\varepsilon(a, x) = 8 \left( 2 - \cos \left( 2\pi \frac{x}{\varepsilon} \right) \right) (1+a).$$

We have compared the numerical results obtained with two different birth rates, which are coherent with the birth rates of the previous simulations.

In the first numerical test with fixed  $\varepsilon$ , we have used the birth rate

$$(43) \quad \sigma_b^\varepsilon(a, x) = 8 \left( 2 - \cos \left( 2\pi \frac{x}{\varepsilon} \right) \right) a(1-a)\mathbb{1}_{a(1-a) \geq 0},$$

whereas for the second numerical test with fixed  $\varepsilon$ , we have used the birth rate

$$(44) \quad \sigma_b^\varepsilon(a, x) = 8 \left( 2 - \sin \left( 2\pi \frac{x}{\varepsilon} \right) \right) a(1-a)\mathbb{1}_{a(1-a) \geq 0},$$

The results, plotted in Figure 3, are in agreement with the asymptotic solutions obtained before.

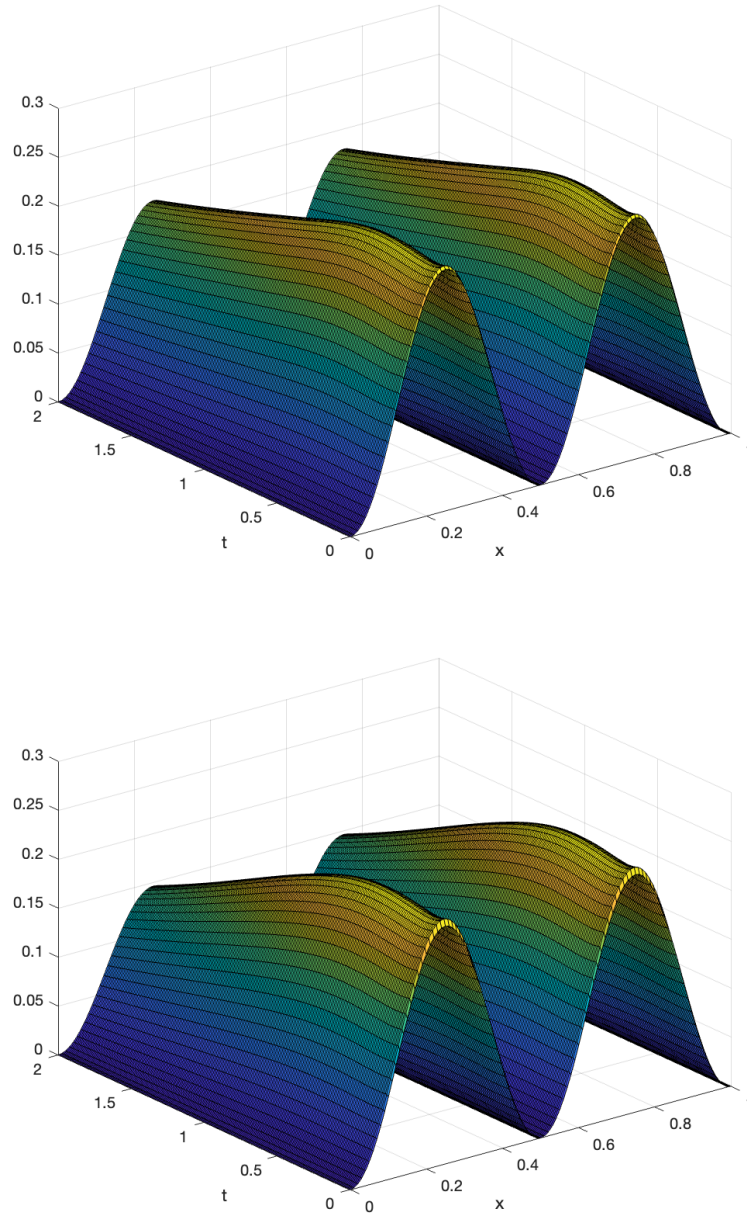


FIGURE 1. Profile of the solution of the initial-boundary value problem (10)-(12) with birth and death rates (38)-(39) (top) and of the solution of the problem (1)-(3) with averaged birth and death rates (40) (bottom), integrated with respect to  $a$ .

