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# Crack front instability in mixed-mode I+III: the influence of non-singular stresses

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## Abstract

In a previous paper (Leblond et al., 2011), a theoretical instability threshold was derived for the currently observed phenomenon of crack front fragmentation under mixed-mode I+III loading conditions. Instability modes were shown to emerge when the mode mixity ratio  $K_{\text{III}}^0/K_{\text{I}}^0$  exceeds some critical value  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}$  that only depends on Poisson's ratio. Unfortunately, the predicted threshold was found to be much larger than that observed in general. Numerical simulations of crack front fragmentation (Chen et al., 2015), based on a phase-field model, subsequently evidenced an important role of the specimen size on the non-coplanar instability. Here, we explore theoretically the influence of existence of some finite characteristic length(s), arising from the loading, by accounting for the presence of non-singular stresses  $T_{ij}^0$  in the unperturbed configuration of the crack. By re-examining the linear stability analysis of Leblond et al. (2011) with these more general assumptions, we show that a negative non-singular stress  $T_{xx}^0$  in the direction of propagation does not affect crack front fragmentation, while a negative non-singular  $T_{zz}^0$  in the direction of the crack front strongly hampers it. On the contrary, positive non-singular stresses  $T_{xx}^0$  and  $T_{zz}^0$ , or non-zero non-singular (antiplane shear) stresses  $T_{xz}^0$ , promote the fragmentation process, sometimes through the formation of facets that drift along the front as propagation proceeds. Large values of all three of these non-singular stresses may result in a significant lowering of the threshold value  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}$  of the mode mixity ratio for instability, even possibly down to *zero*; which stresses out the potential existence of the fragmentation instability even in pure mode I. Yet, our results cannot explain the often observed formation of non-coplanar facets at extremely low mode mixity ratios; indeed the wavelength of the instability modes predicted are comparable to the finite characteristic length(s) introduced, of at least centimetric order of magnitude, while the formation of facets has been observed experimentally at scales as low as some tens of microns.

**Keywords:** Brittle failure, three-dimensional fracture, configurational stability, mode I+III, non-singular stresses

## 1. Introduction

Crack front fragmentation is an ubiquitous phenomenon in tensile fracture of solids. It is generally observed when, in addition to the tensile mode I loading, some mode III antiplane shear is applied. Fragmented cracks display facets that grow parallel to each other, and leave behind fracture surfaces with factory-roof-like morphologies. Since the seminal work of Sommer (1969) on glass, fragmentation patterns have been observed in nearly all types of materials, including rocks (Pollard et al., 1982), metallic alloys (Hourlier and Pineau, 1979; Eberlein et al., 2017), single crystals (Kermode et al., 2008), polymers (Knauss, 1970; Lazarus et al., 2008) and gels (Baumberger et al., 2008). In recent years, crack front fragmentation has been observed in some special materials under conditions of almost pure mode I,  $K_{\text{I}}^0 > 0$  and  $K_{\text{III}}^0 \approx 0$  where  $K_{\text{I}}^0$  and  $K_{\text{III}}^0$  denote the stress intensity factors in the initial planar configuration (Lin et al., 2010; Ronsin et al., 2014; Vasudevan, 2018; Kolvin et al., 2018). This suggests that mode I crack configurations in these materials may be bi-stable, either simple or faceted.

Despite the ubiquity of crack fragmentation during failure of

materials, a comprehensive understanding of, and a fully predictive criterion for, the transition of a propagating crack from a smooth surface to a fragmented one is still missing. LEFM (linear elastic fracture mechanics), applied to slightly perturbed cracks, has proved to be the most promising theoretical framework to describe and predict the phenomenon. In their pioneering work, Leblond et al. (2011), inspired by the results of Pons and Karma (2010)'s numerical simulations based on a phase-field-model, showed that helical perturbations to an initially straight crack front may grow without bound, if the mode mixity ratio  $K_{\text{III}}^0/K_{\text{I}}^0$  exceeds some critical value  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}^{\text{ref}}$ , that only depends on Poisson's ratio. It was then shown by Chen et al. (2015) that such a bifurcation is subcritical, so that fragmented crack fronts may exist even at *low* mixity ratios  $K_{\text{III}}^0/K_{\text{I}}^0 \ll 1$ , and so may co-exist with planar crack surfaces under the same loading conditions. However, such a scenario does not account for the emergence of facets in propagating cracks at *extremely low* levels of anti-plane shear,  $K_{\text{III}}^0/K_{\text{I}}^0 \ll 1$  (very much lower than the theoretical value  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}^{\text{ref}}$ ), which have been reported in brittle polymers and gels (Lin et al., 2010; Ronsin et al., 2014; Vasudevan, 2018; Kolvin et al., 2018).

To reconcile the onset of instability predicted by LEFM and the (material dependent) threshold observed experimentally, Leblond et al. (2019) and Vasudevan et al. (2020) investigated

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46 the effect of a mode-dependent fracture energy, and showed<sup>101</sup>  
 47 that such a dependence may dramatically lower the fragmen-  
 48 tation threshold, provided that the shear-mode-dependence of<sup>102</sup>  
 49  $G_c$  is strong enough. Last but not least, they also showed that<sup>103</sup>  
 50 in the presence of some additional global mode II component,  
 51  $K_{II}^0 \neq 0$ , facets can gradually drift along the crack front as it<sup>104</sup>  
 52 propagates, a behavior which is indeed observed experimen-<sup>105</sup>  
 53 tally (Lin et al., 2010; Ronsin et al., 2014; Vasudevan, 2018;  
 54 Kolvin et al., 2018).

55 The role of the presence of some finite characteristic lengths,<sup>106</sup>  
 56 arising from the finite dimensions of the structure and/or the  
 57 spatial variation of the loading, upon the fragmentation insta-<sup>108</sup>  
 58 bility has been largely neglected so far. However numerical  
 59 simulations of crack front fragmentation, based on a phase-field<sup>109</sup>  
 60 model, have shown that reducing the specimen size results in<sup>110</sup>  
 61 a significant decrease of the instability threshold (Chen et al.,<sup>111</sup>  
 62 2015). In addition, theoretical investigations of the effect of<sup>112</sup>  
 63 non-singular stresses  $T_{xx}^0$  (parallel to the direction of propaga-<sup>113</sup>  
 64 tion),  $T_{zz}^0$  (parallel to the crack front) and  $T_{xz}^0$  (simple shear in<sup>114</sup>  
 65 the crack plane), due to Cotterell and Rice (1980), Gao (1992)<sup>115</sup>  
 66 and Xu et al. (1994), have shown that large values of these quan-<sup>116</sup>  
 67 tities may favor out-of-plane crack excursions, even under pure<sup>117</sup>  
 68 more I conditions. The presence of such non-singular stresses<sup>118</sup>  
 69 in the unperturbed configuration of the crack is intimately tied<sup>119</sup>  
 70 to the presence of one or several finite characteristic lengths.

71 In this work, we explore theoretically the role of the presence  
 72 of some finite characteristic lengths upon the fragmentation insta-  
 73 bility. These lengths, naturally defined as  $\mathcal{L}_{ij} = (K_I^0/T_{ij}^0)^2$   
 74 where  $T_{ij}^0$  denotes any of the three non-singular stresses  $T_{xx}^0$ ,  
 75  $T_{zz}^0$ ,  $T_{xz}^0$ , will arise only from the spatial variation of the load-  
 76 ing, in the absence of any characteristic length in the infinite  
 77 model-geometry considered. The structural length scales  $\mathcal{L}_{ij}$   
 78 may interact with the length scale characterizing the unstable  
 79 mode of crack propagation, as shown in a broader context by  
 80 Favier et al. (2006); Vasoya et al. (2016). We shall thus seek  
 81 to determine the impact of the presence of some non-singular  
 82 stresses, or equivalently some length scales  $\mathcal{L}_{ij}$ , upon the transi-  
 83 tion from a planar to a fragmented crack; and whether or not this  
 84 presence may explain the low fragmentation threshold observed  
 85 in some fracture experiments. We shall also explore the effect  
 86 of non-singular stresses upon the morphology of the facets, and  
 87 in particular (i) their possible drift along the crack front, and (ii)  
 88 the possible selection of the characteristic size of the facets at  
 89 the fragmentation threshold.

90 The paper is organized as follows:

- 91 • In Section 2, we first provide the expressions of the local  
 92 variations  $\delta K_I$ ,  $\delta K_{II}$ ,  $\delta K_{III}$  of the stress intensity factor  
 93 along a perturbed crack front, under mixed mode I+III  
 94 loading conditions with additional non-singular stresses  
 95  $T_{xx}^0$ ,  $T_{zz}^0$  and  $T_{xz}^0$ .
- 96 • These expressions are then applied in Section 3 to predict  
 97 the fragmentation threshold and the properties of insta-  
 98 bility modes. The derivation is based on the assumption  
 99 that both Goldstein and Salganik (1974)'s principle of local  
 100 symmetry and Griffith (1921)'s energetic condition are

satisfied at all instants and all positions along the front, .

