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# An extended Bueckner-Rice theory for arbitrary geometric perturbations of cracks

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## Abstract

The aim of this paper is to define a new, simple and efficient method of solution of elasticity problems for 3D bodies containing cracks slightly perturbed *out of their original plane or surface*. This method extends that proposed by Rice (1985, 1989), from a re-formulation of Bueckner (1987)'s 3D weight function theory, for the treatment of *coplanar* crack perturbations. First, a general formula is derived for the variation of total energy of a cracked elastic body, resulting from some small but otherwise arbitrary geometric perturbation of the embedded crack(s). This formula, which involves integrals over both the front and the surface of the crack, is derived from an expression of deLorenzi (1982) and Destuynder *et al.* (1983) in the form of a volumic integral, originally limited to purely tangential crack perturbations but duly extended here to arbitrary perturbations having both tangential *and* normal components. It is then used to derive a general expression of the variation of displacement arising from a general perturbation of the crack(s), anywhere in the cracked body. The reasoning here basically follows the same lines as in the works of Rice (1985, 1989), but for the presence of an additional normal component of the crack perturbation. The possible use of the formalism developed to treat problems of out-of-plane, or out-of-surface perturbations of cracks is finally briefly evoked; the straightforwardness of the new method proposed is hoped to permit future applications to non-coplanar crack problems too complex to be accessible by more conventional methods.

*Keywords : extended Bueckner-Rice theory, out-of-plane crack perturbations, variation of energy, variation of displacement field*

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## 1 Introduction

Problems of perturbations of cracks in 3D elastic bodies have gradually attracted more and more attention in time, following the development of 3D linear elastic fracture mechanics. The perturbations envisaged may be purely tangential (the front of the crack is perturbed but not its surface) or include a component normal to the crack surface (both the crack front *and* surface are perturbed). The fundamental theoretical study of the first kind of perturbations started with the seminal work of Rice (1985); no less than 13 years, however, then elapsed before the first fully correct theoretical treatment of perturbations of the second, more complex type was proposed by Movchan *et al.* (1998). Many papers, too numerous to be all cited here, were then devoted to applications. Among these, those related to coplanar perturbations of plane cracks essentially pertained to (i) the study of the deformation of crack fronts due to inhomogeneities of the fracture properties; and (ii) the problem of evaluating the resulting “overall fracture toughness” - see, among many others, the works of Rice (1985); Gao and Rice (1986, 1987a,b); Gao (1988); Rice (1989); Bower and Ortiz (1991); Gao (1992); Leblond *et al.* (1996); Lazarus and Leblond (2002a,b); Roux *et al.* (2003); Favier *et al.* (2006a,b); Legrand *et al.* (2010a); Pindra *et al.* (2010a); Leblond *et al.* (2012); Patinet *et al.* (2013); Vasoya *et al.* (2013); Démery *et al.* (2014); Vasoya *et al.* (2016); Lebihain (2021). Applications related to out-of-plane perturbations were mainly devoted to (i) the theoretical interpretation of the well-documented instability of coplanar crack propagation under mixed-mode I+III loading conditions, leading to formation of tilted fracture facets; and (ii) the influence of the circumventing of fracture obstacles through off-plane excursions of cracks upon the overall toughness in pure mode I - see notably the works of Pons and Karma (2010); Leblond *et al.* (2011); Leblond and Lazarus (2015); Leblond *et al.* (2018); Vasudevan *et al.* (2019); Lebihain *et al.* (2020, 2021, 2023).

In order to apply suitable propagation criteria - generally Griffith (1920)’s energetic condition combined with Goldstein and Salganik (1974)’s principle of local symmetry - to predict crack propagation, all works of this type require, as a necessary prerequisite, theoretical expressions of the stress intensity factors (SIFs) in the perturbed crack configurations considered. The elasticity problems on perturbed cracked geometries implied are inevitably complex. Indeed they are of the *singular perturbation* type, in the sense that the expansions of the mechanical fields in powers of the small perturbation parameter cannot converge uniformly - as is evident from the fact that the successive terms of these expansions exhibit different singularities near the crack front. A *boundary layer*, analogous to those encountered in fluid mechanics, must necessarily exist in the vicinity of the crack front, and consideration of this layer is compulsory to get the perturbed SIFs.

There are two basic types of methods to obtain the distribution of the SIFs along the perturbed crack front.

- *General* methods of solution consist of fully solving the elasticity problems implied,

that is of calculating the mechanical fields (displacements and stresses) everywhere in the body, through various approaches.

- For *in-plane* perturbations of plane cracks, the first, prototype treatment of this kind, proposed by Rice (1985), was devoted to in-plane perturbation of the front of a semi-infinite, mode I crack in an infinite body. Use was made in this work of the representation of elastic fields through Papkovitch-Neuber potentials (Papkovitch, 1932; Neuber, 1934). Rice (1985)'s first-order perturbation analysis was extended to the second order by Adda-Bedia *et al.* (2006), again by some general method. Another example of such a method was Legrand *et al.* (2011)'s study of coplanar perturbation of a crack lying on the mid-plane of a plate of arbitrary thickness. The basis of the treatment was the Love-Kirchhoff theory of plates in the limiting case of an infinitesimal thickness, supplemented by Rice (1985)'s results in the opposite case of an infinite body (equivalent to a plate of infinite thickness), and completed by finite element computations for intermediary thicknesses. Other related plate problems were treated by Adda-Bedia and Mahadevan (2006), Xia *et al.* (2013) and Budzik and Jensen (2014), with applications to problems of peeling of thin films. Finally one must mention in this context the work of Piccolroaz *et al.* (2007) on the in-plane perturbation of an *interface crack* (lying between different elastic materials).
- For *out-of-plane* perturbations of cracks, the only fully correct theoretical treatment - again based on a general method - seems to be that of Movchan *et al.* (1998), devoted to the case of a semi-infinite, initially plane crack in an infinite body. Extension of this treatment to other crack configuration does not appear trivial, inasmuch as it made a fundamental use of the Fourier transform, which is well-suited to infinite geometries only.
- *Special* methods of solution represent an appealing alternative to general ones. Instead of targeting a full solution to the elasticity problems implied, they concentrate on the sole main quantities of interest, essentially the SIFs along the perturbed crack front. This results in a generally much simpler treatment. In the case of *coplanar* perturbations of the crack front, an approach of this type was proposed by Rice (1985, 1989).<sup>1</sup> The basis was a re-formulation of the theory of Bueckner (1987)'s 3D weight functions, leading to a general expression of the first-order variation of displacement at any point of a cracked body due to some small, but otherwise arbitrary in-plane perturbation of the crack front. Inspection of the variation of the displacement discontinuity across the crack surface near some arbitrary point of the front then led to expressions of the corresponding local variations of the three SIFs. Using such a method, Rice (1985) treated the problem of in-plane perturbation of a semi-infinite crack in an infinite body subjected to some mode I loading. His work was extended to the same geometry but arbitrary mixed-mode loadings by Gao and Rice (1986); to a penny-shaped crack loaded in mode I or mixed-mode I+II+III by Gao and Rice (1987a) and Gao (1988); and to a mode I external circular crack by Gao and Rice (1987b). Basically using the same approach,

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<sup>1</sup> Rice (1985)'s paper thus included *two* methods of solution of the problem considered, one general mentioned above, and one special referred to here.

Leblond *et al.* (1996) treated the problem of in-plane perturbation of a tunnel-crack loaded in mode I in an infinite body, and Lazarus and Leblond (2002a,b) extended their work to arbitrary loadings. Similarly, Pindra *et al.* (2010b) and Legrand *et al.* (2010b) considered the in-plane perturbation of a system of two parallel tunnel-cracks loaded in pure mode I. The case of an interface crack (between different elastic materials) was also considered by Lazarus and Leblond (1998a,b) using an adapted version of Rice (1985, 1989)'s re-formulation of Bueckner (1987)'s weight function theory. In addition, Rice (1985, 1989)'s method was used by Leblond *et al.* (2012) and Vasoya *et al.* (2013, 2016) in a *second-order* analysis of the in-plane perturbation of a semi-infinite crack in a plate of arbitrary thickness, or in some infinite body. Finally, it is worth mentioning that the method also led to the definition of efficient numerical methods for the simulation of the propagation of planar cracks of arbitrary contour in infinite bodies; see notably the works of Bower and Ortiz (1990), Lazarus (1999) and Favier *et al.* (2006a).

For *non-coplanar* perturbations of cracks, such a special method of solution does not exist at present. This situation is unfortunate, since the formal simplicity of such approaches potentially permits to envisage a greater variety of problems than with general methods - the difficulty of extending Movchan *et al.* (1998)'s solution for the out-of-plane perturbation of a semi-infinite crack in an infinite body has already been mentioned. This is what motivates the development, in the present paper, of a *special method of solution for problems of out-of-plane or out-of-surface perturbations of cracks in arbitrary 3D elastic bodies*, based on some extension of Rice (1985, 1989)'s approach for coplanar perturbations.

The development of the new method will follow three successive steps, which naturally define the organization of the paper:

- In Section 2, we begin by providing a general formula for the first-order variation of the total energy of a 3D elastic body resulting from some small, but otherwise arbitrary geometric perturbation of an embedded *geometrically regular hole*, or *system of such holes*. The treatment is largely inspired from those of deLorenzi (1982) and Destuynder *et al.* (1983) devoted to *tangential* perturbations of *cracks* (with the numerical calculation of the energy-release-rate in mind). The fresh proof provided here permits to extend deLorenzi (1982)'s and Destuynder *et al.* (1983)'s formula to general perturbations of holes of arbitrary shape (though geometrically regular at this stage). A comparison is offered with Eshelby (1951)'s seminal work on the closely related topic of perturbation of an ensemble of point singularities or inhomogeneities.
- In Section 3, using the formula obtained in Section 2 for holes, we derive a similar formula for the variation of energy due to perturbation of a *crack* or *ensemble of cracks*. This is done by “thickening” the crack(s) into geometrically regular holes, and then letting their thickness go to zero. (The reason for using such a detour instead of directly dealing with cracks is that direct application of the reasonings of Section 2 to such geometrically singular objects would lead to difficulties and ambiguities arising from

momentarily divergent integrals). The expression obtained stands as an extension of deLorenzi (1982)'s and Destuynder *et al.* (1983)'s formula, limited in their works to *tangential* crack perturbations, to *general* perturbations including a normal component. It is first derived in the form of a volume integral rather unfit to theoretical analyses, but an integration by parts permits to put it into a more convenient format. The links with the famous works of Rice (1968) and Knowles and Sternberg (1972) are finally explored.

- In Section 4, building on the results obtained previously, we show how the variation of displacement induced at any point of the body by a small, but otherwise arbitrary perturbation of the crack, may be deduced from that of the total energy. This is done by adapting Rice (1985, 1989)'s reasoning, initially limited to tangential crack perturbations, to general perturbations. The treatment makes a fundamental use of a Legendre transform of the energy, and Bueckner (1987)'s concept of 3D weight functions. It leads to a general expression of the first-order variation of the displacement at any point of the cracked body, in the form of a sum of integrals over the crack front and surface, respectively.

## 2 Variation of energy due to arbitrary perturbation of a system of geometrically regular holes

As a necessary prerequisite, this Section is devoted to the derivation of a formula for the variation of energy of an elastic body resulting from some small but otherwise arbitrary geometrical perturbation of an embedded ensemble of geometrically regular holes. It is essentially based on deLorenzi (1982)'s and Destuynder *et al.* (1983)'s approach of the problem, limited in the works of these authors to the case of *tangential* perturbation of a *crack or system of cracks*. The extension of their approach to the situation considered here is seized as an occasion to provide a fresh, hopefully more natural and simpler derivation.

### 2.1 Presentation of the problem - Notations

We thus consider (Fig. 1) a homogeneous, linearly elastic 3D body  $\Omega$  - not necessarily isotropic at this stage - with *external* boundary  $\partial\Omega$ .<sup>2</sup> This body is subjected to standard mixed boundary conditions:

$$\begin{cases} \mathbf{u} = \mathbf{u}^p & \text{on } \partial\Omega_u \\ \mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{T}^p & \text{on } \partial\Omega_T \end{cases} \quad (1)$$

<sup>2</sup> By definition, we do *not* include within  $\partial\Omega$  the *internal* boundary consisting of the union of the boundaries of the enclosed voids.

where  $\mathbf{u}$  and  $\boldsymbol{\sigma}$  denote the displacement vector and stress tensor,  $\mathbf{T}$  and  $\mathbf{n}$  the traction vector and outward unit normal vector to the external boundary  $\partial\Omega$ ,  $\partial\Omega_u$  and  $\partial\Omega_T$  complementary parts of  $\partial\Omega$ , and  $\mathbf{u}^p$  and  $\mathbf{T}^p$  prescribed values of  $\mathbf{u}$  and  $\mathbf{T}$  on  $\partial\Omega_u$  and  $\partial\Omega_T$ . There are no body forces.

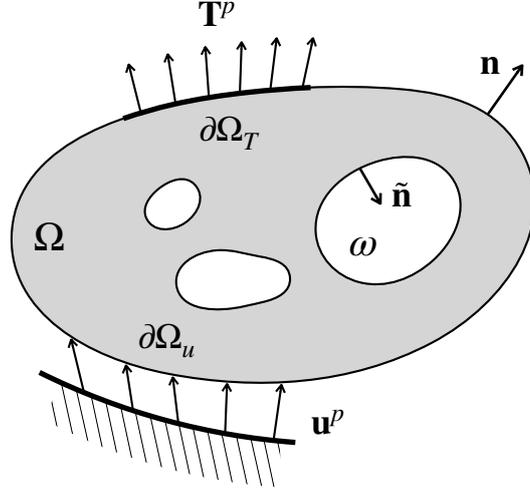


Fig. 1. Arbitrary perturbation of an ensemble of geometrically regular holes in a homogeneous elastic body.

