About the structural stability of Maxwell fluids: convergence toward elastodynamics

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Abstract Maxwell's models for viscoelastic flows are famous for their potential to unify elastic motions of solids with viscous motions of liquids in the continuum mechanics perspective. But rigorous proofs are lacking. The present note is a contribution toward well-defined viscoelastic flows proved to encompass both solid and (liquid) fluid regimes. In a first part, we consider the structural stability of *particular viscoelastic flows*: 1D shear waves solutions to damped wave equations. We show the convergence toward purely elastic 1D shear waves solutions to standard wave equations, as the relaxation time λ and the viscosity $\dot{\mu}$ grow unboundedly $\lambda \equiv \frac{1}{G}\dot{\mu} \to \infty$ in Maxwell's constitutive equation

$$\lambda \overset{\diamond}{\boldsymbol{\tau}} + \boldsymbol{\tau} = 2\dot{\mu}\boldsymbol{D}(\boldsymbol{u})$$

for the stress $\boldsymbol{\tau}$ of viscoelastic fluids with velocity $\boldsymbol{u}$. In a second part, we consider the structural stability of general *multi-dimensional viscoelastic flows*. To that aim, we embed Maxwell's constitutive equation in a symmetric-hyperbolic system of PDEs which we proposed in our previous publication [ESAIM:M2AN 55 (2021) 807-831] so as to define multi-dimensional viscoelastic flows unequivocally. Next, we show the continuous dependence of multi-dimensional viscoelastic flows on $\lambda \equiv \frac{1}{G}\dot{\mu}$ using the relative-entropy tool developped for symmetric-hyperbolic systems after C. M. Dafermos. It implies convergence of the viscoelastic flows defined in [ESAIM:M2AN 55 (2021) 807-831] toward compressible neo-Hookean elastodynamics when $\lambda \to \infty$.

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1 Maxwell fluids as links between solids and Newtonian fluids

We consider the viscoelastic motions of a fluid body occupying on times $t \in [0,T)$ a subset $\Omega \subset \mathbb{R}^3$ of the Euclidean ambiant space equipped with a Cartesian system of coordinates $\{x^i, i = 1 \ldots 3\}$.

Denoting $\boldsymbol{u} = u^i \boldsymbol{e}_i$ the velocity field of the fluid, ρ the mass density (a scalar field), $\boldsymbol{f}$ a bulk force field, we assume the following mass and momentum balances

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0 \tag{1}$$

$$\partial_t (\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\sigma}) = \rho \boldsymbol{f} \tag{2}$$

using a model of Maxwell type [9] for the extra-stress $\boldsymbol{\tau}$ in Cauchy 2-tensor $\boldsymbol{\sigma} = -p\boldsymbol{\delta} + \boldsymbol{\tau}$, $p(\rho)$ being a pressure in the fluid and $\boldsymbol{\delta}$ the identity 2-tensor, i.e.:

$$\lambda \overset{\diamond}{\boldsymbol{\tau}} + \boldsymbol{\tau} = 2\acute{\mu} \boldsymbol{D}(\boldsymbol{u}) \,. \tag{3}$$

In Maxwell's constitutive equation (3), $\acute{\mu} > 0$ is a viscosity parameter, $\lambda > 0$ is a relaxation-time parameter, and $\overset{\diamond}{\boldsymbol{\tau}}$ is an *objective* time-rate operator see e.g. [12].

It is widely admitted that Maxwell fluid models (i.e. those using (3)) can link fluids where $\boldsymbol{\tau} \xrightarrow{\lambda \to 0} 2\acute{\mu} \boldsymbol{D}(\boldsymbol{u})$ in the Newtonian limit, denoting $\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} \boldsymbol{u}^T\right)$ as usual, with solids governed by elastodynamics when $\lambda \equiv \frac{1}{G}\acute{\mu} \to \infty$. Besides, in applications, one often considers the Upper-Convected Maxwell (UCM) model, with objective time-rate $\overset{\diamond}{\boldsymbol{\tau}}$ in (3) defined by the Upper-Convected (UC) derivative