- The impact of non-singular stresses upon the fragmenta-  
 tion instability is analyzed in depth in Section 4.
- Finally Section 5 presents a synthesis of our main findings  
 and their application to experimental situations.

## 2. First-order order perturbation of a semi-infinite crack in an infinite body

### 2.1. Unperturbed geometry and loading - Generalities

We consider a semi-infinite crack embedded in an infinite body made of some isotropic linearly elastic material. In the initial reference configuration  $\Gamma$ , the crack is planar and its front is straight (see Fig. 1a). We adopt the usual convention of LEFM, and thus use a Cartesian frame  $Oxyz$  with origin  $O$  chosen arbitrarily within the crack plane, axis  $(Ox)$  parallel to the direction of crack propagation,  $(Oy)$  parallel to the direction orthogonal to the crack plane, and  $(Oz)$  parallel to the crack front (see Fig. 1a). The associated unit vectors are denoted  $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ . In the following, the notations  $f_{,x}$ ,  $f_{,y}$ ,  $f_{,z}$  denote the partial derivatives of the function  $f$  with respect to the coordinates  $x, y, z$ .

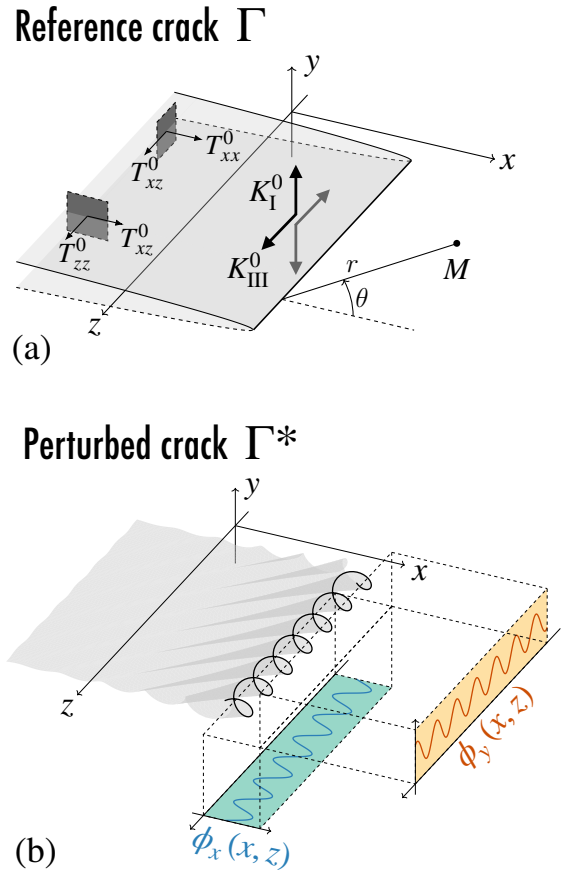


Figure 1: (a) The reference semi-infinite crack  $\Gamma$  with a straight front is loaded in mixed-mode I+III with non-zero non-singular stresses. (b) The crack front and its surface are perturbed both within and out of the original plane, by small amounts noted  $\phi_x$  and  $\phi_y$  respectively.

The unperturbed crack is loaded under mixed-mode I+III conditions through some system of forces, such that the unperturbed SIFs  $K_I^0$  and  $K_{III}^0$  be independent of the position  $x$  of the crack front and the position  $z$  of their point of observation within this front. The SIF  $K_{III}^0$  may harmlessly be assumed to be non-negative. In addition the unperturbed stress field is characterized by non-singular stresses ( $T_{xx}^0, T_{xz}^0, T_{zz}^0$ ) also independent of  $x$  and  $z$ . These non-singular stresses correspond to the constant terms in the asymptotic expansion of the stress field in the vicinity of the crack front, that reads (Williams, 1952):

$$\begin{cases} \sigma_{xx}^0(r, \theta) = \frac{K_I^0}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right)\right] + T_{xx}^0 + O(r^{1/2}) \\ \sigma_{yy}^0(r, \theta) = \frac{K_I^0}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \left[1 + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right)\right] + O(r^{1/2}) \\ \sigma_{xy}^0(r, \theta) = \frac{K_I^0}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{3\theta}{2}\right) + O(r^{1/2}) \\ \sigma_{xz}^0(r, \theta) = -\frac{K_{III}^0}{\sqrt{2\pi r}} \sin\left(\frac{\theta}{2}\right) + T_{xz}^0 + O(r^{1/2}) \\ \sigma_{yz}^0(r, \theta) = \frac{K_{III}^0}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) + O(r^{1/2}) \\ \sigma_{zz}^0(r, \theta) = 2\nu \frac{K_I^0}{\sqrt{2\pi r}} \cos\left(\frac{\theta}{2}\right) + T_{zz}^0 + O(r^{1/2}) \end{cases} \quad (1)$$

where  $(r, \theta)$  denote the polar coordinates of the point of observation  $M$  in the local plane orthogonal to the crack front (see Fig. 1a), and  $\nu$  Poisson's ratio. The non-singular stresses  $T_{ij}^0$  typically emerge from the finite dimensions defined by the loading, in the absence of any characteristic length defined by the geometry itself. Their magnitudes  $|T_{ij}^0|$  define some characteristic lengths

$$\mathcal{L}_{ij} = \left( \frac{K_I^0}{T_{ij}^0} \right)^2 \quad (2)$$

that describe the spatial extent of a  $K_I$ -dominated zone where the terms in  $K_I^0$  of Eq. (1) prevail over those in  $T_{ij}^0$ .

## 2.2. First-order expressions of the stress intensity factors along the crack front in a slightly perturbed configuration

In a perturbed configuration  $\Gamma^*$ , the crack front is displaced within the original crack plane by a small amount  $\phi_x(x, z)$  in the direction ( $Ox$ ), and the crack surface is displaced out of the original crack plane by a small amount  $\phi_y(x, z)$  in the direction ( $Oy$ ) (see Fig. 1.b); the argument  $x$  here represents the average position of the perturbed crack front in the direction of propagation, in the first function, and the first coordinate of the point of observation of the out-of-plane perturbation, in the second.

The  $p$ -th SIF  $K_p$  can be expressed as the sum of the SIF  $K_p^0$  associated to the reference configuration  $\Gamma$ , and the perturbation  $\delta K_p$  arising from the displacements of the crack front and surface:

$$K_p(x, z) = K_p^0 + \delta K_p(x, z). \quad (3)$$

Furthermore because of linearity, each SIF perturbation  $\delta K_p$  can be decomposed as the sum of a term  $\delta_K K_p$  resulting from the

macroscopic mixed-mode I+III loading, and a term  $\delta_T K_p$  resulting from the presence of non-zero non-singular stresses:

$$\delta K_p(x, z) = \delta_K K_p(x, z) + \delta_T K_p(x, z) \quad (4)$$

The detailed expressions of the perturbations  $\delta_K K_p$  and  $\delta_T K_p$  are provided in the works of Gao and Rice (1986) for the in-plane perturbation  $\phi_x$ , and Movchan et al. (1998) for the out-of-plane perturbation  $\phi_y$ ; they are given in Eqs (A.1) and (A.9) of Appendix A respectively. Some simplifications have been made here according to the following hypotheses:

1. The characteristic length defined by the spatial variations of the loading is supposed to be much larger than the typical distances of variation of the perturbations  $\phi_x(x, z)$  and  $\phi_y(x, z)$  of the crack front and surface. (This is the assumption that permits to consider the  $K_p^0$ 's and  $T_{ij}^0$ 's as independent of  $x$ ).
2. The characteristic length associated to the coefficients  $A_q$  appearing in the terms proportional to  $r^{1/2}$  in Eq. (1) – of the order of  $K_p/A_q$  – is much larger than the lengths  $\mathcal{L}_{ij}$  associated to the non-singular stresses.

It is worth noting here that:

1. The expressions of the variations  $\delta_K K_p$  involve three types of terms: *local* terms that relate to the local crack front orientation defined by the derivatives  $\phi_{i,j}$ , *semi-local* terms that involve integrals of the displacements  $\phi_i$  along the whole crack front, and fully *non-local* terms that contain integrals of the displacements  $\phi_i$  over the entire crack surface.
2. The variations SIFs  $\delta_T K_p$  were derived by Movchan et al. (1998) only for a sinusoidal perturbation  $\phi_y(z) = A \cos(kz)$  independent of  $x$ . Their expressions for an arbitrary perturbation  $\phi_y(x, z)$  are derived in Appendix A.
3. The variations  $\delta_T K_p$  involve only non-local terms related to the ‘‘morphology’’ of the crack surface, in Movchan et al. (1998)'s terminology.