Hence, a procedure based on the two-scale description can be useful for avoiding the treatment of the very involved and complicated equation (15).



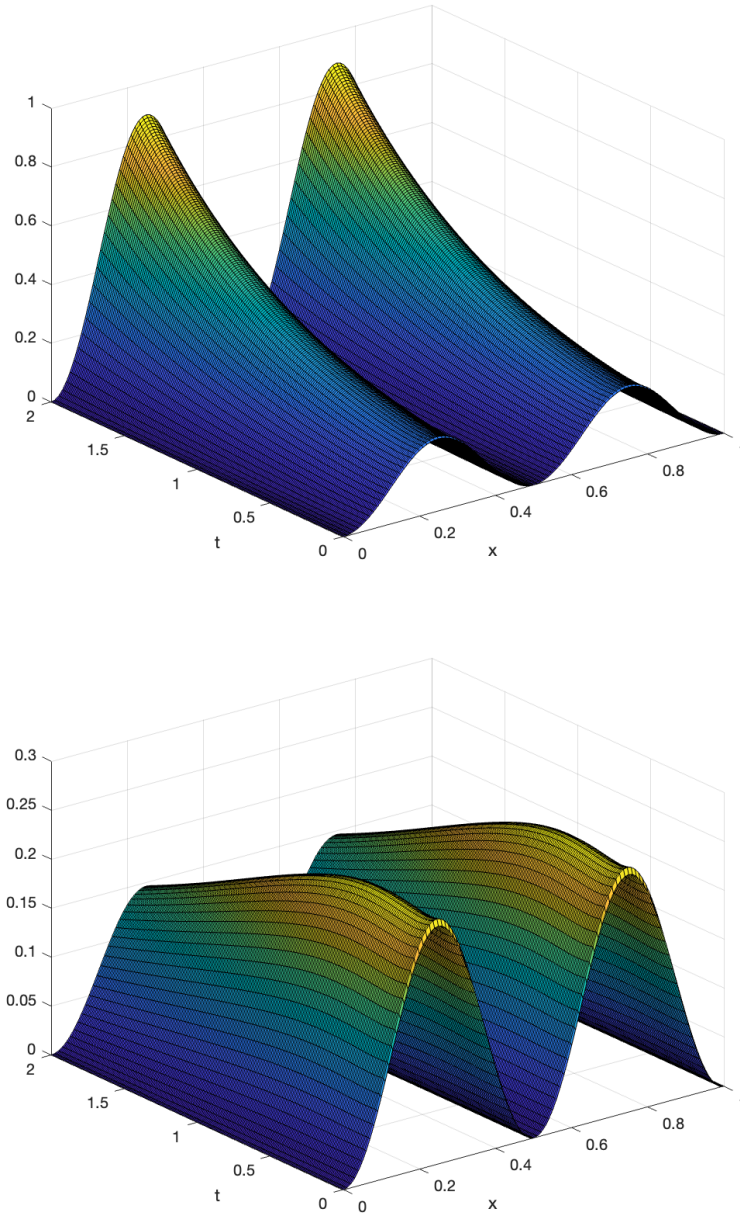


FIGURE 2. Profile of the solution of the initial-boundary value problem (10)-(12) with birth rate (41) and death rate (39) (top) and of the solution of the problem (1)-(3) with the corresponding averaged birth and death rates (bottom). Both densities are integrated with respect to  $a$ .

The integration with respect to  $a$  has been done only for visualization purposes. Our codes give in output  $a$ -dependent densities.