$$\overset{\triangledown}{\boldsymbol{\tau}} := \partial_t \boldsymbol{\tau} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\tau} - (\boldsymbol{\nabla} \boldsymbol{u})\boldsymbol{\tau} - \boldsymbol{\tau}(\boldsymbol{\nabla} \boldsymbol{u})^T \tag{4}$$

since the formal limit of (3) when $\lambda \equiv \frac{1}{G}\acute{\mu} \to \infty$ i.e. $\overset{\triangledown}{\boldsymbol{\tau}} = 2G\boldsymbol{D}(\boldsymbol{u})$ is compatible with Hookean elastodynamics, i.e. the case when $\boldsymbol{\tau} = \frac{\acute{\mu}}{\lambda}\left(\boldsymbol{F}\boldsymbol{F}^T - \boldsymbol{I}\right)$ and $\boldsymbol{F}$ is the deformation gradient associated with $\boldsymbol{u}$, governed by

$$(\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{F} = (\boldsymbol{\nabla} \boldsymbol{u})\boldsymbol{F} \,. \tag{5}$$

But we are not aware of a rigorous proof of such a *structural stability* result for Maxwell fluid models that would link general multi-dimensional motions of (Hookean) solids with Newtonian fluids.

One-dimensional (1D) shear waves can be defined unequivocally with (1)–(2)–(3). The Newtonian fluid limit $\boldsymbol{\tau} \xrightarrow{\lambda \to 0} 2\acute{\mu} \boldsymbol{D}(\boldsymbol{u})$ of such 1D shear waves is established in [10] as a consequence of the structural stability of *linear* UCM models. But we are not aware of a rigorous proof of the elastodynamics limit $\lambda \equiv \frac{1}{G}\acute{\mu} \to \infty$, though.

Multi-dimensional time-continuous motions cannot be defined unequivocally with (1)–(2)–(3) in general, even on a small time interval using smooth initial conditions on the whole space $\mathbb{R}^3$, because the quasilinear system (1)–(2)–(3) may not be hyperbolic. So the question of structural stability cannot be properly addressed for general multi-dimensional viscoeastic flows as such with (1)–(2)–(3), i.e. as one usually "defines" viscoeastic flows of Maxwell type in the literature.

In Sec. 3, we extend the study of [10] (for linear Maxwell equations) to the convergence toward solid elastodynamics when $\lambda \equiv \frac{1}{G}\dot{\mu} \to \infty$, specifically for the *1D shear waves* solutions to (1)–(2)–(3) which are recalled in Sec. 2. Such specific studies are a first step toward the structural stability of more general (nonlinear) models of Maxwell type, and to a fully rigorous link between solid and fluid regimes using Maxwell models.

Next, to address the structural stability of physically-relevant (nonlinear) Maxwell models, a further step is to first unequivocally define *multi-dimensional* motions through solutions to (1)–(2)–(3). We propose here to build upon our former work [2], thus to consider the structural stability of *unequivocal viscoelastic flows* defined as solutions to a quasilinear system of PDEs with a symmetric-hyperbolic reformulation that implies (3).

In a nutshell, our reformulation of Maxwell flows interprets the extra-stress as $\boldsymbol{\tau} = \rho G(\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^T - \boldsymbol{I})$, with a view to extending to *Maxwell fluids with finite parameters* $\dot{\mu}, \lambda \equiv \frac{1}{G}\dot{\mu} > 0$ an elastodynamics system where $\boldsymbol{F}$ is the deformation gradient associated with $\boldsymbol{u}$ and ρ. That is the reason why our reformulation [2] requires

$$\partial_t (\rho \boldsymbol{F}) - \boldsymbol{\nabla} \times \left(\rho \boldsymbol{F}^T \times \boldsymbol{u}\right) = \boldsymbol{0} \tag{6}$$

like in elastodynamics, along with the famous involution termed Piola's identity

$$\operatorname{div}(\rho \boldsymbol{F}^T) = \boldsymbol{0} \,. \tag{7}$$

Notice that (6)–(7)–(1) together imply (5). Moreover, we assume

$$\lambda(\partial_t + \boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{A} = \boldsymbol{F}^{-1}\boldsymbol{F}^{-T} \tag{8}$$