## 3. Linear stability analysis of coplanar propagation

### 3.1. Generalities

Like in our previous works (Leblond et al., 2011, 2019; Vasudevan et al., 2020), we are looking for configurations of the crack and its front  $(\phi_x, \phi_y)$  other than the trivial one (planar crack with straight front), satisfying a ‘‘double’’ propagation criterion enforced at all points of the crack front and all instants of propagation. This criterion consists of:

- Goldstein and Salganik (1974)'s principle of local symmetry, stipulating that the local SIF  $K_{II}(x, z)$  of mode II must be zero;
- Griffith (1921)'s criterion  $G(x, z) = G_c$ , which states that the local energy release rate  $G(x, z)$  must be equal to the local fracture energy  $G_c$ , assumed here to be uniform.<sup>1</sup>

<sup>1</sup>It is interesting to note here that the validity of Griffith (1921)'s criterion at small scales, that is in its *local* form  $G(x, z) = G_c$ , was recently tested experimentally by Wang et al. (2022) by investigating the balance of energy during the tensile failure of a brittle gel.

The analysis will be based on the consideration of instabil-198  
ity modes associated with perturbations of the crack front and199  
surface of the following form (Vasudevan et al., 2020): 200

$$\begin{cases} \phi_x(x, z) = \text{Re} \left[ e^{\lambda x} \psi_x(z) \right] \\ \phi_y(x, z) = \text{Re} \left[ e^{\lambda x} \psi_y(z) \right] \end{cases} \quad (5) \quad \begin{matrix} 201 \\ 202 \\ 203 \end{matrix}$$

177 where  $\lambda$  is the *complex growth rate* of the mode, and  $\psi_x(z)$  and 204  
178  $\psi_y(z)$  two unknown complex functions. Instability of coplanar 205  
179 propagation is characterized by the condition  $\text{Re}(\lambda) > 0$ . Also, 206  
180 as noted by (Vasudevan et al., 2020) and briefly re-explained be- 207  
181 low, a non-zero imaginary part of  $\lambda$  indicates a *drifting motion*  
182 *of the instability mode along the crack front as it propagates*.

Use will be made of Fourier transforms in the direction  $z$   
of the crack front. The definition adopted here for the Fourier  
transform  $\hat{\chi}(k)$  of an arbitrary function  $\chi(z)$  is:

$$\hat{\chi}(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(z) e^{-ikz} dz \Leftrightarrow \chi(z) = \int_{-\infty}^{+\infty} \hat{\chi}(k) e^{ikz} dk. \quad (6)$$

183 One can see here that the imaginary part of  $\lambda$  governs the drift-  
184 ing motion of instability modes: indeed writing  $\lambda$  as  $\lambda_1 + i\lambda_2$   
185 with real  $\lambda_1$  and  $\lambda_2$ , one sees that the expressions of the per-  
186 turbations  $\phi_x$  and  $\phi_y$  involve, for each Fourier component, the  
187 factor  $e^{\lambda x} e^{ikz} = e^{\lambda_1 x} e^{i(\lambda_2 x + kz)}$  depicting an exponential growth  
188 combined with a “drift velocity”  $dz/dx = -\lambda_2/k$  - see (Vasude-  
189 van et al., 2020) for details.

For each Fourier component, we define a *normalized com-  
plex growth rate*  $\xi$  by the formula

$$\xi = \lambda/|k|. \quad (7) \quad 208$$

This dimensionless ratio compares the complex growth rate  $\lambda$ 209  
of the perturbations in the direction  $x$  of propagation to their210  
wavenumber  $|k|$  in the direction  $z$  of the crack front. The value  
of  $\xi$  is restricted to the half-plane  $\text{Re}(\xi) > -1$  for convergence211  
of the integrals in Eqs. (A.1) and (A.9), which provide the vari-  
ations of the SIFs due to the perturbations of the crack front and  
surface. Note also that the imaginary part of  $\xi$  is directly con-  
nected to the angle  $\alpha$  characterizing the drifting motion of the  
mode (see the inset of Fig.3.b) through the relation

$$\alpha = \arctan[-\text{Im}(\xi)]. \quad (8)$$

190 The next two sections aim at deriving the condition on  $\xi$  that 212  
191 results from the double propagation criterion, when the crack  
192 is loaded in mixed-mode I+III with additional non-zero non-  
193 singular stresses. This will permit, upon examination of the  
194 real and imaginary parts of  $\xi$ , to discuss both the conditions  
195 governing instability of coplanar propagation, and the possible  
196 drifting motion of instability modes.

### 197 3.2. Application of the principle of local symmetry

We first make use of Goldstein and Salganik (1974)’s princi-  
ple of local symmetry, stipulating that the stress intensity factor  
of mode II must be uniformly zero at all points of the crack  
front and all instants. In the absence of macroscopic mode II  
( $K_{II}^0 = 0$ ), this condition reduces to:

$$\delta K_{II}(x, z) = 0 \quad (9)$$

The expression of  $\delta K_{II}$  displayed in Eqs. (A.1) and (A.9) in-  
volves complex integrals of the perturbations ( $\phi_x, \phi_y$ ) over the  
entire crack front and the entire crack surface. Fortunately,  
these integrals take a rather “simple” form when expressed  
in Fourier’s space, for instabilities of the type considered in  
Eq. (5). The expressions of the variations  $\delta \widehat{K}_p$  related to the  
unperturbed SIFs were derived by Leblond et al. (2019), and  
are recalled in Eq. (B.1) of Appendix B.a. The expressions of  
the variations  $\delta_T \widehat{K}_p$  associated to the unperturbed non-singular  
stresses are derived in Appendix B.b, and given in Eq. (B.4).

Inserting Eqs. (B.1) and (B.4) in (9) yields, for every  
wavenumber  $k$ :

$$4i\rho_0 s \hat{\psi}_x(k) = \hat{\psi}_y(k) \left\{ 2 - 3\nu + (2 - \nu)\xi \right. \\ \left. - \frac{2\sqrt{2}}{(\xi + 1)^{3/2}} \left[ ((2 - \nu)\xi + 2)\xi q_{xx}^0 + \nu q_{zz}^0 + 2is((1 - \nu)\xi + 1)q_{xz}^0 \right] \right\}. \quad (10)$$

In this equation the quantity

$$\rho_0 = \frac{K_{III}^0}{K_I^0} \quad (\geq 0) \quad (11)$$

denotes the unperturbed mixity ratio,  $s$  the sign of the  
wavenumber  $k$ , and the quantities

$$q_{ij}^0 = \frac{T_{ij}^0}{\sqrt{|k|}K_I^0} = \frac{\epsilon_{ij}}{\sqrt{|k|}\mathcal{L}_{ij}} \quad , \quad \epsilon_{ij} = \text{sgn}(T_{ij}^0) \quad (12)$$

are related to the ratios of the wavelength  $\ell_{\text{pert}} = 2\pi/|k|$  of the  
perturbations to the lengthscales  $\mathcal{L}_{ij}$  set by the non-singular  
stresses  $T_{ij}$ , and defined by Eq. (2).

### 3.3. Application of Griffith’s criterion

We now use Griffith (1921)’s criterion, which states that the  
crack propagates when the local energy release rate  $G(x, z)$  is  
equal to the local fracture energy  $G_c$ . We consider here the  
case where the field of fracture energy  $G_c$  is uniform. Then  
necessarily

$$\delta G(x, z) = 0 \quad (13)$$

at all points of the crack front and all instants.