This body contains an arbitrary, geometrically regular, traction-free hole, or an ensemble of such holes, collectively denoted  $\omega$ . The boundary  $\partial\omega$  of this hole or ensemble of holes is slightly but otherwise arbitrarily displaced by the quantity

$$\boldsymbol{\phi}(\mathbf{x}) = \eta\boldsymbol{\theta}(\mathbf{x}) \quad (\forall \mathbf{x} \in \partial\omega) \quad (2)$$

where  $\eta$  is a small parameter and  $\boldsymbol{\theta}$  a given, fixed, smooth function. We wish to evaluate the variation of total energy resulting from such a perturbation of the hole or ensemble of holes, under *constant loading*  $\{\mathbf{u}^p, \mathbf{T}^p\}$  applied on  $\partial\Omega_u \cup \partial\Omega_T$ .

The principle of deLorenzi (1982)'s and Destuynder *et al.* (1983)'s approach of the problem consists in performing a *Lagrangian* calculation of the energy in the *perturbed* configuration  $\Omega^\eta$ , by expressing it as an integral over the *original* (unperturbed) configuration  $\Omega$ . The aim is to thus permit to evaluate the derivative of an integral taken over an *invariable* domain. This procedure makes it necessary to define a one-to-one correspondence between the original and perturbed domains. This is done by considering the following transformation (change of variable):

$$\mathbf{x} \in \Omega \quad \mapsto \quad \mathbf{x}^\eta = \mathbf{x} + \boldsymbol{\phi}(\mathbf{x}) = \mathbf{x} + \eta\boldsymbol{\theta}(\mathbf{x}) \in \Omega^\eta \quad (3)$$

where  $\boldsymbol{\phi} = \eta\boldsymbol{\theta}$  is some extension, over the entire domain  $\Omega$ , of the function  $\boldsymbol{\phi} = \eta\boldsymbol{\theta}$  initially defined only on  $\partial\omega$ . This extension is requested to be smooth and vanish over the external

boundary  $\partial\Omega$ , so as to correctly map the original domain  $\Omega$  onto the perturbed one  $\Omega^\eta$ , but may otherwise be chosen arbitrarily.

In the sequel mechanical quantities like the displacement vector, stress tensor etc. defined over the perturbed configuration  $\Omega^\eta$  will be denoted with an upper index  $\eta$ :  $\mathbf{u}^\eta$ ,  $\boldsymbol{\sigma}^\eta$  etc., in order to distinguish them from those defined over the original configuration  $\Omega$ , devoid of such a symbol.

## 2.2 Lagrangian calculation of the variation of energy

First, the connection between the elementary volume elements  $d\Omega$  and  $d\Omega^\eta$  in the original and perturbed configurations is easily derived from equation (3) defining the transformation  $\mathbf{x} \mapsto \mathbf{x}^\eta$ :

$$d\Omega^\eta = \det \left( \frac{\partial \mathbf{x}^\eta}{\partial \mathbf{x}} \right) d\Omega = \det \left( \mathbf{1} + \eta \frac{\partial \boldsymbol{\theta}}{\partial \mathbf{x}} \right) d\Omega = \left[ 1 + \eta \operatorname{div} \boldsymbol{\theta} + O(\eta^2) \right] d\Omega. \quad (4)$$

The next task is to relate the densities of elastic energy in the original and perturbed configurations,

$$w = \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \quad \text{and} \quad w^\eta = \frac{1}{2} \boldsymbol{\sigma}^\eta : \boldsymbol{\varepsilon}^\eta \quad (5)$$

where  $\boldsymbol{\varepsilon}$  and  $\boldsymbol{\varepsilon}^\eta$  denote the strain tensors in the two configurations. We begin by noting that at every point  $\mathbf{y}$ ,

$$w^\eta(\mathbf{y}) = w(\mathbf{y}) + \eta \frac{\partial w^\eta}{\partial \eta}(\mathbf{y}) + O(\eta^2)$$

where the derivative with respect to  $\eta$ , like all similar derivatives in the sequel, is taken at  $\eta = 0$ . At the point  $\mathbf{y} = \mathbf{x}^\eta$ , this relation yields

$$w^\eta(\mathbf{x}^\eta) = w(\mathbf{x}^\eta) + \eta \frac{\partial w^\eta}{\partial \eta}(\mathbf{x}^\eta) + O(\eta^2) = w(\mathbf{x}^\eta) + \eta \frac{\partial w^\eta}{\partial \eta}(\mathbf{x}) + O(\eta^2) \quad (6)$$

where the last equality stems from the fact that the difference  $\mathbf{x}^\eta - \mathbf{x}$  being  $O(\eta)$ , so is also the difference  $\frac{\partial w^\eta}{\partial \eta}(\mathbf{x}^\eta) - \frac{\partial w^\eta}{\partial \eta}(\mathbf{x})$ . We then remark that the material being assumed to be homogeneous, the density of elastic energy has no *explicit* dependence upon position; that is,  $w$  depends on  $\mathbf{x}$  only because it is a function of  $\boldsymbol{\varepsilon}$  which itself depends on  $\mathbf{x}$ . It follows that

$$\begin{aligned} w(\mathbf{x}^\eta) &= w(\mathbf{x}) + \frac{\partial w}{\partial \varepsilon_{ij}}(\boldsymbol{\varepsilon}(\mathbf{x})) \varepsilon_{ij,k}(\mathbf{x}) (x_k^\eta - x_k) + O(\eta^2) \\ &= w(\mathbf{x}) + \eta \sigma_{ij}(\mathbf{x}) u_{i,jk}(\mathbf{x}) \theta_k(\mathbf{x}) + O(\eta^2) \end{aligned} \quad (7)$$

where use has been made of the symmetry of the tensor  $\boldsymbol{\sigma}$ ; the indices  $i, j, k$  here refer to

some arbitrary Cartesian coordinate system. Combining equations (6) and (7), we get

$$w^\eta(\mathbf{x}^\eta) = w(\mathbf{x}) + \eta \sigma_{ij}(\mathbf{x}) u_{i,jk}(\mathbf{x}) \theta_k(\mathbf{x}) + \eta \frac{\partial w^\eta}{\partial \eta}(\mathbf{x}) + O(\eta^2). \quad (8)$$

Note that as clearly evidenced by this equation, there are *two* origins to the difference between  $w^\eta(\mathbf{x}^\eta)$  and  $w(\mathbf{x})$ : first the densities of elastic energy are different in the two configurations ( $w^\eta \neq w$ ), because the mechanical fields change when the voids are perturbed; second they are not observed at the same point ( $\mathbf{x}^\eta \neq \mathbf{x}$ ), as required by the Lagrangian calculation of the variation of the total energy.

We may now compare the total elastic energies

$$W = \int_{\Omega} w(\mathbf{x}) d\Omega \quad \text{and} \quad W^\eta = \int_{\Omega^\eta} w(\mathbf{x}^\eta) d\Omega^\eta \quad (9)$$

in the original and perturbed configurations. If, in the expression of  $W^\eta$ , one integrates over  $\Omega$  instead of  $\Omega^\eta$ , one gets from equations (4) and (8):

$$\begin{aligned} W^\eta &= \int_{\Omega} \left[ w(\mathbf{x}) + \eta \sigma_{ij}(\mathbf{x}) u_{i,jk}(\mathbf{x}) \theta_k(\mathbf{x}) + \eta \frac{\partial w^\eta}{\partial \eta}(\mathbf{x}) + O(\eta^2) \right] \left[ 1 + \eta \operatorname{div} \boldsymbol{\theta}(\mathbf{x}) + O(\eta^2) \right] d\Omega \\ &= W + \eta \int_{\Omega} \left[ w(\mathbf{x}) \operatorname{div} \boldsymbol{\theta}(\mathbf{x}) + \sigma_{ij}(\mathbf{x}) u_{i,jk}(\mathbf{x}) \theta_k(\mathbf{x}) + \frac{\partial w^\eta}{\partial \eta}(\mathbf{x}) \right] d\Omega + O(\eta^2). \end{aligned} \quad (10)$$

The next task is to compare the opposites of the potential energies of the prescribed forces in the two configurations,

$$\Phi = \int_{\partial\Omega_T} \mathbf{T}^p(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) dS \quad \text{and} \quad \Phi^\eta = \int_{\partial\Omega_T} \mathbf{T}^p(\mathbf{x}) \cdot \mathbf{u}^\eta(\mathbf{x}) dS. \quad (11)$$

Note that in these two expressions: (i) the prescribed traction-vectors  $\mathbf{T}^p$  are the same (since the variation of the total energy is to be evaluated under constant loading); (ii) there is no distinction between  $\mathbf{x}$  and  $\mathbf{x}^\eta$ , nor between  $dS$  and  $dS^\eta$  (since  $\boldsymbol{\theta}$  is taken nil over  $\partial\Omega$ ). It follows that  $\Phi^\eta$  differs from  $\Phi$  only because of the variation of displacement induced on  $\partial\Omega_T$  by the perturbation of the holes:

$$\begin{aligned} \Phi^\eta &= \int_{\partial\Omega_T} \mathbf{T}^p(\mathbf{x}) \cdot \left[ \mathbf{u}(\mathbf{x}) + \eta \frac{\partial \mathbf{u}^\eta}{\partial \eta}(\mathbf{x}) + O(\eta^2) \right] dS \\ &= \Phi + \eta \int_{\partial\Omega_T} \mathbf{T}^p(\mathbf{x}) \cdot \frac{\partial \mathbf{u}^\eta}{\partial \eta}(\mathbf{x}) dS + O(\eta^2). \end{aligned} \quad (12)$$

We may now compare the total potential energies

$$\mathcal{P} = W - \Phi \quad \text{and} \quad \mathcal{P}^\eta = W^\eta - \Phi^\eta \quad (13)$$

in the two configurations. By equations (10) and (12),

$$\begin{aligned} \mathcal{P}^\eta &= \mathcal{P} + \eta \int_{\Omega} (w \operatorname{div} \boldsymbol{\theta} + \sigma_{ij} u_{i,jk} \theta_k) d\Omega \\ &\quad + \eta \left( \int_{\Omega} \frac{\partial w^\eta}{\partial \eta} d\Omega - \int_{\partial\Omega_T} \mathbf{T}^p \cdot \frac{\partial \mathbf{u}^\eta}{\partial \eta} dS \right) + O(\eta^2) \end{aligned} \quad (14)$$

where indications of dependence upon  $\mathbf{x}$  are henceforward discarded in order to lighten the notations. But *the last term  $\eta(\dots)$  in the right-hand side here is zero*. To establish this property, let us first note that  $w^\eta$  depends on  $\eta$  only because it is a function of the strain tensor  $\boldsymbol{\varepsilon}^\eta$  which itself depends on the perturbation of the holes; hence

$$\frac{\partial w^\eta}{\partial \eta} = \frac{\partial w^\eta}{\partial \boldsymbol{\varepsilon}^\eta} : \frac{\partial \boldsymbol{\varepsilon}^\eta}{\partial \eta} = \boldsymbol{\sigma}^\eta : \frac{\partial \boldsymbol{\varepsilon}^\eta}{\partial \eta} = \boldsymbol{\sigma} : \frac{\partial \boldsymbol{\varepsilon}^\eta}{\partial \eta}$$

at  $\eta = 0$ . Also,

$$\int_{\partial\Omega_T} \mathbf{T}^p \cdot \frac{\partial \mathbf{u}^\eta}{\partial \eta} dS = \int_{\partial\Omega_T} \mathbf{T} \cdot \frac{\partial \mathbf{u}^\eta}{\partial \eta} dS = \int_{\partial\Omega_T \cup \partial\Omega_u} \mathbf{T} \cdot \frac{\partial \mathbf{u}^\eta}{\partial \eta} dS = \int_{\partial\Omega} \mathbf{T} \cdot \frac{\partial \mathbf{u}^\eta}{\partial \eta} dS$$

since on  $\partial\Omega_u$ ,  $\mathbf{u}^\eta = \mathbf{u}^p$  which is independent of  $\eta$ . It follows from these elements that

$$\int_{\Omega} \frac{\partial w^\eta}{\partial \eta} d\Omega - \int_{\partial\Omega_T} \mathbf{T}^p \cdot \frac{\partial \mathbf{u}^\eta}{\partial \eta} dS = \int_{\Omega} \boldsymbol{\sigma} : \frac{\partial \boldsymbol{\varepsilon}^\eta}{\partial \eta} d\Omega - \int_{\partial\Omega} \mathbf{T} \cdot \frac{\partial \mathbf{u}^\eta}{\partial \eta} dS = 0 \quad (15)$$

by the principle of virtual work, applied in the unperturbed configuration to the stress field  $\{\boldsymbol{\sigma}\}$  and the “velocity” field  $\{\frac{\partial \mathbf{u}^\eta}{\partial \eta}\}$ ; this establishes the property announced.

Equation (15) implies that the expression (14) of  $\mathcal{P}^\eta$  reduces to

$$\mathcal{P}^\eta = \mathcal{P} + \eta \int_{\Omega} (w \operatorname{div} \boldsymbol{\theta} + \sigma_{ij} u_{i,jk} \theta_k) d\Omega + O(\eta^2). \quad (16)$$

The disappearance in this expression of all derivatives of mechanical fields with respect to  $\eta$  (the parameter characterizing the perturbation of the holes), although surprising at first sight, is not altogether fortuitous. It means that to first order in  $\eta$ , *the variation of the total potential energy depends on the mechanical fields only through their values on the original, unperturbed configuration of the body*, independently of the manner in which they vary (under constant loading) when the holes are perturbed. This property is well-known at least for *tangential* perturbations of *cracks* - being conspicuous for instance in Irwin (1958)’s famous formula relating the energy-release-rate to the stress intensity factors in the unperturbed configuration.