for the symmetric positive definite 2-tensor $\boldsymbol{A}$. Then, a constitutive equation of Maxwell-type (3) holds (for smooth compressible flows [1]), and solutions to the system (1)–(2)–(6)–(7)–(8) without source term, where $\boldsymbol{\tau} = \rho G(\boldsymbol{F}\boldsymbol{A}\boldsymbol{F}^T - \boldsymbol{I})$ and $p = -\partial_{\rho^{-1}} e_0$ with e_0 convex in ρ^{-1}, additionally satisfy a conservation law for a scalar quantity that is convex in a conserved variable $\mathrm{U}(\boldsymbol{u}, \boldsymbol{F}, \rho, \boldsymbol{A})$ (see [2]):

$$\eta := \frac{\rho}{2}|\boldsymbol{u}|^2 + \rho e_0 + \rho\frac{G}{2}\boldsymbol{F}\boldsymbol{A} : \boldsymbol{F} \,. \tag{9}$$

That is, using notations of [6, Chap. V] and denoting $\xi = \frac{1}{\lambda} > 0$, there exists a variable change in a convex domain $\mathrm{U} \in \mathcal{O}$ such that our system rewrites [1]

$$\partial_t \mathrm{U} + \partial_\alpha \mathrm{G}_\alpha(\mathrm{U}) = \xi \Pi(\mathrm{U}) \tag{10}$$

with smooth vector fluxes G_α, and smooth fluxes $\mathrm{Q}_\alpha(\mathrm{U})$ exist so that $\eta(\mathrm{U})$ satisfies

[1] Involutions are keys here, we refer to the short summary [4] for instance.

$$\partial_t \eta(\mathrm{U}) + \partial_\alpha \mathrm{Q}_\alpha(\mathrm{U}) = \xi \mathrm{D_U}\eta(\mathrm{U}) \cdot \Pi(\mathrm{U})\,. \tag{11}$$

Consequently, our system (1)–(2)–(6)–(7)–(8), equiv. (10) after a variable change, admits a symmetric-hyperbolic formulation, see [2], and one can define unequivocally time-continuous flows of Maxwell fluids on small time intervals given general smooth initial conditions. So the question of structural stability can be considered for our reformulation of Maxwell fluids, in particulat using standard results for symmetric-hyperbolic systems [6]. Note to that aim that, in the hyperbolicity domain $\mathcal{O} \ni \mathrm{U}_1, \mathrm{U}_2$, the source term $\Pi(\mathrm{U})$ is such that

$$|\Pi_\mathrm{m}(\mathrm{U}_1) - \Pi_\mathrm{m}(\mathrm{U}_2)| \le \mathrm{C_m}\|\mathrm{U}_1 - \mathrm{U}_2\|^2 \tag{12}$$

for each component $\mathrm{m} = 1 \ldots 1 + \mathrm{d} + \mathrm{d}^2 + \frac{\mathrm{d(d+1)}}{2}$ of the system (10).

In Sec. 4, we show the continuous dependence on $\lambda \equiv \frac{1}{G}\acute{\mu}$, of general multi-dimensional viscoelastic flows defined unequivocally following [2], using the relative-entropy tool developped for symmetric-hyperbolic systems after C. M. Dafermos. It implies the following structural stability result: convergence of our viscoelastic flows toward compressible neo-Hookean elastodynamics when $\lambda \equiv \frac{1}{G}\acute{\mu} \to \infty$.