The energy-release-rate may be calculated from the SIFs us-  
ing Irwin (1958)’s well-known formula:

$$G(x, z) = \frac{1 - \nu^2}{E} \left[ K_I^2(x, z) + K_{II}^2(x, z) \right] + \frac{1 + \nu}{E} K_{III}^2(x, z) \quad (14)$$

where  $E$  denotes Young’s modulus. The first-order contribution  
of the perturbations  $\phi_x, \phi_y$  of the crack front and surface to the  
energy-release-rate reads:

$$\delta G(x, z) = 2 \frac{1 - \nu^2}{E} K_I^0 \left[ \frac{\delta K_I(x, z)}{K_I^0} + \frac{\rho_0}{1 - \nu} \frac{\delta K_{III}(x, z)}{K_I^0} \right]. \quad (15)$$

Griffith (1921)’s criterion (Eq. (13)) then yields:

$$\delta K_I(x, z) + \frac{\rho_0}{1 - \nu} \delta K_{III}(x, z) = 0. \quad (16)$$

Inserting Eqs. (B.1) and (B.4) into (16) yields for every wavenumber  $k$ :

$$\begin{aligned} & \left[ (1-\nu)(2-\nu) + (2+\nu)\rho_0^2 \right] \hat{\psi}_x(k) \\ &= 2i\rho_0 s \hat{\psi}_y(k) \left\{ -2(1-\nu) + \frac{(1-2\nu)(2-\nu)}{\sqrt{2(\xi+1)}} \right. \\ & \quad + \frac{\sqrt{2}}{\sqrt{\xi+1}} \left[ \nu \frac{\xi}{\xi+1} q_{xx}^0 - \left( 2-\nu - \frac{\nu}{\xi+1} \right) q_{zz}^0 \right. \\ & \quad \left. \left. + is \left( (2-\nu)\xi - \nu \frac{\xi-1}{\xi+1} \right) q_{xz}^0 \right] \right\}. \end{aligned} \quad (17)$$

### 3.4. Combination of the two parts of the criterion

Combination of Eqs.(10) and (17) yields:

$$F(\rho_0, \xi) + f_{xx}(\rho_0, \xi)q_{xx}^0 + f_{zz}(\rho_0, \xi)q_{zz}^0 + is f_{xz}(\rho_0, \xi)q_{xz}^0 = 0 \quad (18)$$

where

$$\begin{cases} F(\rho_0, \xi) &= (1-\nu)(2-3\nu) - 3(2-\nu)\rho_0^2 \\ & \quad + \left[ (1-\nu)(2-\nu) + (2+\nu)\rho_0^2 \right] \xi \\ & \quad + 4 \frac{\sqrt{2}}{\sqrt{\xi+1}} (1-2\nu)\rho_0^2 \\ f_{xx}(\rho_0, \xi) &= -\frac{2\sqrt{2}\xi}{(\xi+1)^{3/2}} \left[ (1-\nu)((2-\nu)\xi+2) \right. \\ & \quad \left. + ((2+\nu)\xi+2)\rho_0^2 \right] \\ f_{zz}(\rho_0, \xi) &= -\frac{2\sqrt{2}}{(\xi+1)^{3/2}} \left[ \nu(1-\nu) + (4\xi+4-\nu)\rho_0^2 \right] \\ f_{xz}(\rho_0, \xi) &= -\frac{4\sqrt{2}}{(\xi+1)^{3/2}} \left[ (1-\nu)((1-\nu)\xi+1) \right. \\ & \quad \left. - (2\xi^2 + (1-\nu)\xi - 1)\rho_0^2 \right] \end{cases} \quad (19)$$

Equation (18) implicitly defines the value of the normalized growth rate  $\xi$  for a given set of loading parameters  $(\rho_0, q_{xx}^0, q_{xz}^0, q_{zz}^0)$ ; our interest is in values of  $\xi$  having a positive real part, corresponding to exponentially growing perturbations. The values of  $\xi$  satisfying Eq. (18) can be computed numerically relatively easily, this equation being equivalent to a polynomial equation of the fifth degree in the variable,  $\zeta = 1/\sqrt{\xi+1}$ . The numerical procedure is described in Appendix C.

It is worth noting that in the absence of non-singular stresses,  $T_{ij}^0$ , Eq. (18) reduces to condition (20) of Leblond et al. (2011), assessing configurational stability in mixed-mode I+III.

## 4. Influence of non-singular stresses on the configurational stability of a plane crack

### 4.1. The stability analysis of Leblond et al. (2011) without non-singular stresses

We first consider the *reference case* of mixed mode I+III in the absence of non-singular stresses, which corresponds to the situation examined by Leblond et al. (2011). They showed that – for values of Poisson’s ratio greater than  $\nu_c \approx 0.03$  – admissible solutions are found along the real line  $\text{Im}(\lambda) = 0$ , so

that no drift of instability modes is predicted. Moreover, the growth rate  $\text{Re}(\lambda) = \lambda$  is negative (decaying perturbations) for low values of the mode mixity ratio  $\rho^0 = K_{\text{I}}^0/K_{\text{III}}^0$ , but becomes positive (increasing perturbations) for ratios larger than some critical value  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}^{\text{ref}}$ . Both this critical value, and the evolution of  $\text{Re}(\lambda) = \lambda$  with  $K_{\text{I}}^0/K_{\text{III}}^0$ , depend on Poisson’s ratio  $\nu$ . In particular, the value of  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}^{\text{ref}}$  is given by

$$[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}^{\text{ref}} = \sqrt{\frac{(1-\nu)(2-3\nu)}{3(2-\nu) - 4\sqrt{2}(1-2\nu)}} \quad (20)$$

The results of the stability analysis are given in Fig. 2 for  $\nu = 0.3$ , in the form of a plot of the “reduced” growth rate  $\text{Re}(\lambda)/|k| = \lambda/|k|$  versus the mixity ratio  $\rho^0 = K_{\text{I}}^0/K_{\text{III}}^0$ . For the value of  $\nu$  considered,  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}^{\text{ref}} \approx 0.52$ . (As noted by Leblond et al. (2011), another branch of solutions exists near  $\text{Re}(\lambda) \approx -1$ , but it corresponds to exponentially decaying perturbations of little interest).

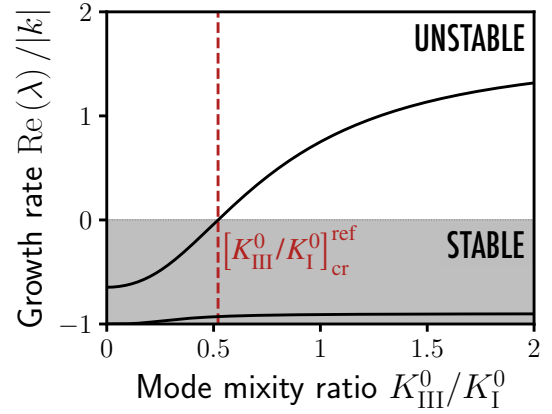


Figure 2: Leblond et al. (2011)’s configurational stability analysis of a planar crack loaded on mode I+III, for a Poisson ratio  $\nu = 0.3$ : when the mode mixity ratio  $K_{\text{III}}^0/K_{\text{I}}^0$  is larger than some critical value  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}^{\text{ref}}$  (vertical red line), the perturbations grow exponentially ( $\text{Re}(\lambda) > 0$ ), whatever their wavelength  $\ell_{\text{pert}}$ .

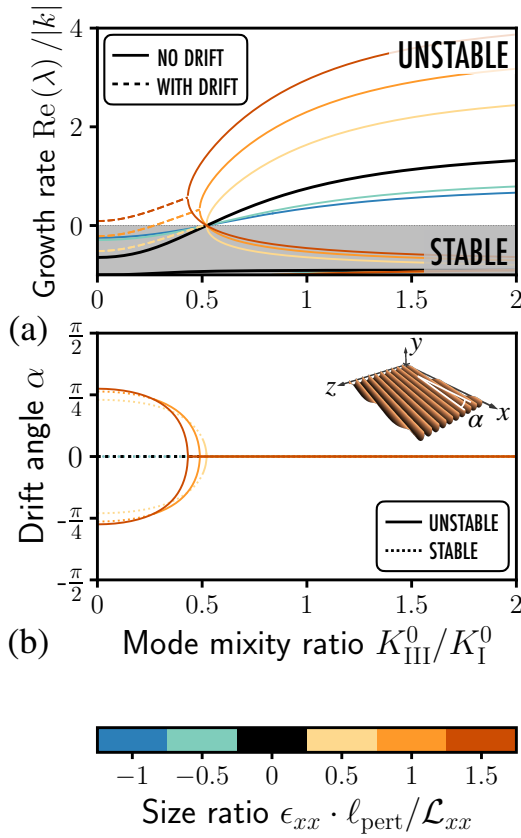
Yet, in phase-field simulations of brittle fracture (Pons and Karma, 2010; Chen et al., 2015) or experiments of crack initiation in mixed mode I+III (Lin et al., 2010; Pham and Ravi-Chandar, 2014), non-zero non-singular stresses are bound to emerge from the finiteness of the crack dimensions with respect to the size of the fracture specimen, and from the boundary conditions. We shall now extend the study of Leblond et al. (2011) by building on equation (18) to explore the influence of such non-singular stresses upon the stability of the planar configuration of the crack.

### 4.2. Influence of $T_{xx}^0$

We first explore the influence of  $T_{xx}^0$ . The sign  $\epsilon_{xx}$  of this non-singular stress has been shown to control the stability of the crack path in both 2D (Cotterell and Rice, 1980) and 3D situations (Gao, 1992; Xu et al., 1994), positive values of  $T_{xx}^0$  being associated with instability. However to the best of the authors’ knowledge, the influence of  $T_{xx}^0$  on the configurational instability in mode I+III has not been studied in the literature.