The final step is to transform part of the right-hand side in equation (16) through some integration by parts:

$$\int_{\Omega} \sigma_{ij} u_{i,jk} \theta_k d\Omega = \int_{\partial\Omega} \sigma_{ij} u_{i,k} \theta_k n_j dS + \int_{\partial\omega} \sigma_{ij} u_{i,k} \theta_k \tilde{n}_j dS - \int_{\Omega} (\sigma_{ij} \theta_k)_{,j} u_{i,k} d\Omega$$

where  $\mathbf{n}$  and  $\tilde{\mathbf{n}}$  denote the unit normal vectors to  $\partial\Omega$  and  $\partial\omega$ , oriented outwards and toward the holes, respectively (see Fig. 1). Now  $\theta_k = 0$  on  $\partial\Omega$  and  $\sigma_{ij}\tilde{n}_j = 0$  on  $\partial\omega$ ; hence the surface integrals here are zero. Furthermore within  $\Omega$ ,  $(\sigma_{ij}\theta_k)_{,j} = \sigma_{ij}\theta_{k,j}$  since  $\sigma_{ij,j} = 0$ . It follows that

$$\int_{\Omega} \sigma_{ij} u_{i,jk} \theta_k d\Omega = - \int_{\Omega} \sigma_{ij} u_{i,k} \theta_{k,j} d\Omega.$$

Insertion of this result into equation (16) yields the final expression of the total potential energy in the perturbed configuration:

$$\mathcal{P}^\eta = \mathcal{P} + \eta \int_{\Omega} (w \operatorname{div} \boldsymbol{\theta} - \sigma_{ij} u_{i,k} \theta_{k,j}) d\Omega + O(\eta^2) = \mathcal{P} + \eta \int_{\Omega} P_{jk} \theta_{k,j} d\Omega + O(\eta^2) \quad (17)$$

where the quantities

$$P_{jk} = w \delta_{jk} - \sigma_{ij} u_{i,k} \quad (18)$$

are the components of Eshelby (1951)'s energy-momentum tensor. This formula may alternatively be written in the following equivalent forms:

$$\frac{\partial \mathcal{P}}{\partial \eta} = \int_{\Omega} P_{jk} \theta_{k,j} d\Omega \quad \Leftrightarrow \quad \delta \mathcal{P} = \int_{\Omega} P_{jk} \phi_{k,j} d\Omega; \quad (19)$$

in these expressions and subsequent ones, indications of dependence of quantities upon  $\eta$  in derivatives  $\frac{\partial}{\partial \eta}|_{\eta=0}$  are discarded for simplicity, and the symbol  $\delta$  applied to some quantity represents its variation evaluated at first order in the perturbation  $\boldsymbol{\phi} = \eta \boldsymbol{\theta}$ .

### 2.3 Connection with the work of Eshelby (1951)

Consider now, following Eshelby (1951), the special case of a perturbation  $\boldsymbol{\phi}$  corresponding to a mere small translatory motion of the ensemble of holes:

$$\boldsymbol{\phi}(\mathbf{x}) = \eta \mathbf{e}_\ell \quad \Leftrightarrow \quad \theta_k(\mathbf{x}) = \delta_{k\ell} \quad (\forall \mathbf{x} \in \partial\omega) \quad (20)$$

where  $\mathbf{e}_\ell$  is the unit vector parallel to the direction  $x_\ell$  of the Cartesian frame considered. For such a *uniform* perturbation, one may choose an extension of the function  $\boldsymbol{\theta}(\mathbf{x})$  taking the *constant* value  $\mathbf{e}_\ell$  between the boundary of the holes and some closed surface  $\mathcal{S}$  enclosing all of them (Fig. 2).

For such an extension of  $\boldsymbol{\theta}$ , since this function appears in equation (19)<sub>1</sub> only through its gradient which is zero in the domain enclosed within  $\mathcal{S}$ , the domain of integration  $\Omega$  may be reduced to that domain  $\mathcal{D}$  extending between  $\mathcal{S}$  and the external boundary  $\partial\Omega$ . Integration by parts then yields

$$\frac{\partial \mathcal{P}}{\partial \eta} = \int_{\mathcal{D}} P_{jk} \theta_{k,j} d\Omega = \int_{\partial\Omega} P_{jk} \theta_k n_j dS + \int_{\mathcal{S}} P_{jk} \theta_k \tilde{n}_j dS - \int_{\mathcal{D}} P_{j,k,j} \theta_k d\Omega$$

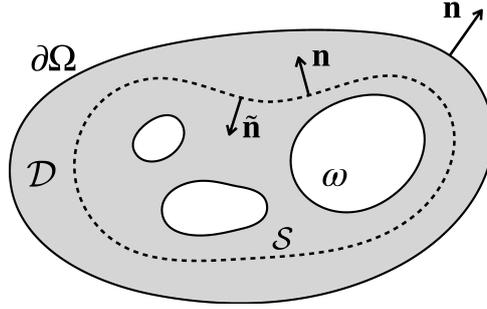


Fig. 2. A closed surface completely surrounding the ensemble of holes.

where again  $\mathbf{n}$  and  $\tilde{\mathbf{n}}$  denote the unit normal vectors to  $\partial\Omega$  and  $\mathcal{S}$ , oriented outwards and inwards, respectively (see Fig. 2). But  $\boldsymbol{\theta} = \mathbf{0}$  on  $\partial\Omega$ , and Eshelby's tensor  $\mathbf{P}$  obeys the well-known property  $P_{jk,j} = 0$  within  $\Omega$  (for a homogeneous material). It follows that

$$\frac{\partial\mathcal{P}}{\partial\eta} = \int_{\mathcal{S}} P_{jk} \theta_k \tilde{n}_j dS = - \int_{\mathcal{S}} P_{jk} \theta_k n_j dS = - \int_{\mathcal{S}} P_{j\ell} n_j dS$$

since  $\theta_k = \delta_{k\ell}$  on  $\mathcal{S}$ ;  $\mathbf{n} = -\tilde{\mathbf{n}}$  now represents the unit normal vector to  $\mathcal{S}$  oriented outwards (see Fig. 2). Equivalently, the variation of energy induced by the perturbation amounts to

$$\delta\mathcal{P} = -\eta \int_{\mathcal{S}} P_{j\ell} n_j dS. \quad (21)$$

Equation (21) is identical to the result obtained by Eshelby (1951) by a completely different method - for an ensemble of point singularities or inhomogeneities instead of holes, but the difference with the situation considered here is only superficial.<sup>3</sup> It should be recalled, however, that Eshelby (1951)'s result is recovered only in the case considered by this author of a *translatory motion* of the ensemble of defects. It was logical to focus in his work on this sole case, insofar as what he was interested in was the “configurational force” acting on the ensemble of defects, related to the variation of total potential energy resulting from a global, rigid-body displacement of these defects.<sup>4</sup> However, with the aim we have in view here, detailed in the Introduction, such a reduction to very special perturbations of the holes will not be possible, and use of the more general formulae (19) will be indispensable.

<sup>3</sup> As is obvious from the very fact that the variation of total potential energy due to some translatory motion of the defects can be expressed as an integral over some distant enclosing surface, independently of their physical nature.

<sup>4</sup> The interpretation of certain variations of total potential energy due to specific perturbations as components of a “configurational force” will not be needed here, and will not be referred to any further.

### 3 Variation of energy due to arbitrary perturbation of a system of cracks

Our main interest in this paper lies in the case of a crack or system of cracks. However, as detailed below, directly dealing with such a situation using the approach of Section 2 would lead to mathematical difficulties, tied to momentarily divergent integrals. This was the reason for first considering, in Section 2, the case of *geometrically regular holes*, for which such issues do not arise. The case of an ensemble of *cracks* is now envisaged as a limiting situation in which the holes become infinitely thin. The formula for cracks is obtained by directly taking the limit in that previously derived for regular holes, thus circumventing the intermediary appearance of divergent integrals.

#### 3.1 The extended deLorenzi-Destuynder formula

We thus consider (Fig. 3) the same problem as before, but now for an ensemble of cracks instead of geometrically regular holes.

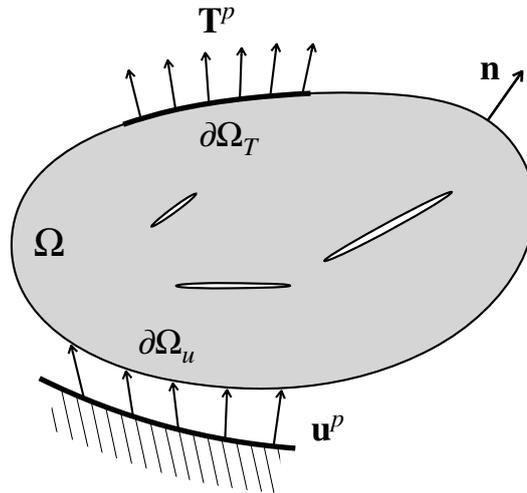


Fig. 3. Arbitrary perturbation of an ensemble of cracks in a homogeneous elastic body.

The first task is to study the asymptotic behaviour of some quantities near the crack front  $\mathcal{F}$ , so as to examine the convergence or divergence of some integrals encountered previously, when now considered for cracks instead of geometrically regular holes. Let  $r$  denote the orthogonal distance from the observation point of mechanical fields to the crack front. Then near  $\mathcal{F}$ , as is well-known,  $\boldsymbol{\sigma}$ ,  $u_{i,j}$  and  $\boldsymbol{\varepsilon}$  behave like  $r^{-1/2}$ , consequently  $w$  and  $\mathbf{P}$  like  $r^{-1}$ ,  $u_{i,jk}$  like  $r^{-3/2}$ . Also,  $w^\eta$  behaves like  $(r^\eta)^{-1}$  where  $r^\eta$  is the distance to the perturbed crack front, depending on  $\eta$ ; hence  $\partial w^\eta / \partial \eta$  behaves like  $r^{-2}$ . Finally the convergence or divergence of a volume integral near  $\mathcal{F}$  is to be assessed using local

cylindrical coordinates  $r, \theta, z$ , for which the infinitesimal volume element  $d\Omega = rd\theta dz$  is proportional to  $r$ . It follows that several of the integrals encountered in Subsection 2.2, if considered for cracks instead of holes, diverge logarithmically at the crack front  $\mathcal{F}$  because the radial integral is of the type  $\int_0 \frac{dr}{r}$ . This notably occurs in the integral  $\int_{\Omega} \frac{\partial w^\eta}{\partial \eta} d\Omega$  in the left-hand side of equation (15), and in the integral  $\int_{\Omega} \sigma_{ij} u_{i,jk} \theta_k d\Omega$  in the right-hand side of equation (16). *Such divergences prohibit direct application of the reasoning of Subsection 2.2 to cracks.*

However such a divergence does *not* occur in the integrals appearing in the final result (19), because the integrand in the radial integral asymptotically behaves (accounting for the infinitesimal volume element  $d\Omega$ ) like  $r^0 = 1$  instead of  $r^{-1}$ , as a consequence of the preceding integration by parts. Hence the solution to the problem just raised is simple: (i) apply the final result (19) to an ensemble of *elongated, thin but geometrically regular* holes; and then (ii) let the thickness of these holes go to zero. In this limit the quantities  $\frac{\partial \mathcal{P}}{\partial \eta}$  and  $\delta \mathcal{P}$  in the left-hand sides go to the derivative and first-order variation of energy corresponding to an ensemble of cracks; and the integrals in the right-hand sides tend toward the same - still convergent - integrals but with an Eshelby tensor  $\mathbf{P}$  corresponding to cracks instead of holes. The natural conclusion of this procedure is that *equation (19) applies unmodified to an ensemble of cracks.*

Equation (19), as applied to cracks, was first established almost simultaneously and independently by deLorenzi (1982) and Destuynder *et al.* (1983), for *purely tangential* perturbations.<sup>5</sup> But the derivation presented here unambiguously shows that it in fact applies to arbitrary perturbations of the cracks including *both tangential and normal components*. For this reason it will be referred to as the *XdLD (eXtended deLorenzi-Destuynder) formula* in the sequel.

### 3.2 *Expression of the variation of energy as a sum of integrals over the crack front and surface*

From a *numerical* point of view, the XdLD formula (19) possesses many virtues, the discussion of which is not the topic of this paper. From a *theoretical* point of view, however, its very format in the form of a volume integral makes it cumbersome to use, because it hides some important features of the derivative or first-order variation of energy:

- the fact that it depends on the perturbation function  $\phi$  or  $\theta$  only through its values on the crack surface  $\mathcal{S}$  and front  $\mathcal{F}$ , and not on its (largely arbitrary) extension over the entire body  $\Omega$ ;

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<sup>5</sup> This restriction on the perturbations envisaged was clearly stated in both works, but it remains unclear at what step of the derivations it was used, or even if it was really needed at all.

- the similar fact that - as will be detailed below - it also depends upon the mechanical fields only through their values on  $\mathcal{S}$  and  $\mathcal{F}$ .

The obvious remedy to this shortcoming of the XdLD formula is to perform an integration by parts of the volume integral. When doing so, however, one must beware of the geometric singularity introduced by the cracks. Dealing with this additional difficulty requires introducing some notations. First, let  $s$  denote some curvilinear abscissa along the crack front  $\mathcal{F}$ . Then for every point  $s$  on  $\mathcal{F}$  and every small, positive number  $\epsilon$ , let  $\mathcal{D}(s; \epsilon)$  denote the circular disk lying within the plane orthogonal to  $\mathcal{F}$  at the point  $s$ , of centre  $s$  and radius  $\epsilon$ . Let  $\omega_\epsilon$  denote the union of such disks when the point  $s$  sweeps the entire front;  $\omega_\epsilon$  is the “circular tube” centered on  $\mathcal{F}$ , of radius  $\epsilon$ . Finally let us define the “shrunk domain”  $\Omega_\epsilon = \Omega - \omega_\epsilon$ . All these geometric notions are illustrated in Fig. 4, which provides a schematic view of a planar section of the geometry.