2 Setting of the problem for 1D viscoelastic shear waves

A shear wave $\boldsymbol{u} = u(t,y)\boldsymbol{e}_x$, $\boldsymbol{\tau} = \tau^{xy}(t,y)\boldsymbol{e}_x \otimes \boldsymbol{e}_y$ solution to (1)–(2)–(3) on $\{t \ge 0, x^2 \equiv y \in \Omega\} := (y_{\min}, y_{\max}) \subset \mathbb{R}$, can be built unequivocally given initial conditions

$$u(t=0,y) = u^0(y) \quad \tau^{xy}(t=0,y) = \tau^0(y)$$

plus boundary conditions at $y \in \partial\Omega$ when necessary (when $y_{\min}, y_{\max}$ finite), as briefly recalled below. Indeed, (1)–(2)–(3) reduces to:

$$\partial_t u = \partial_y \tau^{xy} + f^x\,, \tag{13}$$

$$\lambda \partial_t \tau^{xy} + \tau^{xy} = \acute{\mu}\partial_y u\,, \tag{14}$$

when one assumes ρ constant (this is natural for the 1D motion along $\boldsymbol{e}_x \equiv \boldsymbol{e}_{x^1}$ of a 2D body with a Lagrangian description using material coordinates $a = x - X(t,y)$, $b = y \equiv x^2$ in a Cartesian frame, such that $u \equiv \partial_t X$, $\tau^{xy} \equiv \partial_y X$, as it is the case for our reformulation of Maxwell fluids in [2]).

When $\Omega \equiv \{y > 0\}$ for instance, the Stokes first problem for (13–14), with $u^0 \equiv 0 \equiv \tau^0$, $u(t,y=0) = UH(t)$ ($U \in \mathbb{R}^+_*$, H denoting Heaviside's function), can be solved *analytically* by particular 1D shear waves [11]. Next, for those particular 1D shear waves, the Newtonian fluid limit $\lambda \to 0$ can be established directly using an analytical expression of the solution, see e.g. [8, (4.3)–(4.4)].

However, the same structural stability result can be established much more generally, i.e. for a class of well-defined solutions, by an *energy method*, i.e. an analysis using energy estimates satisfed by the solutions. For instance, the Newtonian fluid limit of well-defined solutions to (13–14) is a direct consequence of the structural stability established in [10], for a large class of solutions to *linear* Maxwell models (with limited physical relevance, though).

In the sequel, we use arguments similar to [10] (i.e. energy estimates satisfied by the solutions) to analyze the structural stability, when $\lambda \equiv \frac{1}{G}\dot{\mu}$ or equivalently when $\xi := \frac{1}{\lambda} \to 0$ keeping G fixed, of the damped wave system (13–14) which we rewrite using $\tau := \tau^{xy}(t,y)$, $f := f^x$ for simplicity:

$$\partial_t u - \partial_y \tau = f \tag{15}$$

$$\partial_t \tau - G\partial_y u = -\xi\tau\,. \tag{16}$$

Note that contrary to the limit $\lambda \to 0$ studied in [10], the limit $\xi := \frac{1}{\lambda} \to 0$ studied here is "easier" because it is a *non-singular* limit: only a *lower-order* term vanishes when $\xi \to 0$, which does not change the hyperbolic type of (15–16) in the limit. A detailed proof is nevertheless useful with a view to extending the structural stability result to general multi-dimensional viscoelastic flows solutions to a complex quasilinear system of PDEs. Introducing the variables $w^\pm = \tau \pm \sqrt{G}u$, it is also useful to rewrite (15–16) as

$$\partial_t w^\pm \mp \sqrt{G}\partial_y w^\pm = \pm\sqrt{G}f - \frac{\xi}{2}(w^+ + w^-)\,. \tag{17}$$

3 Structural stability of 1D shear waves when $\frac{1}{\lambda} \equiv \xi \to 0$

Given $\xi \geq 0$ and an open subset $\Omega := (y_{\min}, y_{\max}) \subset \mathbb{R}$, time-continuous solutions $u(t,y)$, $\tau(t,y)$ to (13–14) on $t \geq 0$, with value in $L^2(\Omega)$, are well-defined given

$$u(t=0,y) = u^0(y) \in L^2(\Omega) \quad \tau(t=0,y) = \tau^0(y) \in L^2(\Omega) \quad f(y) \in L^2\left((0,T)\times\Omega\right)$$

when $y \in \Omega \equiv \mathbb{R}$. We recall that $u(t,y)$, $\tau(t,y)$ in fact take values in $H^1(\Omega) \subset L^2(\Omega)$ then, see [6].