255 We shall show here that a positive  $T_{xx}^0$  favors both pure mode 279  
 256 I and mixed-mode I+III instabilities through formation of drift-280  
 257 ing perturbations of large wavelength; while a negative  $T_{xx}^0$  pre-281  
 258 vents instabilities in pure mode I, and does not have any impact 282  
 259 on the onset of the faceting instability in mode I+III.

260 In contrast to the work of Leblond et al. (2011) where no 283  
 261 length scale was involved, a non-zero non-singular stress  $T_{xx}^0$  284  
 262 yields a characteristic length  $\mathcal{L}_{xx} = (K_I^0/T_{xx}^0)^2$  which can be 285  
 263 compared to the wavelength  $\ell_{\text{pert}} = 2\pi/|k|$  of the perturbation in 286  
 264 the direction of the crack front. Combined with the sign  $\epsilon_{xx}$  of 287  
 265  $T_{xx}^0$ , the ratio of these two lengths sets the values of the growth 288  
 266 rate  $\text{Re}(\lambda)$  and the drift angle  $\alpha$ , for any mode mixity ratio  $\rho^0 =$  289  
 267  $K_{\text{III}}^0/K_I^0$ . We show in Fig. 3 the values of  $\text{Re}(\lambda)/|k|$  and  $\alpha$  versus 290  
 268 the mode mixity ratio, for several values of the non-singular 291  
 269 stress  $T_{xx}^0$  and a Poisson ratio  $\nu = 0.3$ .



270 Figure 3: Configurational stability of a planar crack front for a Poisson ratio 321  
 271  $\nu = 0.3$ : influence of the non-singular stress  $T_{xx}^0$  on (a) the growth rate  $\text{Re}(\lambda)$  322  
 272 of the perturbation and (b) its drift angle  $\alpha$ . The black curves correspond to the 323  
 273 case  $T_{xx}^0 = 0$  considered by Leblond et al. (2011). 324

270 Prior to commenting on these results, a definition and some 325  
 271 remarks are in order. We define  $[K_{\text{III}}^0/K_I^0]_{\text{cr}}$  as the *smallest value* 326  
 272 *of the (non-negative) mixity ratio*  $\rho^0 = K_{\text{III}}^0/K_I^0$  *for which the* 327  
 273 *real part of the reduced growth rate*  $\xi = \lambda/|k|$  *becomes non-* 328  
 274 *negative*. It is important to note here that at the “threshold” 329  
 275  $[K_{\text{III}}^0/K_I^0]_{\text{cr}}$ : 330

- 276 •  $\text{Re}(\xi)$  may be zero, if  $[K_{\text{III}}^0/K_I^0]_{\text{cr}}$  is positive;  $\text{Re}(\xi)$  332  
 277 then crosses the value 0 when  $K_{\text{III}}^0/K_I^0$  crosses the value 333  
 278  $[K_{\text{III}}^0/K_I^0]_{\text{cr}}$ ; 334

- but  $\text{Re}(\xi)$  may also be positive, if  $[K_{\text{III}}^0/K_I^0]_{\text{cr}}$  is zero;  $\text{Re}(\xi)$  335  
 is then positive for all values of  $K_{\text{III}}^0/K_I^0$ ;
- $\text{Im}(\xi)$  need not necessarily be zero, and may take arbitrary 336  
 values.

292 We observe in Fig. 3.a that negative non-singular stresses 323  
 293 do not *qualitatively* change the stability behavior of the crack 324  
 294 with respect to the reference case considered by Leblond 325  
 295 et al. (2011): stability prevails for mode mixity ratios below 326  
 296  $[K_{\text{III}}^0/K_I^0]_{\text{cr}}$  and instability above it, and  $[K_{\text{III}}^0/K_I^0]_{\text{cr}}$  corresponds 327  
 297 to the value  $\xi = 0$  and is independent of  $T_{xx}^0$  - and thus remains 328  
 298 equal to  $[K_{\text{III}}^0/K_I^0]_{\text{cr}}^{\text{ref}}$ . This means that a negative non-singular 329  
 299 stress does not have any stabilizing influence on the onset of the 330  
 300 mode I+III instability.

301 The case of a positive  $T_{xx}$  is also, and probably more, inter- 323  
 302 esting: for low values of the mode mixity ratio, typically 324  
 303  $K_{\text{III}}^0/K_I^0 \lesssim 0.5$ , drifting perturbations ( $\alpha \neq 0$ , dotted lines in 325  
 304 Fig. 3.a) may develop along the crack surface, a feature that is 326  
 305 absent for  $T_{xx}^0 = 0$ . Note that both perturbations with a positive 327  
 306 and a negative drift angle can form, owing to the invariance of 328  
 307 the problem in a rotation of the geometry and loading of  $180^\circ$  329  
 308 around the  $x$ -axis (see Fig. 3.b). The critical value  $[K_{\text{III}}^0/K_I^0]_{\text{cr}}$  330  
 309 may also now depend on the non-singular stress  $T_{xx}^0$ . For low 331  
 310 length scale ratios  $\ell_{\text{pert}}/\mathcal{L}_{xx} \lesssim 0.5$ , the drifting perturbations 332  
 311 decay in time ( $\text{Re}(\xi) < 0$ ), and exponentially growing, non- 333  
 312 drifting perturbations ( $\text{Re}(\xi) > 0$ ,  $\alpha = 0$ ) emerge when  $K_{\text{III}}^0/K_I^0$  334  
 313 becomes larger than the value  $[K_{\text{III}}^0/K_I^0]_{\text{cr}} = [K_{\text{III}}^0/K_I^0]_{\text{cr}}^{\text{ref}}$ , which 335  
 314 still corresponds to  $\xi = 0$  and remains independent of  $T_{xx}^0$  or 336  
 315  $\ell_{\text{pert}}/\mathcal{L}_{xx}$ . For large length scale ratios  $\ell_{\text{pert}}/\mathcal{L}_{xx} \gtrsim 0.5$  (i.e. large 337  
 316 values of  $T_{xx}^0$ ), drifting perturbations emerge below the refer- 338  
 317 ence critical value  $[K_{\text{III}}^0/K_I^0]_{\text{cr}}^{\text{ref}} \approx 0.5$  obtained for  $T_{xx}^0 = 0$  (see 339  
 318 Section 4.1). In other words, a large non-singular stress  $T_{xx}^0 > 0$  340  
 319 results in a *decrease* of the critical mode mixity  $[K_{\text{III}}^0/K_I^0]_{\text{cr}}$  for 341  
 320 fragmentation. For even larger values of  $\ell_{\text{pert}}/\mathcal{L}_{xx}$ ,  $[K_{\text{III}}^0/K_I^0]_{\text{cr}}$  342  
 321 becomes zero so that perturbations grow even in the absence of 343  
 322 mode III. These behaviors are qualitatively similar to those depic- 344  
 323 ted by Cotterell and Rice (1980), Gao (1992) and Xu et al. 345  
 324 (1994), who predicted that a positive  $T_{xx}^0$  favors out-of-plane 346  
 325 crack excursions. Note that the behavior observed in Fig. 3 for 347  
 326 large mode mixity ratios,  $K_{\text{III}}^0/K_I^0 \gtrsim 0.5$ , is rather similar to 348  
 327 that in the reference case  $T_{xx}^0 = 0$ : out-of-plane perturbations 349  
 328 grow in time, and propagate parallel to the mean direction of 350  
 329 crack propagation ( $\alpha = 0$ ), irrespective of the length scale ratio 351  
 330  $\ell_{\text{pert}}/\mathcal{L}_{xx}$ .