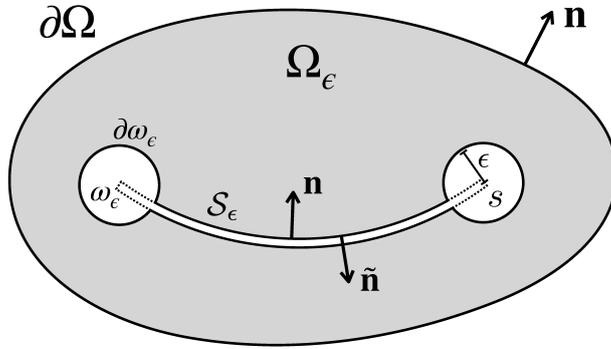


Fig. 4. 2D view (planar section) of the geometry, with a small “circular tube” enclosing the crack front.

The boundary  $\partial\Omega_\epsilon$  of the shrunk domain  $\Omega_\epsilon$  consists of three parts (see Fig. 4): (i) the external boundary  $\partial\Omega$ ; (ii) the boundary  $\partial\omega_\epsilon$  of the tube  $\omega_\epsilon$ , consisting of the union of the boundaries  $\partial\mathcal{D}(s; \epsilon)$  of the disks  $\mathcal{D}(s; \epsilon)$ ; (iii) that “shrunk” part  $\mathcal{S}_\epsilon$  of the crack surface  $\mathcal{S}$  consisting of points located at a distance larger than  $\epsilon$  from the crack front  $\mathcal{F}$ . Like before, the outward unit normal vector to  $\partial\Omega$  is denoted  $\mathbf{n}$ , and the unit normal vector to  $\partial\omega_\epsilon \cup \mathcal{S}_\epsilon$  oriented toward the crack,  $\tilde{\mathbf{n}}$ .

The convergence of the integral  $\int_\Omega P_{jk} \theta_{k,j} d\Omega$  noted above, combined with the fact that the volume of the tube  $\omega_\epsilon$  goes to zero with  $\epsilon$ , implies that

$$\int_\Omega P_{jk} \theta_{k,j} d\Omega = \lim_{\epsilon \rightarrow 0^+} \int_{\Omega_\epsilon} P_{jk} \theta_{k,j} d\Omega.$$

It then follows from the XdLD formula (19)<sub>1</sub> and integration by parts that

$$\begin{aligned} \frac{\partial \mathcal{P}}{\partial \eta} &= \lim_{\epsilon \rightarrow 0^+} \int_{\Omega_\epsilon} P_{jk} \theta_{k,j} d\Omega \\ &= \lim_{\epsilon \rightarrow 0^+} \left( \int_{\partial\Omega} P_{jk} \theta_k n_j dS + \int_{\partial\omega_\epsilon \cup \mathcal{S}_\epsilon} P_{jk} \theta_k \tilde{n}_j dS - \int_{\Omega_\epsilon} P_{jk,j} \theta_k d\Omega \right), \end{aligned}$$

that is, account being taken of the properties  $\boldsymbol{\theta} = \mathbf{0}$  on  $\partial\Omega$  and  $P_{jk,j} = 0$  in  $\Omega$ :

$$\frac{\partial \mathcal{P}}{\partial \eta} = - \lim_{\epsilon \rightarrow 0^+} \left( \int_{\partial\omega_\epsilon} P_{jk} \theta_k n_j dS + \int_{\mathcal{S}_\epsilon} P_{jk} \theta_k n_j dS \right) \quad (22)$$

where now  $\mathbf{n} = -\tilde{\mathbf{n}}$  represents the unit normal vector to  $\partial\omega_\epsilon$  or  $\mathcal{S}_\epsilon$  oriented outwards.

The integrals over  $\partial\omega_\epsilon$  and  $\mathcal{S}_\epsilon$  in equation (22) must be evaluated separately. It is now time to introduce the hypothesis that *the material is isotropic*, with Young's modulus  $E$  and Poisson's ratio  $\nu$ . Also we introduce, at each point  $s$  of the crack front  $\mathcal{F}$ , a local "adapted" Cartesian frame  $(s, x, y, z)$  with axis  $(sx)$  parallel to the direction of crack propagation, axis  $(sy)$  orthogonal to  $\mathcal{S}$ , and axis  $(sz)$  parallel to  $\mathcal{F}$ , oriented in the same way as the curvilinear abscissa  $s$  (Fig. 5).

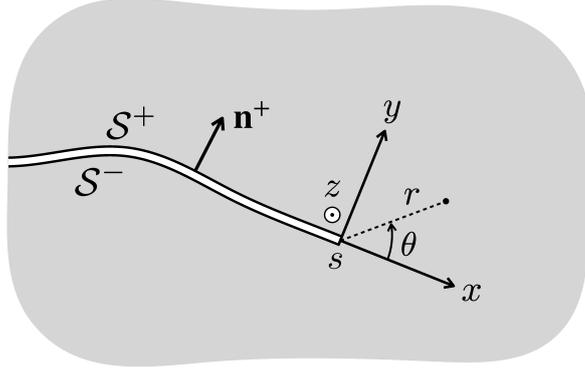


Fig. 5. The local Cartesian frame "adapted" to the crack.

- (1) *Calculation of the integral over  $\partial\omega_\epsilon$ .* When  $\epsilon$  goes to zero, the tube  $\omega_\epsilon$  becomes more and more similar to a circular cylinder. It then follows, by splitting the double integral over  $\partial\omega_\epsilon$  into simple "axial" and "orthoradial" integrals, that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \int_{\partial\omega_\epsilon} P_{jk} \theta_k n_j dS &= \lim_{\epsilon \rightarrow 0^+} \int_{\mathcal{F}} \left( \int_{\partial\mathcal{D}(s;\epsilon)} P_{jk} \theta_k n_j d\ell \right) ds \\ &= \int_{\mathcal{F}} \left( \lim_{\epsilon \rightarrow 0^+} \int_{\partial\mathcal{D}(s;\epsilon)} P_{jk} \theta_k n_j d\ell \right) ds. \end{aligned} \quad (23)$$

To evaluate the integral over the circle  $\partial\mathcal{D}(s;\epsilon)$  in the limit  $\epsilon \rightarrow 0^+$ , one must use the asymptotic expressions of the mechanical fields near the point  $s$  of the crack front,

and integrate with local polar coordinates  $(r, \theta)$  (see Fig. 5). Since the elementary length  $d\ell = \epsilon d\theta$  along the circle is proportional to  $\epsilon$ , it is sufficient to consider only, in these asymptotic expressions, the main terms involving the local SIFs  $K_I(s)$ ,  $K_{II}(s)$ ,  $K_{III}(s)$  generating a contribution proportional to  $\epsilon^{-1}$  in  $P_{jk}$ . A sketch of the calculation is provided in Appendix A. The result is classical and reads

$$\lim_{\epsilon \rightarrow 0^+} \int_{\partial\mathcal{D}(s;\epsilon)} P_{jk} \theta_k n_j d\ell = G(s) \theta_x(s) - 2 \frac{1 - \nu^2}{E} K_I(s) K_{II}(s) \theta_y(s) \quad (24)$$

where  $G(s)$  is the local energy-release-rate, given by Irwin (1958)'s famous formula

$$G(s) = \frac{1 - \nu^2}{E} [K_I^2(s) + K_{II}^2(s)] + \frac{1 + \nu}{E} K_{III}^2(s), \quad (25)$$

and  $\theta_x(s)$  and  $\theta_y(s)$  are the local components of the perturbation function  $\boldsymbol{\theta}(s)$  parallel to the direction of propagation and perpendicular to the crack surface, respectively. Using the result (24) in equation (23), we finally get

$$\lim_{\epsilon \rightarrow 0^+} \int_{\partial\omega_\epsilon} P_{jk} \theta_k n_j dS = \int_{\mathcal{F}} \left[ G(s) \theta_x(s) - 2 \frac{1 - \nu^2}{E} K_I(s) K_{II}(s) \theta_y(s) \right] ds. \quad (26)$$

- (2) *Calculation of the integral over  $\mathcal{S}_\epsilon$ .* We first note that in the integral over  $\mathcal{S}_\epsilon$  of equation (22), the term  $-\sigma_{ij} u_{i,k}$  of the expression (18) of  $P_{jk}$  makes no contribution, since  $\sigma_{ij} n_j = 0$  on  $\mathcal{S}$ ; it follows that

$$\int_{\mathcal{S}_\epsilon} P_{jk} \theta_k n_j dS = \int_{\mathcal{S}_\epsilon} w \boldsymbol{\theta} \cdot \mathbf{n} dS.$$

Next we introduce a distinction between the ‘‘upper’’ (+) and ‘‘lower’’ (−) parts of the crack surface  $\mathcal{S}$ , according to the convention and notations shown in Fig. 5. We also define, with obvious notations, the discontinuity  $[[w]](\mathbf{x}) = w^+(\mathbf{x}) - w^-(\mathbf{x})$  of the density of elastic energy at the point  $\mathbf{x}$  of the crack surface  $\mathcal{S}$ . Regrouping the contributions of  $\mathcal{S}_\epsilon^+$  and  $\mathcal{S}_\epsilon^-$  in the preceding integral, and noting that  $\boldsymbol{\theta}$  takes identical values on  $\mathcal{S}_\epsilon^+$  and  $\mathcal{S}_\epsilon^-$  whereas  $\mathbf{n}$  changes sign, we get

$$\int_{\mathcal{S}_\epsilon} P_{jk} \theta_k n_j dS = \int_{\mathcal{S}_\epsilon^+} [[w]](\mathbf{x}) \boldsymbol{\theta}(\mathbf{x}) \cdot \mathbf{n}^+(\mathbf{x}) dS. \quad (27)$$

It remains to take the limit  $\epsilon \rightarrow 0^+$  in the last integral. It would seem at first sight that this integral diverges in this limit, because the principal terms in the asymptotic expression of the mechanical fields, involving the SIFs, generate a term proportional to  $r^{-1}$  in the density of elastic energy  $w$ , and therefore, apparently, a logarithmically divergent radial integral of the form  $\int_0^{\frac{dr}{r}}$ . This is not so, however, because as shown in Appendix B, the contribution of these principal terms in  $w$  takes identical values on  $\mathcal{S}_\epsilon^+$  and  $\mathcal{S}_\epsilon^-$ , so that the corresponding discontinuity is zero. This means that the term proportional to  $r^{-1}$  in  $[[w]]$  is in fact zero, so that the integral in the right-hand

side of equation (27) converges in the limit  $\epsilon \rightarrow 0^+$ , and

$$\lim_{\epsilon \rightarrow 0^+} \int_{\mathcal{S}_\epsilon^+} \llbracket w \rrbracket(\mathbf{x}) \boldsymbol{\theta}(\mathbf{x}) \cdot \mathbf{n}^+(\mathbf{x}) dS = \int_{\mathcal{S}^+} \llbracket w \rrbracket(\mathbf{x}) \boldsymbol{\theta}(\mathbf{x}) \cdot \mathbf{n}^+(\mathbf{x}) dS.$$

Using this result in equation (27), we finally get

$$\lim_{\epsilon \rightarrow 0^+} \int_{\mathcal{S}_\epsilon} P_{jk} \theta_k n_j dS = \int_{\mathcal{S}^+} \llbracket w \rrbracket(\mathbf{x}) \boldsymbol{\theta}(\mathbf{x}) \cdot \mathbf{n}^+(\mathbf{x}) dS. \quad (28)$$

Combining equations (22), (26) and (28), we get the expressions of the derivative and first-order variation of the total potential energy in the following final equivalent forms:

$$\begin{cases} \frac{\partial \mathcal{P}}{\partial \eta} = \int_{\mathcal{F}} \left[ -G(s) \theta_x(s) + 2 \frac{1 - \nu^2}{E} K_I(s) K_{II}(s) \theta_y(s) \right] ds - \int_{\mathcal{S}^+} \llbracket w \rrbracket(\mathbf{x}) \boldsymbol{\theta}(\mathbf{x}) \cdot \mathbf{n}^+(\mathbf{x}) dS \\ \delta \mathcal{P} = \int_{\mathcal{F}} \left[ -G(s) \phi_x(s) + 2 \frac{1 - \nu^2}{E} K_I(s) K_{II}(s) \phi_y(s) \right] ds - \int_{\mathcal{S}^+} \llbracket w \rrbracket(\mathbf{x}) \boldsymbol{\phi}(\mathbf{x}) \cdot \mathbf{n}^+(\mathbf{x}) dS. \end{cases} \quad (29)$$

A number of particular cases of these expressions, corresponding to crack perturbations of various special types, have been derived before - see the brief discussion in the next Subsection. However with the degree of generality envisaged here - small but otherwise completely arbitrary crack perturbations - they are, to the best of the authors' knowledge, new.

### 3.3 Connections with the works of Rice (1968) and Knowles and Sternberg (1972)

In this Subsection, in order to emphasize essential similarities and differences with important previous works, it will be sufficient to consider the sole 2D case. In this context the integrals over the crack front  $\mathcal{F}$  in equations (29) reduce to local terms at the crack tip, and the integrals over the upper part  $\mathcal{S}^+$  of the crack surface become integrals over the upper crack lip.

For a 2D *straight* crack, Rice (1968) derived the following expression of the energy-release-rate  $G$ :

$$G = J_x = \int_{\Gamma} P_{jx} n_j d\ell = \int_{\Gamma} (w n_x - \sigma_{ij} u_{i,x} n_j) d\ell$$

where  $\Gamma$  denotes *any* contour with both ends on the crack lips and enclosing the crack tip.<sup>6</sup> Now taking  $\theta_x = 1$  and  $\theta_y = 0$  (tangential crack perturbation) in our equation (24), we get

$$G = \lim_{\epsilon \rightarrow 0^+} \int_{\partial \mathcal{D}(s;\epsilon)} P_{jx} n_j d\ell.$$

<sup>6</sup> The integral  $J_x$ , and the integral  $J_y$  introduced below, are often denoted  $J_1$  and  $J_2$  in the literature because of use of the notations  $x_1, x_2$  instead of  $x, y$  for the local coordinates near the crack tip.