When $y_{\min}, y_{\max}$ are finite, time-continuous solutions $u(t,y)$, $\tau(t,y)$ to (13–14) remain well-defined on additionally specifying boundary conditions at $y \in \partial\Omega$, typically *maximally dissipative* given $g \in L^2\left((0,T)\times\partial\Omega\right)$ as follows

$$z_l^- := c_l^u u + c_l^\tau \tau = g_l \text{ with } c_l^u c_l^\tau < 0,\ c_l^\tau \neq -\sqrt{G}c_l^u \text{ at } y_{\min}\,, \tag{18}$$

$$z_r^+ := c_r^u u + c_r^\tau \tau = g_r \text{ with } c_r^u c_r^\tau > 0,\ c_r^\tau \neq \sqrt{G}c_r^u \text{ at } y_{\max}\,. \tag{19}$$

The latter solutions satisfy the following energy estimate

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\left(|w^+|^2+|w^-|^2\right)+\frac{\sqrt{G}}{2c_l^u c_l^\tau}|z_l^+|^2_{y_{\max}}+\frac{\sqrt{G}}{2c_r^u c_r^\tau}|z_r^-|^2_{y_{\max}}$$
$$+\xi\int_\Omega (w^+ + w^-)^2 = \int_\Omega f(w^+ - w^-)+\frac{\sqrt{G}}{2c_l^u c_l^\tau}g_l^2+\frac{\sqrt{G}}{2c_r^u c_r^\tau}g_r^2 \quad (20)$$

where $z_l^- = c_u u - c_\tau \tau$, $z_r^- = c_u u - c_\tau \tau$, on multiplying (13) by Gu and (14) by τ.

Consider now two solutions (u_1,τ_1) and (u_2,τ_2) for two parameter values $\xi_1 \geq \xi_2 \geq 0$, with same initial conditions and source term. On Ω, they satisfy

$$\partial_t(u_1-u_2)-\partial_y(\tau_1-\tau_2)=0\,, \quad (21)$$
$$\partial_t(\tau_1-\tau_2)-G\partial_y(u_1-u_2)=-\xi_1(\tau_1-\tau_2)-(\xi_1-\xi_2)\tau_2\,, \quad (22)$$

and homogeneous boundary conditions on $\partial\Omega$ i.e. (18–19) with $g_l = 0 = g_r$, hence

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\left(|w_1^+-w_2^+|^2+|w_1^- - w_2^-|^2\right)+\frac{\sqrt{G}}{2c_l^u c_l^\tau}|z_{1,l}^+-z_{2,l}^+|^2_{y_{\max}}+\frac{\sqrt{G}}{2c_r^u c_r^\tau}|z_{1,r}^+-z_{2,r}^+|^2_{y_{\max}}$$
$$+\xi_1\int_\Omega(\tau_1-\tau_2)^2 = -(\xi_1-\xi_2)\int_\Omega \tau_2(\tau_1-\tau_2) \quad (23)$$

with obvious notations, on multiplying (21) by $G(u_1-u_2)$ and (22) by $(\tau_1-\tau_2)$.

Using Cauchy-Schwarz and Young inequalities with (23), one finally obtains:

Proposition 1 *Given two parameter values $\xi_1 \geq \xi_2 \geq 0$, two solutions (u_1,τ_1) and (u_2,τ_2) of the damped wave system (15–16) with same conditions satisfy*

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\left(|w_1^+-w_2^+|^2+|w_1^- - w_2^-|^2\right)+\frac{\sqrt{G}}{2c_l^u c_l^\tau}|z_{1,l}^+-z_{2,l}^+|^2_{y_{\max}}+\frac{\sqrt{G}}{2c_r^u c_r^\tau}|z_{1,r}^+-z_{2,r}^+|^2_{y_{\max}}$$
$$\frac{\xi_1+\xi_2}{2}\int_\Omega(\tau_2-\tau_1)^2 \leq \frac{\xi_1-\xi_2}{2}\int_\Omega \tau_2^2 \quad (24)$$

i.e. continuous dependence on the relaxation parameter

$$(u_1,\tau_1)\xrightarrow{\xi_1\to\xi_2}(u_2,\tau_2)$$

in $L^2(\Omega)$ for all times $t \geq 0$, as well as in $L^2(0,T)$ on the boundary $\partial\Omega$.