331 The evolution of the critical mode mixity ratio  $[K_{\text{III}}^0/K_I^0]_{\text{cr}}$  352  
 332 - as defined above - with the length scale ratio  $\epsilon_{xx} \cdot \ell_{\text{pert}}/\mathcal{L}_{xx}$  353  
 333 depends on Poisson’s ratio, and is displayed in Fig. 4.a for 354  
 334  $\nu \in [0.1, 0.5]$ . Interestingly, as noted above, for all values of  $\nu$ , 355  
 335 above some “critical” length scale ratio  $[\ell_{\text{pert}}/\mathcal{L}_{xx}]_{\text{cr}}$ ,  $[K_{\text{III}}^0/K_I^0]_{\text{cr}}$  356  
 336 becomes zero so that the perturbations grow exponentially no 357  
 337 matter the mode mixity ratio. This critical length scale ratio 358  
 338 is shown in Fig. 4.b and decreases almost linearly with 359  
 339 Poisson’s ratio, from about 1.87 to 0.77 for  $\nu \in [0.1, 0.5]$ . 360  
 340 Unfortunately, no simple analytical expression of  $[\ell_{\text{pert}}/\mathcal{L}_{xx}]_{\text{cr}}$  361  
 341 could be derived from Eq.(18). (The difficulty being that when 362  
 342  $\epsilon_{xx} \cdot \ell_{\text{pert}}/\mathcal{L}_{xx} = [\ell_{\text{pert}}/\mathcal{L}_{xx}]_{\text{cr}}$ ,  $\xi$  is still not completely known 363  
 343 since only its real part is zero). 364

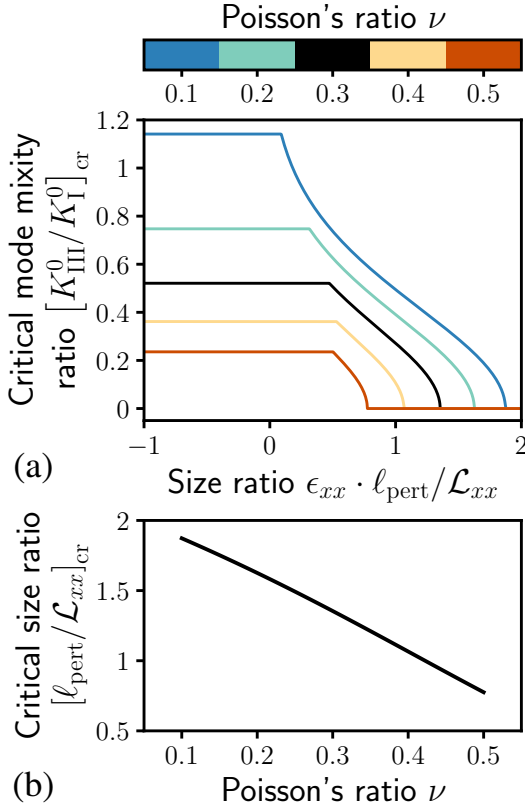


Figure 4: (a) Critical mode mixity ratio  $[K_{III}^0/K_I^0]_{cr}$  versus the perturbation size ratio  $\epsilon_{xx} \cdot \ell_{pert}/\mathcal{L}_{xx}$ , for various values of Poisson's ratio  $\nu$ . (b) Evolution of the critical size ratio  $[\ell_{pert}/\mathcal{L}_{xx}]_{cr}$  with Poisson's ratio  $\nu$ . This is the perturbation size ratio for which the planar crack becomes unstable whatever the value of the mode mixity ratio  $K_{III}^0/K_I^0$ .

It should be noted that even though the existence of such a critical length scale ratio, above which instability prevails whatever the mixity ratio, is interesting, it cannot account for the small values of the mixity ratio at which facets form experimentally in some cases ( $[K_{III}^0/K_I^0]_{cr} \approx 0.001 - 01$  for Homalite and PMMA, see Pham and Ravi-Chandar (2014) and Vasudevan (2018)). Indeed in these experiments, the non-singular stress  $T_{xx}^0$  was rather small, the length  $\mathcal{L}_{xx}$  consequently rather large, so that the length scale ratio  $\ell_{pert}/\mathcal{L}_{xx}$  was smaller than the critical value  $[\ell_{pert}/\mathcal{L}_{xx}]_{cr}$  - especially at the scale  $\ell_{pert} \approx 10 - 100 \mu\text{m}$  at which the facets were observed.

Let us finally investigate the limiting 2D case, corresponding to perturbations independent of  $z$ , having thus a zero wavenumber  $k$  in the direction of the crack front. The reduced growth rate  $\xi = \lambda/|k|$  then becomes ill-defined; but to solve this problem, it suffices to multiply Eq. (18) by  $|k|$  and then let  $|k|$  go to zero. One thus gets upon division by the common factor  $[(1-\nu)(2-\nu) + (2+\nu)\rho_0^2]$ :

$$\lambda - 2\sqrt{2}\lambda \frac{T_{xx}^0}{K_I^0} = 0.$$

For a negative  $T_{xx}^0$ , this equation on  $\lambda$  does not have any solution, that is, there are no increasing perturbations (configura-

tional stability). However, for  $T_{xx}^0 > 0$ , it does admits a solution that reads:

$$\sqrt{\lambda} = 2\sqrt{2} \frac{T_{xx}^0}{K_I^0} \Rightarrow \lambda = 8 \left( \frac{T_{xx}^0}{K_I^0} \right)^2 = 8 \mathcal{L}_{xx}. \quad (21)$$

We thus recover the results of Cotterell and Rice (1980), who showed that in 2D, the decreasing or increasing character of an arbitrary out-of-plane perturbation  $\phi_y$  of the crack surface is controlled by the sole sign of  $T_{xx}^0$ . We also recover, for  $T_{xx}^0 > 0$ , the value of the parameter  $\lambda$  governing the exponential growth of perturbations.

#### 4.3. Influence of $T_{zz}^0$

We now investigate the influence of  $T_{zz}^0$  on the configurational stability of the planar crack. Like in the case where  $T_{xx}^0 \neq 0$ , a non-zero non-singular stress  $T_{zz}^0$  yields a characteristic length  $\mathcal{L}_{zz} = (K_I^0/T_{zz}^0)^2$  which may be compared to the perturbation wavelength  $\ell_{pert} = 2\pi/|k|$ . Gao (1992) and Xu et al. (1994) showed that for pure mode I loading, the influence of  $T_{zz}^0$  on crack path stability is similar to that of  $T_{xx}^0$ : negative values of  $T_{zz}^0$  are associated to stable crack paths, while positive values favor instabilities. Moreover, Gao (1992) suggested that negative non-singular stresses  $T_{zz}^0$  may prevent the formation of *en echelon* cracking patterns associated to mixed mode I+III loadings.

We shall show here that negative non-singular stresses  $T_{zz}^0$  indeed stabilize out-of-plane perturbations in pure mode I, and significantly increase the mode mixity ratio at which the facing (out-of-plane) instability may occur in mode I+III. In contrast, positive values of  $T_{zz}^0$  somehow favor both instabilities in pure mode I and mode I+III for perturbations of large wavelength, albeit in a less marked way than  $T_{xx}^0$ .

We show in Fig. 5 the normalized growth rate  $\text{Re}(\lambda)/|k|$  and the drift angle  $\alpha$  as functions of the mode mixity ratio  $\rho^0 = K_{III}^0/K_I^0$ , for several values of the non-singular stress  $T_{zz}^0$  and a Poisson ratio  $\nu = 0.3$ . A general remark is that drifting perturbations ( $\alpha \neq 0$ ) exist, but are of little interest since they always decrease in time. Negative non-singular stresses  $T_{zz}^0$  strongly stabilize the system, since even mode mixity ratios significantly larger than the reference threshold value  $[K_{III}^0/K_I^0]_{cr}$ , corresponding to  $T_{zz}^0 = 0$ , do not suffice to fragment the crack front. This is confirmed in Fig. 6.a, which shows  $[K_{III}^0/K_I^0]_{cr}$  - as defined in Subsection 4.2 above - as a function of the length scale ratio  $\epsilon_{zz} \cdot \ell_{pert}/\mathcal{L}_{zz}$ . Note that here  $\text{Im}(\xi)$  is always zero at the threshold ( $K_{III}^0/K_I^0 = [K_{III}^0/K_I^0]_{cr}$ ). We observe that even a moderately negative non-singular stress  $T_{zz}^0$  triggers a significant increase of the critical mode mixity ratio, especially for low values of Poisson's ratio. Also, the positiveness of  $[K_{III}^0/K_I^0]_{cr}$  for all negative values of  $T_{zz}^0$  implies that such non-singular stresses preclude any out-of-plane instability in pure mode I (since  $K_{III}^0/K_I^0 = 0$  is then necessarily smaller than  $[K_{III}^0/K_I^0]_{cr} > 0$ ).