The two preceding expressions of  $G$  are identical except for one small difference: the contour  $\Gamma$  in Rice's integral  $J_x$  is arbitrary, whereas it is a circle of vanishingly small radius in the formula just above. The possibility of choosing  $\Gamma$  arbitrarily in Rice (1968)'s definition of  $J_x$  is tied to the path-independence of the integral in the specific case he considered of a straight crack; for a general, curved crack the integral is no longer path-independent (because  $n_x \neq 0$  on the crack lips), and it *must* be considered over some path of infinitesimal length surrounding the crack tip.

Rice and other authors also considered the integral - now necessarily for some infinitesimally small contour  $\Gamma$ , in the absence of any property of path-independence:

$$J_y = \int_{\Gamma} P_{jy} n_j d\ell = \int_{\Gamma} (w n_y - \sigma_{ij} u_{i,y} n_j) d\ell = -2 \frac{1 - \nu^2}{E} K_I K_{II}$$

(where the last equality results from equation (24) with  $\theta_x = 0$  and  $\theta_y = 1$ ). This other integral is currently interpreted as tied to the "variation of energy arising from some displacement of the crack tip perpendicularly to the direction of propagation". This interpretation is not basically incorrect, since  $J_y$  does appear in the expressions (29) of the derivative and first-order variation of the energy, as a cofactor of  $\theta_y$  in the local term at the crack tip. But our results show that it is *incomplete*, insofar as a displacement of the crack tip in the direction  $y$  orthogonal to the crack lips must necessarily be accompanied by a motion of these lips in the same direction, which unavoidably generates an additional variation of energy apparent in the integrals over the upper lip of expressions (29).

It would seem tempting to try and salvage the popular interpretation of  $J_y$  just mentioned, by arguing that the values of the perturbation function  $\phi$  or  $\theta$  at the crack tip and on the lips being largely independent, one may consider perturbations more and more localized within the immediate vicinity of the tip; this would permit to minimize, and ultimately totally erase, the contribution of the integrals over the upper lip in expressions (29). But this temptation must be firmly resisted. Indeed expressions (29) provide the derivative or first-order variation of energy for a perturbation function  $\phi$  of the form (2), with  $\eta$  some small parameter and  $\theta$  some fixed, *smooth* function; confining more and more the perturbation to the vicinity of the crack tip would mean ultimately considering a function  $\theta$  discontinuous at this tip, which would make no sense.

Again in the 2D case, Knowles and Sternberg (1972) defined line integrals providing the variation of total potential energy generated by a global rotation and a uniform expansion (homothetical transformation) of the crack. These integrals again appear as special cases of the general expressions (29).

## 4 Variation of displacement field due to arbitrary perturbation of a crack or system of cracks

In addition to their intrinsic interest, expressions (29) offer the possibility to derive an expression of the variation of the whole displacement field arising from some arbitrary perturbation of the crack(s). The object of this Section is to do this by extending Rice (1985, 1989)'s reformulation of Bueckner (1987)'s weight function theory, initially limited to tangential crack perturbations, to arbitrary ones.

### 4.1 Preliminary considerations

The principle of Rice (1985, 1989)'s reasoning consists of (i) adding to the “primary” loading, resulting from the boundary conditions (1), a “secondary” loading consisting of a point force  $P\mathbf{e}_i$  exerted at some arbitrary point  $\mathbf{x}$  in the body; and (ii) studying the variations of the total potential energy resulting from simultaneous variations of the position of the crack(s) (described by some perturbation function of the form (2)) and this point force, the primary loading remaining fixed.

In fact, as remarked by Rice (1985), strict application of such a force would generate an indeterminate/infinite displacement at the point  $\mathbf{x}$ , which would raise difficulties and ambiguities in the mathematical reasoning to follow. For this reason, in a first step the force  $P\mathbf{e}_i$  will be uniformly spread over a spherical ball  $\mathcal{B}(\mathbf{x}; \epsilon)$  of center  $\mathbf{x}$  and small radius  $\epsilon$ , thus eliminating the occurrence of an infinite displacement at  $\mathbf{x}$ ; note that the elementary work done by the force is then  $Pd\bar{u}_i^\epsilon(\mathbf{x})$ , where  $\bar{u}_i^\epsilon(\mathbf{x})$  is a short notation for the average value  $\langle u_i \rangle_{\mathcal{B}(\mathbf{x}; \epsilon)}$  of the component  $u_i$  of the displacement over the ball  $\mathcal{B}(\mathbf{x}; \epsilon)$ . In a second step, it will be harmless to let  $\epsilon$  go to zero in the final formula obtained.

### 4.2 Detailed derivation

We envisage infinitesimal variations of both the parameter  $\eta$  characterizing the perturbation of the crack(s) and the intensity  $P$  of the “point” force. We claim that for any such variations, the differential of the total potential energy is given by

$$dW = d\Phi + Pd\bar{u}_i^\epsilon(\mathbf{x}) + \left\{ \int_{\mathcal{F}} \left[ -G(s)\theta_x(s) + 2\frac{1-\nu^2}{E}K_I(s)K_{II}(s)\theta_y(s) \right] ds \right\} d\eta - \left\{ \int_{S^+} \llbracket w \rrbracket(\mathbf{x}')\boldsymbol{\theta}(\mathbf{x}') \cdot \mathbf{n}^+(\mathbf{x}') dS' \right\} d\eta \quad (30)$$

where  $\Phi$  is the opposite of the potential energy of the sole primary loading, defined by equation (11)<sub>1</sub>. Indeed:

- At constant  $P$  and variable  $\eta$ , equation (30) is equivalent to (29)<sub>2</sub>. (Note that the total potential energy  $\mathcal{P}$  of the body, including the contribution of the slightly spread point force, is  $W - \Phi - P\bar{u}_i^\epsilon(\mathbf{x})$ ).
- At constant  $\eta$  and variable  $P$ , it is a consequence of the principle of virtual work.

We now define the quantity

$$\Psi = P\bar{u}_i^\epsilon(\mathbf{x}) - W + \Phi. \quad (31)$$

Note that  $\Psi$  is simply, as just remarked, the opposite of the total potential energy  $\mathcal{P}$ ; but a different notation is used here because  $\Psi$  will be considered, somewhat unusually, as a function of the force  $P$  instead of the displacement  $\bar{u}_i^\epsilon(\mathbf{x})$ . One may also remark that according to equation (30),  $\Psi$  is the Legendre transform of the energy  $W - \Phi$  with respect to the variable  $\bar{u}_i^\epsilon(\mathbf{x})$  at fixed  $\eta$ .

By equation (30), the differential of  $\Psi$  is given by

$$\begin{aligned} d\Psi &= Pd\bar{u}_i^\epsilon(\mathbf{x}) + \bar{u}_i^\epsilon(\mathbf{x})dP - dW + d\Phi \\ &= \bar{u}_i^\epsilon(\mathbf{x})dP + \left\{ \int_{\mathcal{F}} \left[ G(s)\theta_x(s) - 2\frac{1-\nu^2}{E}K_I(s)K_{II}(s)\theta_y(s) \right] ds \right\} d\eta \\ &\quad + \left\{ \int_{S^+} \llbracket w \rrbracket(\mathbf{x}')\boldsymbol{\theta}(\mathbf{x}') \cdot \mathbf{n}^+(\mathbf{x}') dS' \right\} d\eta. \end{aligned}$$

It follows,  $\Psi$  being considered as a function of  $\eta$  and  $P$ , that

$$\begin{cases} \frac{\partial \Psi}{\partial P} = \bar{u}_i^\epsilon(\mathbf{x}) \\ \frac{\partial \Psi}{\partial \eta} = \int_{\mathcal{F}} \left[ G(s)\theta_x(s) - 2\frac{1-\nu^2}{E}K_I(s)K_{II}(s)\theta_y(s) \right] ds + \int_{S^+} \llbracket w \rrbracket(\mathbf{x}')\boldsymbol{\theta}(\mathbf{x}') \cdot \mathbf{n}^+(\mathbf{x}') dS' \end{cases} \quad (32)$$

and therefore, by the symmetry of second cross-derivatives, that

$$\begin{aligned} \frac{\partial \bar{u}_i^\epsilon(\mathbf{x})}{\partial \eta} &= \int_{\mathcal{F}} \left[ \frac{\partial G(s)}{\partial P}\theta_x(s) - 2\frac{1-\nu^2}{E}\frac{\partial [K_I(s)K_{II}(s)]}{\partial P}\theta_y(s) \right] ds \\ &\quad + \int_{S^+} \frac{\partial (\llbracket w \rrbracket(\mathbf{x}'))}{\partial P}\boldsymbol{\theta}(\mathbf{x}') \cdot \mathbf{n}^+(\mathbf{x}') dS'. \end{aligned} \quad (33)$$

When this equality is applied at the point  $(\eta, P) = (0, 0)$ , the derivative  $\frac{\partial}{\partial \eta}$  in the left-hand side is taken in the absence of the point force  $P\mathbf{e}_i$ , which implies that the displacement field is simply that generated by the primary loading, which is perfectly smooth at the point  $\mathbf{x}$ . Hence in the limit  $\epsilon \rightarrow 0^+$  the average value  $\bar{u}_i^\epsilon(\mathbf{x})$  simply becomes identical to

the local displacement  $u_i(\mathbf{x})$  due to this primary loading, and equation (33) becomes

$$\begin{aligned} \frac{\partial u_i(\mathbf{x})}{\partial \eta} = & \int_{\mathcal{F}} \left[ \frac{\partial G(s)}{\partial P} \theta_x(s) - 2 \frac{1 - \nu^2}{E} \frac{\partial [K_I(s) K_{II}(s)]}{\partial P} \theta_y(s) \right] ds \\ & + \int_{S^+} \frac{\partial(\llbracket w \rrbracket)(\mathbf{x}')}{\partial P} \boldsymbol{\theta}(\mathbf{x}') \cdot \mathbf{n}^+(\mathbf{x}') dS' \end{aligned} \quad (34)$$

where it is understood that the derivatives  $\frac{\partial}{\partial P}$  in the right-hand side are taken at the point  $(\eta, P) = (0, 0)$ , and for a true (not spread) point load ( $\epsilon = 0$ ).

It remains to ascribe expressions to the derivatives  $\partial G(s)/\partial P$ ,  $\partial[K_I(s)K_{II}(s)]/\partial P$  and  $\partial(\llbracket w \rrbracket)(\mathbf{x}')/\partial P$  in equation (34). We first note that owing to linearity, the local SIFs in the presence of the point load  $P\mathbf{e}_i$  are given by

$$K_\alpha(s) = K_\alpha^0(s) + Ph_\alpha^{(i)}(\mathbf{x}; s) \quad (\alpha = I, II, III) \quad (35)$$

where  $K_\alpha^0(s)$  is the SIF of mode  $\alpha$  at  $s \in \mathcal{F}$  due to the sole primary loading, and  $h_\alpha^{(i)}(\mathbf{x}; s)$  - a Bueckner 3D weight function - that generated at the same point by application of a *unit* point force in the direction  $\mathbf{e}_i$  at the point  $\mathbf{x} \in \Omega$ , with the boundary conditions  $\mathbf{u} = \mathbf{0}$  on  $\partial\Omega_u$  and  $\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}$  on  $\partial\Omega_T$  (and the crack faces). It then follows from Irwin (1958)'s formula (25) that

$$\frac{\partial G(s)}{\partial P} = 2 \frac{1 - \nu^2}{E} \left[ K_I^0(s) h_I^{(i)}(\mathbf{x}; s) + K_{II}^0(s) h_{II}^{(i)}(\mathbf{x}; s) + \frac{K_{III}^0(s) h_{III}^{(i)}(\mathbf{x}; s)}{1 - \nu} \right]. \quad (36)$$

Similarly,

$$\frac{\partial [K_I(s) K_{II}(s)]}{\partial P} = K_I^0(s) h_{II}^{(i)}(\mathbf{x}; s) + K_{II}^0(s) h_I^{(i)}(\mathbf{x}; s). \quad (37)$$

Moreover, by the symmetries of the elastic stiffness and stress tensors, at any point  $\mathbf{x}' \in \Omega$ ,

$$\frac{\partial w(\mathbf{x}')}{\partial P} = \boldsymbol{\sigma}^0(\mathbf{x}') : \frac{\partial \boldsymbol{\varepsilon}(\mathbf{x}')}{\partial P} = \sigma_{jk}^0(\mathbf{x}') \frac{\partial u_{j,k}(\mathbf{x}')}{\partial P} \quad (38)$$

where  $\boldsymbol{\sigma}^0$  denotes the stress field generated by the sole primary loading. Furthermore the components of the displacement field in the presence of the point load  $P\mathbf{e}_i$  are given by

$$u_j(\mathbf{x}') = u_j^0(\mathbf{x}') + PU_j^{(i)}(\mathbf{x}; \mathbf{x}') \quad (39)$$

where  $u_j^0(\mathbf{x}')$  and  $U_j^{(i)}(\mathbf{x}; \mathbf{x}')$  are the  $j$ -th components of the displacement at  $\mathbf{x}'$  due to the sole primary loading, and to a unit point force exerted in the direction  $\mathbf{e}_i$  at the point  $\mathbf{x}$  (with  $\mathbf{u} = \mathbf{0}$  on  $\partial\Omega_u$  and  $\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}$  on  $\partial\Omega_T$ ), respectively. Combination of equations (38) and (39) yields

$$\frac{\partial w(\mathbf{x}')}{\partial P} = \sigma_{jk}^0(\mathbf{x}') \frac{\partial U_j^{(i)}(\mathbf{x}; \mathbf{x}')}{\partial x'_k}. \quad (40)$$

In this equation and subsequent ones, spatial derivatives are noted explicitly in the form “ $\frac{\partial}{\partial x}$ ” rather than with commas, in order to avoid ambiguities between derivatives with respect to the coordinates of  $\mathbf{x}$  and  $\mathbf{x}'$ .