When $\xi_2 = 0$ in particular, the latter structural stability result of Prop. 1 yields convergence of (u_1,τ_1) solution of the damped wave system (15–16) toward (u_2,τ_2) solution of a standard wave system that coincides with 1D elastodynamics.

So convergence toward elastodynamics is quite a simple structural-stability result for the 1D viscoelastic shear waves solutions to (linear) Maxwell equations – in comparison with a singular limit like convergence toward Newtonian fluids [10].

However, that non-singular limit is not easy anymore when considering non-linear equations for general multi-dimensional motions, defined e.g. through our reformulation in [2].

4 Structural stability of general Maxwell flows when $\frac{1}{\lambda} \equiv \xi \to 0$

To establish convergence toward elastodynamics of general multi-dimensional viscoelastic flows, we consider our reformulation [2] of Maxwell flows using a symmetric-hyperbolic system of conservation laws (10) with variable U, and we use the standard comparison tool introduced by C. M. Dafermos: the *relative entropy* [6, Chap. V]. Precisely, consider two classical solutions $\mathrm{U}_1, \mathrm{U}_2$ using same conditions but two relaxation parameters ξ_1, ξ_2. It holds

$$\begin{aligned}
&\partial_t \left(\eta(\mathrm{U}_1) - \eta(\mathrm{U}_2) - \mathrm{D_U}\eta(\mathrm{U}_2)\cdot(\mathrm{U}_1 - \mathrm{U}_2)\right)\\
&\quad + \partial_\alpha \left(\mathrm{Q}_\alpha(\mathrm{U}_1) - \mathrm{Q}_\alpha(\mathrm{U}_2) - \mathrm{D_U}\eta(\mathrm{U}_2)\cdot(\mathrm{G}_\alpha(\mathrm{U}_1) - \mathrm{G}_\alpha(\mathrm{U}_2))\right)\\
&\quad = \xi_1 \left(\mathrm{D_U}\eta(\mathrm{U}_1) - \mathrm{D_U}\eta(\mathrm{U}_2)\right)\cdot\Pi(\mathrm{U}_1)\\
&\quad - (\partial_t \mathrm{D_U}\eta(\mathrm{U}_2))\cdot(\mathrm{U}_1 - \mathrm{U}_2) - (\partial_\alpha \mathrm{D_U}\eta(\mathrm{U}_2))\cdot(\mathrm{G}_\alpha(\mathrm{U}_1) - \mathrm{G}_\alpha(\mathrm{U}_2))\\
&= \xi_1 \left(\mathrm{D_U}\eta(\mathrm{U}_1) - \mathrm{D_U}\eta(\mathrm{U}_2)\right)\cdot\Pi(\mathrm{U}_1) - \xi_2(\mathrm{U}_1 - \mathrm{U}_2)\cdot \mathrm{D}^2_{\mathrm{UU}}\eta(\mathrm{U}_2)\cdot\Pi(\mathrm{U}_2)\\
&\quad - \partial_\alpha \mathrm{U}_2 \cdot \mathrm{D}^2_{\mathrm{UU}}\eta(\mathrm{U}_2)\cdot\left(\mathrm{G}_\alpha(\mathrm{U}_1) - \mathrm{G}_\alpha(\mathrm{U}_2) - \mathrm{D_U G}_\alpha(\mathrm{U}_2)\cdot(\mathrm{U}_1 - \mathrm{U}_2)\right)
\end{aligned} \tag{25}$$

which can be compared to [6, (5.2.10)]: our relative-entropy equality holds for two *classical* solutions $\mathrm{U}_1, \mathrm{U}_2$, with an additional source term (first line of RHS in (25)).