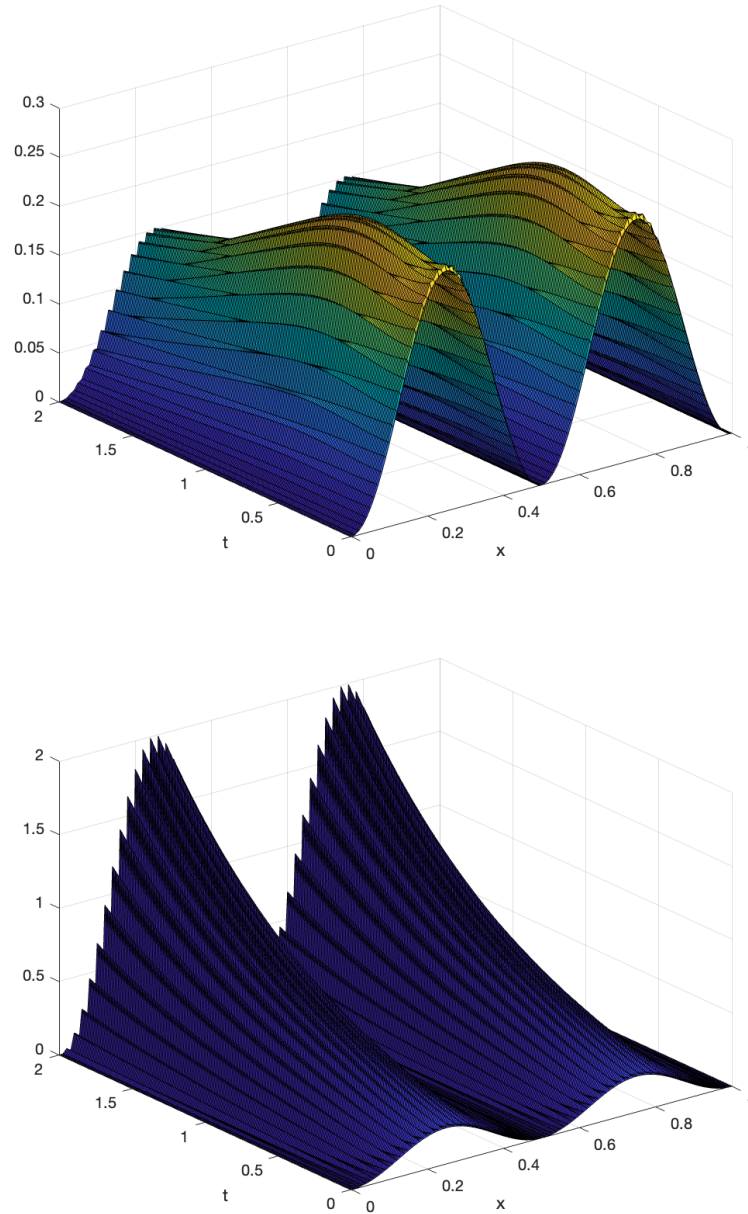


FIGURE 3. Profile of the solution of the initial-boundary value problem (10)-(12) with death rate (42) and birth rate (43) (top) and profile of the solution of the same problem, with death rate (42) and birth rate (44) (bottom). In both cases,  $\varepsilon = 0.02$ . The densities have been integrated with respect to  $a$ .

## 6. CONCLUSION AND PERSPECTIVES

This work may suggest to explore some research directions.

For example, in (1), we have assumed that the individual entities do not move in space. To take into account the spatial movement of individuals, we can add diffusion and/or transport terms to (1)-(3) (see [28] and references therein), and obtain an equation with the following structure

$$\begin{aligned} \partial_t u(a, t, x) + \partial_a u(a, t, x) &= -\sigma_a(a, x)u(a, t, x) + \alpha(a, x)\Delta u(a, t, x) \\ &\quad - \nabla \cdot (\beta(a, x)u(a, t, x)). \end{aligned}$$

A natural question consists to study the homogenization limit of the previous equation taking spatial movement into account. In particular, it would be interesting to understand if the memory term is retained in the presence of the diffusion term – as it is known that a bit of stochasticity can destroy memory terms.

Another possible extension would consist in taking into account the impact of individuals on the environment through consumption and/or nutrient enrichment and to study how and in what forms heterogeneity appears and evolves.

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