Positive non-singular stresses  $T_{zz}^0$  have an opposite impact upon stability of coplanar propagation, as they drive the system towards the mode I+III instability for lower critical mode mixity ratios  $[K_{III}^0/K_I^0]_{cr}$ , see Fig. 5.a and Fig. 6.a. Like for the



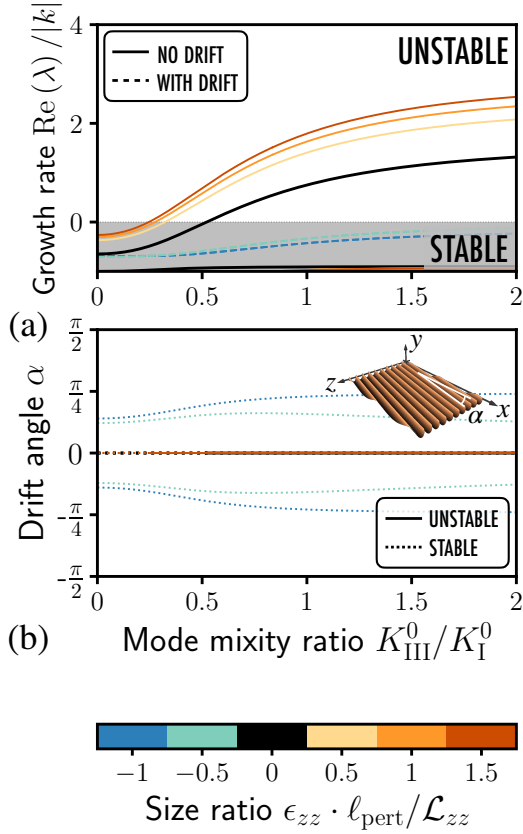


Figure 5: Configurational stability of a planar crack front for a Poisson ratio  $\nu = 0.3$ : influence of the non-singular stress  $T_{zz}^0$  on (a) the growth rate  $\text{Re}(\lambda)$  of the perturbation and (b) its drift angle  $\alpha$ . The black curves correspond to the case  $T_{zz}^0 = 0$  considered by Leblond et al. (2011).

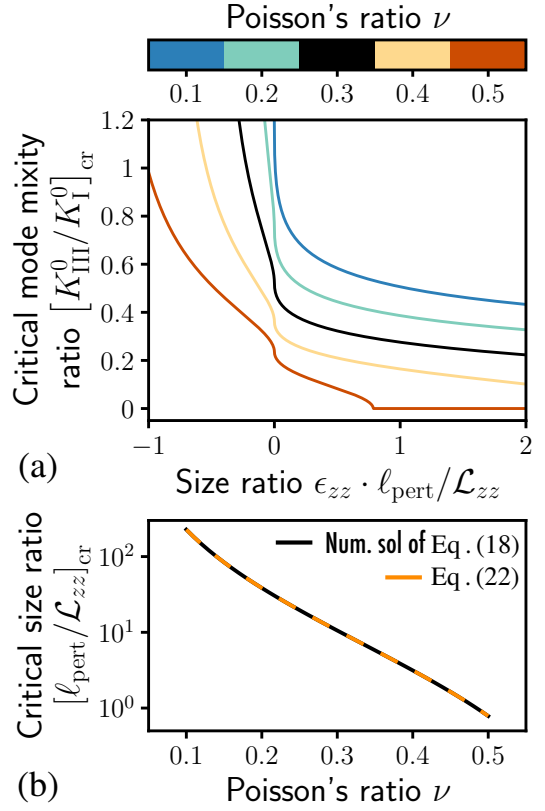


Figure 6: (a) Critical mode mixity ratio  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}$  versus the perturbation size ratio  $\epsilon_{zz} \cdot \ell_{\text{pert}}/\mathcal{L}_{zz}$ , for various values of Poisson's ratio  $\nu$ . (b) Evolution of the critical size ratio  $[\ell_{\text{pert}}/\mathcal{L}_{zz}]_{\text{cr}}$  with Poisson's ratio  $\nu$ . This is the perturbation size ratio for which the planar crack becomes unstable whatever the value of the mode mixity ratio  $K_{\text{III}}^0/K_{\text{I}}^0$ .

non-singular stress  $T_{xx}^0$ , we observe that there exists a critical length scale ratio  $[\ell_{\text{pert}}/\mathcal{L}_{zz}]_{\text{cr}}$  above which  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}$  becomes zero, so that perturbations grow unstably no matter the value of the mode mixity ratio. This means that for a positive non-singular stress  $T_{zz}^0$ , large-wavelength perturbations increase in time even in pure mode I. (In contrast, small-wavelength perturbations still decrease in time).

The critical length scale ratio  $[\ell_{\text{pert}}/\mathcal{L}_{zz}]_{\text{cr}}$ , above which coplanar propagation thus becomes unstable even in pure mode I, is determined by inserting the values  $\rho_0 = K_{\text{III}}^0/K_{\text{I}}^0 = 0$  and  $\xi = 0$  (see Fig. 5.a) in Eq.(18); the result,

$$[\ell_{\text{pert}}/\mathcal{L}_{zz}]_{\text{cr}} = \frac{\pi}{4} \left( \frac{2-3\nu}{\nu} \right)^2 \quad (22)$$

agrees with the values computed numerically (see Fig. 6.b). Equation (22) means that perturbations with a wavelength  $\ell_{\text{pert}} > \frac{\pi}{4} \left( \frac{2-3\nu}{\nu} \right)^2 \mathcal{L}_{zz}$  will grow in time. The  $1/\nu^2$  dependence here accounts for the rapid decrease of  $[\ell_{\text{pert}}/\mathcal{L}_{zz}]_{\text{cr}}$  with increasing values of  $\nu$ : large values of Poisson's ratio favor crack front fragmentation, a behavior that is already clear in the absence of non-singular stresses, as the reference critical threshold ratio  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}^{\text{ref}}$  is also a decreasing function of  $\nu$  (see Eq. 20). Note that the critical length scale ratio  $[\ell_{\text{pert}}/\mathcal{L}_{zz}]_{\text{cr}}$

happens to be much larger than that,  $[\ell_{\text{pert}}/\mathcal{L}_{xx}]_{\text{cr}}$ , associated to the non-singular stress  $T_{xx}^0$ . This implies that the destabilizing effect of  $T_{xx}^0 > 0$  is much stronger than that of  $T_{zz}^0 > 0$ .

Equation (22) is reminiscent of Eq. (21) that provides the growth rate of out-of-plane perturbations of a mode I 2D crack (invariant along the direction of the front), in the presence of some non-singular stress  $T_{xx} > 0$ . The formal similarity between both equations emphasizes the similarity between the phenomena they depict: positive values of both non-singular stresses  $T_{xx}^0$  and  $T_{zz}^0$  imply, in pure mode I, formation of perturbations involving some characteristic length proportional to the square of  $K_{\text{I}}^0/T_{xx}^0$  or  $K_{\text{I}}^0/T_{zz}^0$ . In both cases this reflects the natural tendency of the crack to grow perpendicularly to the direction of far tensile principal stresses.

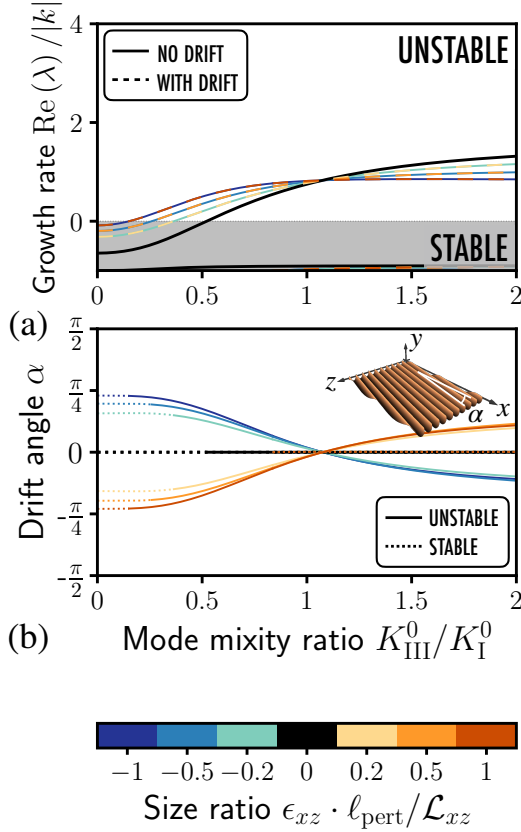
#### 4.4. Influence of $T_{xz}$

Finally, we investigate the influence of the non-singular stress  $T_{xz}^0$  that introduces an additional length scale  $\mathcal{L}_{xz} = (K_{\text{I}}^0/T_{xz}^0)^2$  in the problem. Gao (1992) and Xu et al. (1994) found that it had no influence on crack path stability in pure mode I. Gao (1992) nonetheless suggested that even for such a loading, it may promote the formation of non-coplanar facets, due to the generation of a local mode III component if the crack

434 is slightly perturbed out of its original plane (see the third of 459  
435 Eqs. (A.9)). 460

436 We shall show here that a non-zero  $T_{xz}^0$  actually influences 461  
437 crack path stability in pure mode I, through formation of 462  
438 large-wavelength perturbations that drift along the front as 463  
439 the crack propagates. For mixed mode I+III loadings, the 464  
440 same mechanism also promotes the emergence of the faceting 465  
441 instability toward lower mode mixity ratios. 466