Equation (40) could be used to get an expression of the derivative  $\partial u_i(\mathbf{x})/\partial\eta$  looked for. However, it will reveal more convenient in applications to deal with a point force applied at the integration point  $\mathbf{x}' \in \mathcal{S}^+$ , rather than at the observation point  $\mathbf{x} \in \Omega$  of  $\partial u_i/\partial\eta$ . This may easily be achieved by applying Betti’s theorem to the two loadings consisting of unit point forces parallel to  $\mathbf{e}_i$  and exerted at  $\mathbf{x}$ , and parallel to  $\mathbf{e}_j$  and exerted at  $\mathbf{x}'$  (with  $\mathbf{u} = \mathbf{0}$  on  $\partial\Omega_u$  and  $\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}$  on  $\partial\Omega_T$ ). This yields the “symmetry relation”

$$U_j^{(i)}(\mathbf{x}; \mathbf{x}') = U_i^{(j)}(\mathbf{x}'; \mathbf{x}) \quad (41)$$

which permits to transform the expression (40) of  $\partial w(\mathbf{x}')/\partial P$  into

$$\frac{\partial w(\mathbf{x}')}{\partial P} = \sigma_{jk}^0(\mathbf{x}') \frac{\partial U_i^{(j)}(\mathbf{x}'; \mathbf{x})}{\partial x'_k}. \quad (42)$$

Insertion of expressions (36), (37) and (42) into equation (34) yields the final expression of the derivative of the displacement field with respect to the perturbation parameter  $\eta$ , or equivalently of its first-order variation:

$$\left\{ \begin{array}{l} \frac{\partial u_i(\mathbf{x})}{\partial \eta} = 2 \frac{1-\nu^2}{E} \int_{\mathcal{F}} \left\{ \left[ K_I^0(s) h_I^{(i)}(\mathbf{x}; s) + K_{II}^0(s) h_{II}^{(i)}(\mathbf{x}; s) + \frac{K_{III}^0(s) h_{III}^{(i)}(\mathbf{x}; s)}{1-\nu} \right] \theta_x(s) \right. \\ \quad \left. - \left[ K_I^0(s) h_{II}^{(i)}(\mathbf{x}; s) + K_{II}^0(s) h_I^{(i)}(\mathbf{x}; s) \right] \theta_y(s) \right\} ds \\ \quad + \int_{\mathcal{S}^+} \left[ \sigma_{jk}^0(\mathbf{x}') \frac{\partial U_i^{(j)}(\mathbf{x}'; \mathbf{x})}{\partial x'_k} \right] \boldsymbol{\theta}(\mathbf{x}') \cdot \mathbf{n}^+(\mathbf{x}') dS' \\ \delta u_i(\mathbf{x}) = 2 \frac{1-\nu^2}{E} \int_{\mathcal{F}} \left\{ \left[ K_I^0(s) h_I^{(i)}(\mathbf{x}; s) + K_{II}^0(s) h_{II}^{(i)}(\mathbf{x}; s) + \frac{K_{III}^0(s) h_{III}^{(i)}(\mathbf{x}; s)}{1-\nu} \right] \phi_x(s) \right. \\ \quad \left. - \left[ K_I^0(s) h_{II}^{(i)}(\mathbf{x}; s) + K_{II}^0(s) h_I^{(i)}(\mathbf{x}; s) \right] \phi_y(s) \right\} ds \\ \quad + \int_{\mathcal{S}^+} \left[ \sigma_{jk}^0(\mathbf{x}') \frac{\partial U_i^{(j)}(\mathbf{x}'; \mathbf{x})}{\partial x'_k} \right] \boldsymbol{\phi}(\mathbf{x}') \cdot \mathbf{n}^+(\mathbf{x}') dS'. \end{array} \right. \quad (43)$$

Equation (43) constitutes the output of the *extended Bueckner-Rice theory* developed.

### 4.3 Comments

For a purely tangential perturbation ( $\theta_y = 0$  on  $\mathcal{F}$ ,  $\boldsymbol{\theta} \cdot \mathbf{n} = 0$  on  $\mathcal{S}$ ), the expression (43)<sub>1</sub> of  $\partial u_i(\mathbf{x})/\partial \eta$  or (43)<sub>2</sub> of  $\delta u_i(\mathbf{x})$  reduces to that part of the integral over the crack front  $\mathcal{F}$  involving the tangential component  $\theta_x$  of the perturbation function. The formula then becomes exactly identical to that derived by Rice (1985, 1989) for crack perturbations of this kind. Introduction of a component of the perturbation function orthogonal to the crack surface makes the formula notably more complex, by generating not only an additional term proportional to  $\theta_y$  in the integral over the crack front  $\mathcal{F}$ , but an extra integral over the entire crack surface  $\mathcal{S}$ . (Just like for the variation of the total potential energy, this was to be expected since an arbitrary perturbation of the crack implies a motion of all points of its surface, not only of its front).

In principle, equation (43) provides the full first-order solution to the elasticity problem considered on the perturbed cracked geometry, since it gives the perturbed displacement at all points of the body. (Note however that equation (43) may be put to use *only* provided that one knows the elastic solution for a point force applied anywhere in the original unperturbed geometry - which of course depends upon this entire geometry and the type of boundary conditions). In practice, however, one is generally mainly interested in the asymptotic behaviour of the mechanical fields in the vicinity of the perturbed crack front, and especially in the distribution of the SIFs along this front. Such features of the perturbed solution may be obtained from a detailed study of the first-order variation  $\delta u_i(\mathbf{x})$  of the displacement, in the limit where its point of observation  $\mathbf{x}$  gets close to some arbitrary point on the crack front. This was precisely what Rice and coworkers and followers did for in-plane perturbations of planar cracks, in a number of interesting geometric and mechanical configurations (Gao and Rice, 1986, 1987a,b; Gao, 1988; Leblond *et al.*, 1996; Lazarus and Leblond, 2002a,b; Legrand *et al.*, 2010b; Pindra *et al.*, 2010b; Leblond *et al.*, 2012; Vasoya *et al.*, 2013; Leblond and Lazarus, 2015; Vasoya *et al.*, 2016). (Note again, however, that this was possible *only* because in the cases considered, one knew the distributions of the SIFs - if not the full elastic solution - due to application of a point force on the unperturbed geometry, that is Bueckner (1987)'s weight functions for the geometry and boundary conditions considered). This is also what equation (43) should now permit to do for more general crack perturbations including an out-of-plane component.

What is remarkable in this “special” method of solution - in the terminology of the Introduction - based on the extended Bueckner-Rice theory, is that it permits to totally circumvent the search and definition of a full method of solution for the elasticity problem posed on the perturbed cracked geometry. This is all the more advantageous since there is no systematic method of solution for 3D elasticity problems, and an approach devised for some specific situation will most often *not* apply to others; for instance Movchan *et al.* (1998)'s “general” method of solution for the out-of-plane perturbation of a semi-infinite crack in an infinite body made a heavy use of Fourier transforms, adapted to infinite

geometries only. The special method of solution proposed here reduces the problem to the purely mathematical, perhaps cumbersome and heavy, but basically straightforward task of finding the limits of certain integrals for certain limiting values of their defining parameters.

In a forthcoming paper, we shall illustrate the simplicity and power of the method by applying it to the prototype problem of the out-of-plane perturbation of a semi-infinite crack in an infinite body. (A foretaste of the treatment is given in the next Subsection for a simple special loading). The results will be compared to those of Movchan *et al.* (1998) derived from their general method. It may be appropriate to make here, in anticipation, a few remarks on the solution. According to Movchan *et al.* (1998)'s results, the expressions of the SIFs at some observation point on the perturbed front involve three kinds of terms:

- purely *local* terms involving only the values and spatial derivatives of the mechanical fields and the perturbation function *at the observation point* considered;
- *semi-local* terms in the form of integrals over the crack *front*;
- fully *non-local* terms in the form of integrals over the entire crack *surface*.

It is easy to foresee how results derived from equation (43)<sub>2</sub> will fit into this scheme. Indeed, letting the point of observation of the displacement in this equation go to some point on the crack front, one will obviously get semi-local terms arising from the integral over  $\mathcal{F}$ , and non-local terms arising from the integral over  $\mathcal{S}^+$ . Local terms, on the other hand, will have several, more subtle origins.<sup>7</sup>

Up to now, the infinite body containing a semi-infinite crack is the only configuration for which the out-of-plane crack perturbation problem has been solved. But the method proposed permits to envisage other cracked geometries of practical interest - provided that the solution for the unperturbed geometry subjected to a point force loading is known. A typical example is the penny-shaped crack in an infinite body, under general loading conditions. This configuration is similar, to a large extent,<sup>8</sup> to that used in Sommer (1969)'s classical experimental study of the well-known out-of-plane instability of the crack in mode I+III; and it is hoped that a theoretical analysis of the problem based on the method proposed here will be useful to improve the interpretation of his observations.

#### 4.4 *Illustration of the method in a simple case*

To provide the reader with a foretaste of the method of solution proposed, we shall consider here Movchan *et al.* (1998)'s problem of the out-of-plane perturbation of a semi-infinite

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<sup>7</sup> It suffices to note here, in anticipation, that all local terms will be of *universal* character, that is identical for all conceivable cracked geometries, in contrast to the semi-local and fully non-local terms.

<sup>8</sup> The difference residing in the finite dimensions of the specimens in the experiments.

crack in an infinite body, for a special type of loading allowing for an appealingly simple treatment. The unperturbed geometry is depicted schematically in Fig. 6; the loading is special in that it consists only of uniform far stresses  $\sigma_{xx}^\infty$ ,  $\sigma_{zz}^\infty$ ,  $\sigma_{xz}^\infty$  which do not generate any SIFs on this geometry, but only non-singular stresses  $T_{xx}^0 = \sigma_{xx}^\infty$ ,  $T_{zz}^0 = \sigma_{zz}^\infty$ ,  $T_{xz}^0 = \sigma_{xz}^\infty$  uniform along the crack front. (This type of situation may correspond to the stable propagation of shear ruptures along frictional interfaces, governed by Coulomb's friction law: in such a case  $K_I^0 = 0$  as the crack is loaded in compression, and  $K_{II}^0 = K_{III}^0 = 0$  as required by energy balance (Sáez *et al.*, 2022)).

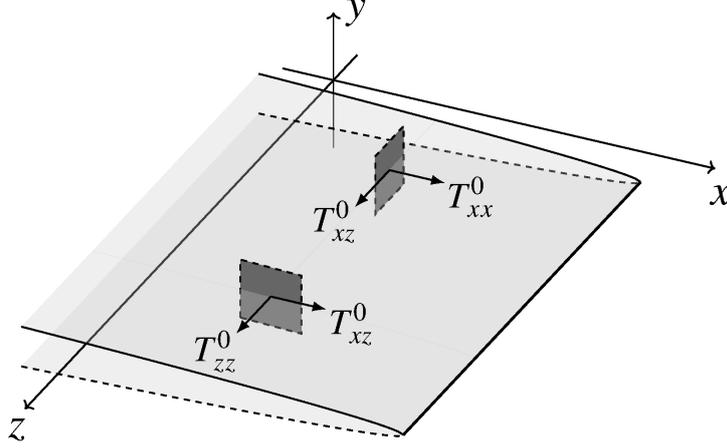


Fig. 6. A semi-infinite crack in an infinite body, loaded through far stresses generating only non-singular stresses.