Then, we suggest to compare two well-defined classical solutions $\mathrm{U}_1, \mathrm{U}_2$ that use the same conditions in the hyperbolicity domain $\mathcal{O}$ on complementing [6, (5.2.14)] as follows to take into account the additional source term. In (25) we use:

$$\begin{aligned}
&\xi_1 \left(\mathrm{D_U}\eta(\mathrm{U}_1) - \mathrm{D_U}\eta(\mathrm{U}_2)\right)\cdot\Pi(\mathrm{U}_1) - \xi_2(\mathrm{U}_1 - \mathrm{U}_2)\cdot \mathrm{D}^2_{\mathrm{UU}}\eta(\mathrm{U}_2)\cdot\Pi(\mathrm{U}_2)\\
&\quad = \left(\mathrm{D_U}\eta(\mathrm{U}_1) - \mathrm{D_U}\eta(\mathrm{U}_2)\right)\cdot\left(\xi_1\Pi(\mathrm{U}_1) - \xi_2\Pi(\mathrm{U}_2)\right) + \xi_2 \mathrm{Z}(\mathrm{U}_1, \mathrm{U}_2)\cdot\Pi(\mathrm{U}_2)\\
&\quad = \xi_1 \left(\mathrm{D_U}\eta(\mathrm{U}_1) - \mathrm{D_U}\eta(\mathrm{U}_2)\right)\cdot\left(\Pi(\mathrm{U}_1) - \Pi(\mathrm{U}_2)\right)\\
&\quad + (\xi_2 - \xi_1)\left(\mathrm{D_U}\eta(\mathrm{U}_1) - \mathrm{D_U}\eta(\mathrm{U}_2)\right)\cdot\Pi(\mathrm{U}_2) + \xi_2 \mathrm{Z}(\mathrm{U}_1, \mathrm{U}_2)\cdot\Pi(\mathrm{U}_2)
\end{aligned} \tag{26}$$

where $Z(U_1, U_2)$ is quadratic in $U_1 - U_2$.

We conclude using (12) with Cauchy-Schwarz and Young inequalities for any $t \in (0, T), r > 0$, to complement [6, (5.2.16)] in our case with source terms. Note that by contrast with the linear case of Sec. 3, coercivity does not hold i.e.

$$\mathrm{c}_0 \left|\mathrm{D_U}\eta(\mathrm{U}_1) - \mathrm{D_U}\eta(\mathrm{U}_2)\right|^2 + \left(\mathrm{D_U}\eta(\mathrm{U}_1) - \mathrm{D_U}\eta(\mathrm{U}_2)\right)\cdot\left(\Pi(\mathrm{U}_1) - \Pi(\mathrm{U}_2)\right)$$

cannot be guaranteed non-positive for all $\mathrm{U}_1, \mathrm{U}_2$ whatever $c_0 > 0$! Still, $\forall \sigma \in (0, t)$

$$\begin{aligned}
\int_{|\boldsymbol{x}|\le r + c(t-\sigma)} |U_1(t) - U_2(t)|^2 \le \int_0^\sigma ds \int_{|\boldsymbol{x}|\le r + c(t-s)} d\boldsymbol{x} \Bigg(& C(\xi_1 - \xi_2)^2 |U_2(s)|^2 \\
& + C'\left(1 + \xi_1^2\right)|U_1(s) - U_2(s)|^2 \Bigg)
\end{aligned} \tag{27}$$

holds given positive constants c, C, C' depending solely on the initial conditions and U_2, ξ_2 but not ξ_1. So Gromwall's inequality allows one to conclude about the structural stability of general multidimensional Maxwell flows:

Proposition 2 *Given two parameter values $\xi_1 \geq \xi_2 \geq 0$ bounded above, consider two smooth solutions U_1, U_2 to (10) on $[0,T) \times \mathbb{R}^3$ with same initial condition of bounded support. There exists $C_T(U_2)$ such that*

$$\|U_1(t) - U_2(t)\|_{L^2(\mathbb{R}^3)} \leq C_T|\xi_1 - \xi_2| \quad \forall t \in (0,T) .$$

When $\xi_2 = 0$ in particular, the structural stability result of Prop. 2 yields convergence of U_1 solution to our formulation in [2] of viscoelastic Maxwell flows toward U_2 solution of an elastodynamics system for compressible hyperelastic materials of Neo-Hooken type.