In this situation, the expression (43)<sub>2</sub> of the variation of the displacement field simply becomes (since the initial stresses are merely identical everywhere to the non-singular stresses):

$$\delta u_i(\mathbf{x}) = \int_{\mathcal{S}^+ \cup \mathcal{S}^-} T_{jk}^0 \frac{\partial U_i^{(j)}(\mathbf{x}'; \mathbf{x})}{\partial x'_k} \phi(\mathbf{x}') \cdot \mathbf{n}(\mathbf{x}') dS'$$

where the upper  $\mathcal{S}^+$  and lower  $\mathcal{S}^-$  crack surfaces over which one integrates have been distinguished again for clarity, and the indices  $j$  and  $k$  only take the values  $x$  and  $z$ . Now for any point  $\mathbf{x}$  of coordinates  $(x, y, z)$ , let  $\mathbf{x}^s$  denote the symmetric point with respect to the unperturbed crack surface, of coordinates  $(x, -y, z)$ . Subtracting from the preceding equation the same equation but taken at this symmetric point, one gets

$$\delta v_i(\mathbf{x}) = \int_{\mathcal{S}^+ \cup \mathcal{S}^-} T_{jk}^0 \frac{\partial V_i^{(j)}(\mathbf{x}'; \mathbf{x})}{\partial x'_k} \phi(\mathbf{x}') \cdot \mathbf{n}(\mathbf{x}') dS'$$

where

$$\delta v_i(\mathbf{x}) = \delta u_i(\mathbf{x}) - \delta u_i(\mathbf{x}^s) \quad \text{and} \quad V_i^{(j)}(\mathbf{x}'; \mathbf{x}) = U_i^{(j)}(\mathbf{x}'; \mathbf{x}) - U_i^{(j)}(\mathbf{x}'; \mathbf{x}^s).$$

A double passage to the limit is now in order:

- First we apply the preceding result to some point  $\bar{\mathbf{x}}$  located above the crack surface  $\mathcal{S}^+$ , and let this point go to the point  $\mathbf{x}^+$  of this surface (the symmetric point  $\bar{\mathbf{x}}^s$  goes to the point  $\mathbf{x}^-$  of the surface  $\mathcal{S}^-$ ). The quantities  $\delta v_i(\bar{\mathbf{x}})$  and  $V_i^{(j)}(\mathbf{x}'; \bar{\mathbf{x}})$  go to the discontinuities (with respect to  $\mathbf{x}$ )  $\delta v_i(\mathbf{x}) = \delta u_i(\mathbf{x}^+) - \delta u_i(\mathbf{x}^-)$  and  $V_i^{(j)}(\mathbf{x}'; \mathbf{x}) = U_i^{(j)}(\mathbf{x}'; \mathbf{x}^+) - U_i^{(j)}(\mathbf{x}'; \mathbf{x}^-)$ .
- Second, we rename  $\bar{\mathbf{x}}$  the point  $\mathbf{x} \in \mathcal{S}$  in the result just obtained, and then let  $\bar{\mathbf{x}}$  go to some point of the crack front  $\mathcal{F}$ , again noted  $\mathbf{x}$ . We replace the discontinuities  $\delta v_i(\bar{\mathbf{x}})$  by their asymptotic expressions in terms of the local perturbed SIFs  $\delta K_\alpha(\mathbf{x})$ , and the discontinuities  $V_i^{(j)}(\mathbf{x}'; \bar{\mathbf{x}})$  by their asymptotic expressions in the terms of the crack-face weight functions  $k_\alpha^{(j)}(\mathbf{x}'; \mathbf{x})$  -  $k_\alpha^{(j)}(\mathbf{x}'; \mathbf{x})$  is the SIF of mode  $\alpha$  at the point  $\mathbf{x} \in \mathcal{F}$  generated through application of a unit point force in the direction  $\mathbf{e}_j$  at the point  $\mathbf{x}' \in \mathcal{S}^+ \cup \mathcal{S}^-$ .

This procedure yields the expressions of the local perturbed SIFs  $\delta K_\alpha(\mathbf{x})$ :

$$\delta K_\alpha(\mathbf{x}) = \int_{\mathcal{S}^+ \cup \mathcal{S}^-} T_{jk}^0 \frac{\partial k_\alpha^{(j)}(\mathbf{x}'; \mathbf{x})}{\partial x'_k} \phi(\mathbf{x}') \cdot \mathbf{n}(\mathbf{x}') dS' = \int_{\mathcal{S}^+} T_{jk}^0 \frac{\partial \llbracket k_\alpha^{(j)}(\mathbf{x}'; \mathbf{x}) \rrbracket}{\partial x'_k} \phi_y(\mathbf{x}') dS'.$$

In the last expression the contributions of the crack surfaces  $\mathcal{S}^+$  and  $\mathcal{S}^-$  have been grouped again, using the notation  $\llbracket . \rrbracket$  for a discontinuity with respect to  $\mathbf{x}'$  - note that  $\llbracket k_\alpha^{(j)}(\mathbf{x}'; \mathbf{x}) \rrbracket$  thus represents the  $\alpha$ -th SIF at the point  $\mathbf{x} \in \mathcal{F}$  generated by a *pair of opposite* unit point forces in the direction  $\mathbf{e}_j$  applied at the points  $\mathbf{x}'^+ \in \mathcal{S}^+$  and  $\mathbf{x}'^- \in \mathcal{S}^-$ .

In a final step, we use the expressions of the crack-face weight functions  $\llbracket k_\alpha^{(j)}(\mathbf{x}'; \mathbf{x}) \rrbracket$  for a semi-infinite crack in an infinite body, provided for instance in (Gao and Rice, 1986); the final results read, after some integrations by parts aimed at simplifying expressions:

$$\left\{ \begin{array}{l} \delta K_I(x, z) = 0 \\ \delta K_{II}(x, z) = -\frac{\sqrt{2}}{\pi^{3/2}} \int_{-\infty}^x dx' \int_{-\infty}^{+\infty} \frac{\sqrt{x-x'}}{(x-x')^2 + (z-z')^2} \times \\ \quad \left\{ \left[ 1 + \frac{2\nu}{2-\nu} \frac{(x-x')^2 - (z-z')^2}{(x-x')^2 + (z-z')^2} \right] \left[ T_{xx}^0 \frac{\partial \phi_y}{\partial x'}(x', z') + T_{xz}^0 \frac{\partial \phi_y}{\partial z'}(x', z') \right] \right. \\ \quad \left. + \frac{4\nu}{2-\nu} \frac{(x-x')(z-z')}{(x-x')^2 + (z-z')^2} \left[ T_{xz}^0 \frac{\partial \phi_y}{\partial x'}(x', z') + T_{zz}^0 \frac{\partial \phi_y}{\partial z'}(x', z') \right] \right\} dz' \\ \delta K_{III}(x, z) = -\frac{\sqrt{2}}{\pi^{3/2}} \int_{-\infty}^x dx' \int_{-\infty}^{+\infty} \frac{\sqrt{x-x'}}{(x-x')^2 + (z-z')^2} \times \\ \quad \left\{ \frac{4\nu}{2-\nu} \frac{(x-x')(z-z')}{(x-x')^2 + (z-z')^2} \left[ T_{xx}^0 \frac{\partial \phi_y}{\partial x'}(x', z') + T_{xz}^0 \frac{\partial \phi_y}{\partial z'}(x', z') \right] \right. \\ \quad \left. + \left[ 1 - \frac{2\nu}{2-\nu} \frac{(x-x')^2 - (z-z')^2}{(x-x')^2 + (z-z')^2} \right] \left[ T_{xz}^0 \frac{\partial \phi_y}{\partial x'}(x', z') + T_{zz}^0 \frac{\partial \phi_y}{\partial z'}(x', z') \right] \right\} dz'. \end{array} \right.$$

In these expressions  $x$  denotes the position of the unperturbed crack front within the crack plane, and  $z$  the position of the point of observation of the SIFs on this front. These results coincide (in a more explicit form) with those obtained by Movchan *et al.* (1998) by their general method of solution, and confirmed by Lebihain *et al.* (2023) by a third method (differing from that used here).

Consider for instance, following Movchan *et al.* (1998), an out-of-plane crack perturbation independent of  $x'$  and sinusoidal in  $z'$  (so that the perturbed crack surface assumes the shape of a corrugated sheet),

$$\phi_y(x', z') = A \cos(kz')$$

where  $A$  and  $k$  are parameters, with  $k > 0$ . Then calculation of the integrals yields

$$\begin{cases} \delta K_I(x, z) &= 0 \\ \delta K_{II}(x, z) &= A \frac{\sqrt{2k}}{2 - \nu} \left[ 2T_{xz}^0 \sin(kz) - \nu T_{zz}^0 \cos(kz) \right] \\ \delta K_{III}(x, z) &= A \frac{\sqrt{2k}}{2 - \nu} \left[ -\nu T_{xz}^0 \cos(kz) + 2(1 - \nu) T_{zz}^0 \sin(kz) \right]. \end{cases}$$

Again, these results coincide with those of Movchan *et al.* (1998), as corrected for some errors of sign by Lebihain *et al.* (2023).

## 5 Concluding summary

Bueckner-Rice's classical theory for 3D cracked elastic bodies, which was developed by Rice (1985, 1989) using a re-formulation of Bueckner (1987)'s theory of 3D weight functions, provided a general expression of the first-order variation of the displacement field arising from some small, but otherwise arbitrary *tangential* perturbation of the crack or cracks considered. Its major output lied in the possibility to evaluate the perturbed SIFs along the crack front from inspection of the asymptotic form of the perturbed displacement near this front; this paved the way to applications to problems of crack propagation in various configurations of practical interest.

The object of this paper was to extend the theory to general perturbations of the crack(s) including a component perpendicular to the crack surface. The essential aim was to enlarge the possibility of evaluating the distribution of the SIFs along the perturbed front from the asymptotic behaviour of the perturbed displacement, to small but otherwise completely arbitrary perturbations; and thus offer the perspective, in the future, of a much simpler and more direct treatment of out-of-plane crack perturbation problems than before.

The first step, expounded in Section 2, consisted of deriving a general formula for the first-order variation of the total potential energy of a 3D elastic body, resulting from some arbitrary perturbation of an embedded ensemble of geometrically regular holes. The motivation for dealing with such geometric objects, rather than cracks directly, was to avoid ambiguities tied to the momentary occurrence of divergent integrals in the mathematical analysis. The treatment, although new, was inspired from those of deLorenzi (1982) and Destuynder *et al.* (1983) for *tangential* perturbations of *cracks*, in that it adopted their strategy of performing a *Lagrangian* calculation of the energy in the perturbed configuration. The final result obtained stood as an extension of that of Eshelby (1951) for a translatory (rigid-body) motion of an ensemble of defects, to general perturbations.

Section 3 was devoted to the second step wherein the same problem was considered, but for embedded cracks instead of geometrically regular holes. In order to avoid divergent integrals which would result from repetition of the entire reasoning of Section 2 for cracks, these geometrically singular objects were dealt with by applying the final result of this Section to elongated voids, and letting their thickness go to zero. The formula obtained for the variation of energy was identical to that derived by deLorenzi (1982) and Destuynder *et al.* (1983), except for the extension of its domain of validity, initially limited to tangential crack perturbations, to completely arbitrary ones. The volume integral found was transformed through some integration by parts into a sum of integrals over the crack front and surface, better fit to theoretical analyses; careful attention was paid in the derivation to the geometric singularity at the crack front. Comments on the connections with the works of Rice (1968) and Knowles and Sternberg (1972) devoted to special cracked geometries and perturbations were finally provided, with special emphasis on the somewhat controversial mechanical interpretation of Rice's second integral,  $J_y$  or  $J_2$ .

In a third and final step, in Section 4, we drew inspiration from the works of Rice (1985, 1989) to show how the first-order variation of the entire displacement field may be deduced from that of the total energy. In spite of the greater complexity of the expression of the variation of energy than in the previous works, due to consideration of general crack perturbations instead of purely tangential ones, Rice (1985, 1989)'s arguments based on a Legendre transform of the energy and Bueckner (1987)'s theory of 3D weight functions, could be adapted without any major additional difficulty. It led to a more complex final expression of the local variation of displacement than that found by Rice (1985, 1989), including not only a complementary term in the integral over the crack front, but also an extra integral over the entire crack surface.

Forthcoming papers will be devoted to the application of the *extended Bueckner-Rice theory* thus defined to the evaluation of the distribution of the SIFs along the front of slightly perturbed cracks, in various configurations of practical interest. We shall consider first the simplest case of an initially plane, semi-infinite crack in an infinite body, for which a solution of the out-of-plane perturbation problem is available from the work of Movchan *et al.* (1998). The aim here will be to confirm - and extend somewhat - the results

of Movchan *et al.* (1998), and illustrate how use of the extended Bueckner-Rice theory can lead to a simpler and more versatile treatment of such problems than general methods of solution of elasticity problems. In a second step, we shall envisage more complex cracked configurations for which no solution of the out-of-plane perturbation problem is known at present - first of all that of a penny-shaped crack embedded in an infinite body, of special interest in view of Sommer (1969)'s seminal experimental study of the out-of-plane instability of a mode I+III crack in a very similar geometric configuration.

A final additional remark pertains to the potential use of Bueckner-Rice's theory, in both its original and extended versions, for applications other than the evaluation of the SIFs along the perturbed crack front. The expressions of the variation of the full displacement field, derived in Rice (1985, 1989)'s works and in a more general context here, obviously provide much more than the distribution of the perturbed SIFs. Little use has however been made of this displacement field up to now - notable exceptions including the works of Gao (1992) and Leblond and Lazarus (2015) on the distribution of the *non-singular stresses* along the front of a crack slightly perturbed within its plane. But much more could be extracted from the variation of the displacement field; the perturbation of the *displacement discontinuity across the crack surface*, for instance, could be of major interest in crack studies using cohesive zone models - see for instance the very recent work of Lebihain *et al.* (2022).

## Dedication

This paper is dedicated to our dear friend and colleague, Professor James R. Rice of Harvard University, as a token of our deep affection and admiration.

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**A Appendix - Elements for the calculation of the integral  $\int_{\partial\mathcal{D}(s;\epsilon)} P_{jk} \theta_k n_j dl$  in the limit  $\epsilon \rightarrow 0^+$**

In this Appendix, we provide useful indications on how to calculate the limit of the integral  $\int_{\partial\mathcal{D}(s;\epsilon)} P_{jk} \theta_k n_j dl$  for  $\epsilon \rightarrow 0^+$ , without insisting on unnecessary details. We start by providing a number of required formulae.