5 Conclusion and Perpsectives

In this short note, we have completed the structural stability results of [10] for global-in-time *linear* Maxwell flows (Proposition 1 for 1D shear waves) in the "easy" elastodynamics limit case $\lambda \sim \dot{\mu} \to \infty$, as opposed to the singular Newtonian fluid limit $\lambda \to 0$ in [10]. Note that in a particular case with smoother solutions, our elastodynamics solid limit case $\lambda \sim \dot{\mu} \to \infty$ was already covered by a structural stability result from [7].

Moreover, using our recent reformulation of Maxwell's models [2] so as to unequivocally define viscoelastic flows as solutions to symmetric-hyperbolic PDEs, we could extend in Prop. 2 the structural stability result to generic multi-dimensional viscoelastic flows: as widely believed, multi-dimensional viscoelastic flows of Maxwell type can be *rigorously* linked with solid elastodynamics (of compressible hyperelastic Neo-Hooken materials) when the relaxation time grows unboundedly.

The result should still hold in the case of *non-isothermal* Maxwell flows defined similarly in [3] using a symmetric-hyperbolic reformulation which extends our former work [2] (to non-isothermal Maxwell flows, as well as to viscoelastic flows with finite-extensibility effects and various objective time-rates). But a second step in structural stability should then furthermo cover more physical liquid-solid transitions, driven by temperature changes. And it remains a challenge to establish structural stability results for multidimensional solutions of physically-relevant Maxwell models in the *singular* Newtonian-limit case $\lambda \to 0$.

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References

1. P. C. Bollada and T. N. Phillips. On the mathematical modelling of a compressible viscoelastic fluid. *Arch. Ration. Mech. Anal.*, 205(1):1–26, 2012.
2. Sébastien Boyaval. Viscoelastic flows of Maxwell fluids with conservation laws. *ESAIM Math. Model. Numer. Anal.*, 55(3):807–831, 2021.
3. Sébastien Boyaval and Mark Dostalík. Non-isothermal viscoelastic flows with conservation laws and relaxation. *J. Hyperbolic Differ. Equ.*, 19(2):337–364, 2022.
4. Sébastien Boyaval A viscoelastic flow model of Maxwell-type with a symmetric-hyperbolic formulation. Accepted for publication in Comptes Rendus – Mécanique. Preprint hal-03880001
5. Christoforou, Cleopatra and Tzavaras, Athanasios E. Relative Entropy for Hyperbolic–Parabolic Systems and Application to the Constitutive Theory of Thermoviscoelasticity *Arch. Ration. Mech. Anal.*, 229(1):1–52, 2018.
6. Constantine M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, fourth edition, 2016.
7. Gür, Şevket and Güleç, İpek, Structural stability analysis of solutions to the initial boundary value problem for a nonlinear strongly damped wave equation *Turkish J. Math.*, 40(6):1231–1236, 2016.
8. Jordan, P. M. and Puri, Ashok and Boros, G. *On a new exact solution to Stokes' first problem for Maxwell fluids*, Internat. J. Non-Linear Mech., 39(8):1371–1377, 2004.
9. James Clerk Maxwell. IV. On double refraction in a viscous fluid in motion. *Proceedings of the Royal Society of London*, 22(148-155):46–47, 1874.
10. L. E. Payne and B. Straughan. Convergence of the equations for a Maxwell fluid. *Stud. Appl. Math.*, 103(3):267–278, 1999.
11. Luigi Preziosi and Daniel D. Joseph. Stokes' first problem for viscoelastic fluids. *Journal of Non-Newtonian Fluid Mechanics*, 25(3):239 – 259, 1987.
12. Saramito, P. *Complex fluids*, volume 79 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer, Cham, 2016.