- *Asymptotic mechanical fields in cylindrical coordinates and components:*<sup>9</sup>

\* Stresses:

$$\left\{ \begin{array}{l} \sigma_{rr} = \frac{K_I}{4\sqrt{2\pi r}} \left( 5 \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \right) + \frac{K_{II}}{4\sqrt{2\pi r}} \left( -5 \sin \frac{\theta}{2} + 3 \sin \frac{3\theta}{2} \right) \\ \sigma_{\theta\theta} = \frac{K_I}{4\sqrt{2\pi r}} \left( 3 \cos \frac{\theta}{2} + \cos \frac{3\theta}{2} \right) + \frac{K_{II}}{4\sqrt{2\pi r}} \left( -3 \sin \frac{\theta}{2} - 3 \sin \frac{3\theta}{2} \right) \\ \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) = 2\nu \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} - 2\nu \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \\ \sigma_{r\theta} = \frac{K_I}{4\sqrt{2\pi r}} \left( \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right) + \frac{K_{II}}{4\sqrt{2\pi r}} \left( \cos \frac{\theta}{2} + 3 \cos \frac{3\theta}{2} \right) \\ \sigma_{rz} = \frac{K_{III}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \\ \sigma_{\theta z} = \frac{K_{III}}{\sqrt{2\pi r}} \cos \frac{\theta}{2}. \end{array} \right. \quad (\text{A.1})$$

\* Displacements:

$$\left\{ \begin{array}{l} u_r = \frac{1+\nu}{2E} K_I \sqrt{\frac{r}{2\pi}} \left[ (5-8\nu) \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \right] \\ \quad + \frac{1+\nu}{2E} K_{II} \sqrt{\frac{r}{2\pi}} \left[ (-5+8\nu) \sin \frac{\theta}{2} + 3 \sin \frac{3\theta}{2} \right] \\ u_\theta = \frac{1+\nu}{2E} K_I \sqrt{\frac{r}{2\pi}} \left[ (-7+8\nu) \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} \right] \\ \quad + \frac{1+\nu}{2E} K_{II} \sqrt{\frac{r}{2\pi}} \left[ (-7+8\nu) \cos \frac{\theta}{2} + 3 \cos \frac{3\theta}{2} \right] \\ u_z = \frac{4(1+\nu)}{E} K_{III} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2}. \end{array} \right. \quad (\text{A.2})$$

<sup>9</sup> Asymptotic expressions of the stresses and displacements are classically provided using polar coordinates but *Cartesian* components; however cylindrical components are much more adapted to the calculation in question here.

- *Cylindrical components of the gradient of displacement:*

$$\begin{cases} (\nabla u)_{rr} = u_{r,r} \\ (\nabla u)_{\theta r} = u_{\theta,r} \\ (\nabla u)_{zr} = u_{z,r} \end{cases} ; \begin{cases} (\nabla u)_{r\theta} = \frac{1}{r}(u_{r,\theta} - u_\theta) \\ (\nabla u)_{\theta\theta} = \frac{1}{r}(u_{\theta,\theta} + u_r) \\ (\nabla u)_{z\theta} = \frac{1}{r}u_{z,\theta}. \end{cases} \quad (\text{A.3})$$

Components  $(\nabla u)_{\alpha z}$  are not provided here because they are of order  $O(r^{1/2})$ , and therefore negligible with respect to components  $(\nabla u)_{\alpha r}$  and  $(\nabla u)_{\alpha\theta}$  which are of order  $O(r^{-1/2})$ .

- *Expression of the quantity  $P_{jk} \theta_k n_j$  over  $\partial\mathcal{D}(s; \epsilon)$ :*

On the boundary  $\partial\mathcal{D}(s; \epsilon)$  of the circular disk  $\mathcal{D}(s; \epsilon)$ , the unit outward normal vector  $\mathbf{n}$  coincides with the unit radial vector  $\mathbf{e}_r$ , and  $\boldsymbol{\theta} = \theta_x \mathbf{e}_x + \theta_y \mathbf{e}_y = (\theta_x \cos \theta + \theta_y \sin \theta) \mathbf{e}_r + (-\theta_x \sin \theta + \theta_y \cos \theta) \mathbf{e}_\theta$ . It follows, using the definition (18) of Eshelby's tensor  $\mathbf{P}$ , that

$$\begin{aligned} P_{jk} \theta_k n_j &= w \boldsymbol{\theta} \cdot \mathbf{n} - \sigma_{ij} n_j u_{i,k} \theta_k = w \boldsymbol{\theta} \cdot \mathbf{e}_r - T_i u_{i,k} \theta_k \quad (\text{where } \mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{e}_r) \\ &= w(\theta_x \cos \theta + \theta_y \sin \theta) - \mathbf{T} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\theta}, \end{aligned} \quad (\text{A.4})$$

with

$$w = \frac{1}{2} [\sigma_{rr}(\nabla u)_{rr} + \sigma_{\theta\theta}(\nabla u)_{\theta\theta} + \sigma_{r\theta}(\nabla u)_{r\theta} + \sigma_{r\theta}(\nabla u)_{\theta r} + \sigma_{rz}(\nabla u)_{zr} + \sigma_{\theta z}(\nabla u)_{z\theta}] \quad (\text{A.5})$$

and - Greek indices referring to cylindrical components:

$$\begin{aligned} \mathbf{T} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\theta} &= T_\alpha (\nabla u)_{\alpha\beta} \theta_\beta \\ &= [\sigma_{rr}(\nabla u)_{rr} + \sigma_{r\theta}(\nabla u)_{\theta r} + \sigma_{rz}(\nabla u)_{zr}] (\theta_x \cos \theta + \theta_y \sin \theta) \\ &\quad + [\sigma_{rr}(\nabla u)_{r\theta} + \sigma_{r\theta}(\nabla u)_{\theta\theta} + \sigma_{rz}(\nabla u)_{z\theta}] (-\theta_x \sin \theta + \theta_y \cos \theta). \end{aligned} \quad (\text{A.6})$$

Combination of equations (A.4), (A.5) and (A.6) yields

$$\begin{aligned} P_{jk} \theta_k n_j &= \theta_x \left\{ \frac{1}{2} \cos \theta [-\sigma_{rr}(\nabla u)_{rr} + \sigma_{\theta\theta}(\nabla u)_{\theta\theta} + \sigma_{r\theta}(\nabla u)_{r\theta} - \sigma_{r\theta}(\nabla u)_{\theta r} \right. \\ &\quad \left. - \sigma_{rz}(\nabla u)_{zr} + \sigma_{\theta z}(\nabla u)_{z\theta}] + \sin \theta [\sigma_{rr}(\nabla u)_{r\theta} + \sigma_{r\theta}(\nabla u)_{\theta\theta} + \sigma_{rz}(\nabla u)_{z\theta}] \right\} \\ &\quad + \theta_y \left\{ \frac{1}{2} \sin \theta [-\sigma_{rr}(\nabla u)_{rr} + \sigma_{\theta\theta}(\nabla u)_{\theta\theta} + \sigma_{r\theta}(\nabla u)_{r\theta} - \sigma_{r\theta}(\nabla u)_{\theta r} \right. \\ &\quad \left. - \sigma_{rz}(\nabla u)_{zr} + \sigma_{\theta z}(\nabla u)_{z\theta}] - \cos \theta [\sigma_{rr}(\nabla u)_{r\theta} + \sigma_{r\theta}(\nabla u)_{\theta\theta} + \sigma_{rz}(\nabla u)_{z\theta}] \right\}. \end{aligned} \quad (\text{A.7})$$

Equations (A.1), (A.2), (A.3) and (A.7) permit to calculate the integral  $\int_{\partial\mathcal{D}(s;\epsilon)} P_{jk} \theta_k n_j dl$  in the limit  $\epsilon \rightarrow 0^+$ . The calculation is heavy but a number of useful remarks, listed below, permit to simplify it to some extent.

- *About the term proportional to  $\theta_x$ .*

\* Apply a symmetry with respect to the plane ( $sxy$ ) to both the geometry and the loading. In this transformation the quantities  $K_I, K_{II}, K_{III}, \theta_x$  become  $K_I, K_{II}, -K_{III}, \theta_x$  respectively. Furthermore the variation of total energy obviously does not change. It follows that the terms in  $K_I K_{III} \theta_x$  and  $K_{II} K_{III} \theta_x$  in the integral  $\int_{\partial\mathcal{D}(s;\epsilon)} P_{jk} \theta_k n_j d\ell$  must be zero.

\* Now apply a symmetry with respect to the plane ( $sxz$ ). The quantities  $K_I, K_{II}, K_{III}, \theta_x$  become  $K_I, -K_{II}, -K_{III}, \theta_x$  and the variation of energy still does not change; hence the term in  $K_I K_{II} \theta_x$  (and again that in  $K_I K_{III} \theta_x$ ) in the integral must be zero.

\* Hence the term proportional to  $\theta_x$  in the integral can only include contributions in  $K_I^2 \theta_x, K_{II}^2 \theta_x$  and  $K_{III}^2 \theta_x$ . Furthermore accounting for the nonzero stress components in each mode, one sees that to calculate the terms in  $K_I^2 \theta_x$  and  $K_{II}^2 \theta_x$ , it suffices to consider, within the right-hand side of equation (A.7), the expression

$$\theta_x \left\{ \frac{1}{2} \cos \theta [-\sigma_{rr}(\nabla u)_{rr} + \sigma_{\theta\theta}(\nabla u)_{\theta\theta} + \sigma_{r\theta}(\nabla u)_{r\theta} - \sigma_{r\theta}(\nabla u)_{\theta r}] \right. \\ \left. + \sin \theta [\sigma_{rr}(\nabla u)_{r\theta} + \sigma_{r\theta}(\nabla u)_{\theta\theta}] \right\},$$

whereas for the term in  $K_{III}^2 \theta_x$ , the expression

$$\theta_x \left\{ \frac{1}{2} \cos \theta [-\sigma_{rz}(\nabla u)_{zr} + \sigma_{\theta z}(\nabla u)_{z\theta}] + \sin \theta \sigma_{rz}(\nabla u)_{z\theta} \right\}$$

is sufficient.

- *About the term proportional to  $\theta_y$ .*

\* Consideration of a symmetry with respect to the plane ( $sxy$ ), in which the quantities  $K_I, K_{II}, K_{III}, \theta_y$  become  $K_I, K_{II}, -K_{III}, \theta_y$  now yields that the terms in  $K_I K_{III} \theta_y$  and  $K_{II} K_{III} \theta_y$  in the integral  $\int_{\partial\mathcal{D}(s;\epsilon)} P_{jk} \theta_k n_j d\ell$  must be zero.

\* Similarly consideration of a symmetry with respect to the plane ( $sxz$ ), in which  $K_I, K_{II}, K_{III}, \theta_y$  become  $K_I, -K_{II}, -K_{III}, -\theta_y$ , reveals that the terms in  $K_I^2 \theta_y, K_{II}^2 \theta_y, K_{III}^2 \theta_y$  (and again that in  $K_{II} K_{III} \theta_y$ ) must also be zero.

\* Hence the term proportional to  $\theta_y$  in the integral can only include a contribution in  $K_I K_{II} \theta_y$ . Accounting for the nonzero stress components in modes I and II, one sees that to calculate this contribution, it suffices to consider in equation (A.7) the expression

$$\theta_y \left\{ \frac{1}{2} \sin \theta [-\sigma_{rr}(\nabla u)_{rr} + \sigma_{\theta\theta}(\nabla u)_{\theta\theta} + \sigma_{r\theta}(\nabla u)_{r\theta} - \sigma_{r\theta}(\nabla u)_{\theta r}] \right. \\ \left. - \cos \theta [\sigma_{rr}(\nabla u)_{r\theta} + \sigma_{r\theta}(\nabla u)_{\theta\theta}] \right\}.$$

- *Final reduction of the number of terms.*

The problem is reduced to calculating the integral, from  $\theta = -\pi$  to  $\pi$ , of expressions of the form  $\sigma_{\alpha\beta}(\nabla u)_{\gamma\delta} \times (\cos \theta$  or  $\sin \theta)$ . All components  $\sigma_{\alpha\beta}$  and  $(\nabla u)_{\gamma\delta}$  are sums of terms in  $\cos(\theta/2), \sin(\theta/2), \cos(3\theta/2)$  and  $\sin(3\theta/2)$ . Using classical trigonometric formulae, each product  $\sigma_{\alpha\beta}(\nabla u)_{\gamma\delta}$  may be expressed as a sum of terms in  $\cos(n\theta)$  and  $\sin(n\theta)$  where  $n$  is an integer. But, among these terms, one need only consider those

in  $\cos \theta$ , if the product is to be multiplied by  $\cos \theta$  prior to integration, or in  $\sin \theta$ , if it is to be multiplied by  $\sin \theta$ ; indeed all other terms yield a zero contribution in the integral. This greatly reduces the amount of calculation required.

## B Appendix - Convergence of the integral $\int_{S^+} [[w]](\mathbf{x}) \boldsymbol{\theta}(\mathbf{x}) \cdot \mathbf{n}^+(\mathbf{x}) dS$ at the crack front

The aim of this Appendix is to show that the principal term of the expansion of the discontinuity  $[[w]]$  of the density of elastic energy  $w$  on the crack surface, proportional to the inverse of the distance to the crack front, is zero, resulting in convergence of the integral  $\int_{S^+} [[w]](\mathbf{x}) \boldsymbol{\theta}(\mathbf{x}) \cdot \mathbf{n}^+(\mathbf{x}) dS$ .

The principal term of the expansion of  $[[w]]$  arises from the singular terms in the expansion of the stress components, involving the SIFs; hence *it suffices to consider only these singular terms*, given by equations (A.1).

On the crack faces, because of the boundary conditions (in local cylindrical coordinates)  $\sigma_{\theta r} = \sigma_{\theta\theta} = \sigma_{\theta z} = 0$ , there are only three nonzero stress components,  $\sigma_{rr}$ ,  $\sigma_{zz}$  and  $\sigma_{rz}$ . But  $\sigma_{zz}$  is tied to  $\sigma_{rr}$  through the plane strain condition<sup>10</sup>  $\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) = \nu\sigma_{rr}$ ; hence there are in fact only two independent nonzero stress components,  $\sigma_{rr}$  and  $\sigma_{rz}$ . The density of elastic energy  $w$  is a weighted sum of the squares of these two components.

Now for the singular terms of the stress expansion, see equations (A.1):

- $\sigma_{rr}$  is nonzero only in mode II, and then takes opposite values on the two faces of the crack;
- $\sigma_{rz}$  is nonzero only in mode III, and similarly changes sign from one face to the other.

Thus, if one considers only the principal term of the expansion of  $w$ , this quantity takes identical values on the two faces of the crack so that the corresponding discontinuity  $[[w]]$  is zero, as announced.

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<sup>10</sup> This seemingly “purely 2D” condition applies even in 3D to the singular terms of the stress expansion; see Leblond and Torlai (1992).