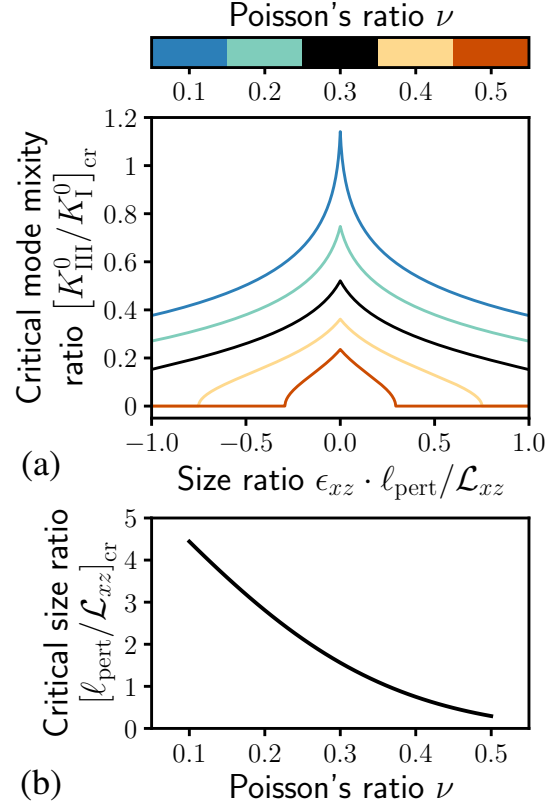
443 In Fig. 7, we display the normalized growth rate  $\text{Re}(\lambda)/|k|$   
444 and the drift angle  $\alpha$  as functions of the mode mixity ratio  
445  $\rho^0 = K_{\text{III}}^0/K_{\text{I}}^0$ , for several values of the non-singular stress  $T_{xz}^0$   
446 and a Poisson ratio  $\nu = 0.3$ . We observe that a non-zero  $T_{xz}^0$   
447 may promote the formation of facets, as it lowers the value of  
448 the critical mode mixity ratio  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}$  above which perturbations  
449 increase in time (see Fig. 7.a). This instability is character-  
450 ized by a drifting motion of the crack front with a positive  
451 drift angle  $\alpha$  when  $T_{xz}^0 < 0$ , and a negative one when  $T_{xz}^0 > 0$   
452 (see Fig. 7.b). Such an ‘‘oriented’’ drift of facets could some-  
453 how be anticipated, as the introduction of a non-zero  $T_{xz}^0$  breaks  
454 the invariance of the problem in a rotation of the geometry and  
455 loading of  $180^\circ$  about the  $x$ -axis.



467 Figure 7: Configurational stability of a planar crack front for a Poisson ratio  
468  $\nu = 0.3$ : influence of the non-singular stress  $T_{xz}^0$  on (a) the growth rate  $\text{Re}(\lambda)$   
469 of the perturbation and (b) its drift angle  $\alpha$ . The black curves correspond to the  
470 case  $T_{xz}^0 = 0$  considered by Leblond et al. (2011). 475

456 Figure 8 shows the critical mode mixity ratio  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}$  as 477  
457 a function of the length scale ratio  $\ell_{\text{pert}}/\mathcal{L}_{xz}$ . We observe a sig- 478  
458 nificant decrease of  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}$  with increasing values of  $|T_{xz}^0|$ , 479

especially for large Poisson ratios. It turns out that for small  
non-singular stresses  $T_{xz}$ , the critical ratio  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}$  may be  
developed at first order in  $q_{xz}^0 \propto \sqrt{\ell_{\text{pert}}/\mathcal{L}_{xz}}$ , so that the thresh-  
old decreases linearly with  $\sqrt{\ell_{\text{pert}}/\mathcal{L}_{xz}}$ ; this explains the cusps  
on the vertical axis observed in Fig. 8. We again note, for  
each value of  $\nu$ , the existence of a critical length scale ratio  
 $[\ell_{\text{pert}}/\mathcal{L}_{xz}]_{\text{cr}}$  above which perturbations grow in time, even in  
pure mode I, no matter the sign of  $T_{xz}^0$ .



467 Figure 8: (a) Critical mode mixity ratio  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}$  versus the perturbation size  
468 ratio  $\epsilon_{xz} \cdot \ell_{\text{pert}}/\mathcal{L}_{xz}$ , for various values of Poisson’s ratio  $\nu$ . (b) Evolution of the  
469 critical size ratio  $[\ell_{\text{pert}}/\mathcal{L}_{xz}]_{\text{cr}}$  with Poisson’s ratio  $\nu$ . This is the perturbation  
470 size ratio for which the planar crack becomes unstable whatever the value of  
471 the mode mixity ratio  $K_{\text{III}}^0/K_{\text{I}}^0$ .

## 467 5. Conclusion

468 The configurational stability of a crack propagating under  
469 mixed-mode I+III loading conditions, with additional non-  
470 singular stresses  $T_{xx}^0, T_{zz}^0, T_{xz}^0$ , was investigated within the per-  
471 turbative framework of linear elastic fracture mechanics, on the  
472 basis of a linear stability analysis. The work stood as a natural  
473 extension of that of Leblond et al. (2011), that did not account  
474 for the influence of non-singular stresses upon the onset of the  
475 faceting instability. We showed that the critical mode mixity  
476 ratio  $[K_{\text{III}}^0/K_{\text{I}}^0]_{\text{cr}}$ , at which the instability emerges, is strongly  
influenced by the presence of non-singular stresses, due to the  
emergence of perturbations of large wavelength which may or  
may not drift along the front as the crack propagates. The

possibility evidenced here of a drifting motion of instability modes has been observed experimentally, see for example Lin et al. (2010) and Vasudevan (2018), and is similar to that predicted for a crack loaded under completely general mixed mode I+II+III conditions (Vasudevan et al., 2020).

More specifically, the main findings of this study are as follows:

- A negative non-singular stress  $T_{xx}^0$  does not promote formation of facets, and leaves unchanged the instability threshold  $[K_{III}^0/K_I^0]_{cr}$  which thus retains the value calculated by Leblond et al. (2011) in the absence of non-singular stresses. On the contrary, a positive non-singular stress  $T_{xx}^0$  lowers the critical mode mixity ratio  $[K_{III}^0/K_I^0]_{cr}$  through formation of drifting facets. The impact of a positive  $T_{xx}^0$  gets stronger as the ratio between the wavelength  $\ell_{pert}$  of the perturbation approaches the structural length scale  $\mathcal{L}_{xx} = (K_I^0/T_{xx}^0)^2$  associated to  $T_{xx}^0$ . For perturbations of large wavelength, or large values of  $T_{xx}^0$ , facets growing in time may even form no matter the value of the mixity ratio ( $[K_{III}^0/K_I^0]_{cr} = 0$ ).
- A negative non-singular stress  $T_{zz}^0$  strongly stabilizes straight crack fronts, as growing facets can only form for mode mixity ratios  $K_{III}^0/K_I^0$  considerably larger than the instability threshold calculated by Leblond et al. (2011) in the absence of non-singular stresses. In contrast, a positive  $T_{zz}^0$  promotes the faceting instability, as the critical mode mixity ratio  $[K_{III}^0/K_I^0]_{cr}$  is a decreasing function of the length scale ratio  $\ell_{pert}/\mathcal{L}_{zz}$ . Like for the non-singular stress  $T_{xx}^0$ , growing facets may even form no matter the mode mixity ratio, if  $\ell_{pert}/\mathcal{L}_{zz}$  is large enough. However the critical length scale ratio  $[\ell_{pert}/\mathcal{L}_{zz}]_{cr}$ , above which this phenomenon occurs, happens to be much larger than that  $[\ell_{pert}/\mathcal{L}_{xx}]_{cr}$ , associated to the non-singular stress  $T_{xx}^0$ .
- Non-zero non-singular stresses  $T_{xz}^0$  promote the mode I+III instability, through formation of “oriented” facets that drift along the front in a specific direction determined by the sign of  $T_{xz}^0$ . Again, we also predict that facets may even form no matter the mode mixity ratio, if the length scale ratio  $\ell_{pert}/\mathcal{L}_{xz}$  is large enough.

The existence, for the ratios of the wavelength  $\ell_{pert}$  of the perturbation and the structural length scales  $\mathcal{L}_{xx}$ ,  $\mathcal{L}_{zz}$ ,  $\mathcal{L}_{xz}$ , of critical values  $[\ell_{pert}/\mathcal{L}_{xx}]_{cr}$ ,  $[\ell_{pert}/\mathcal{L}_{zz}]_{cr}$ ,  $[\ell_{pert}/\mathcal{L}_{xz}]_{cr}$  above which perturbations grow in time whatever the mode mixity ratio  $K_{III}^0/K_I^0$ , has important implications for the stability of crack path under *pure mode I loading* conditions. In two dimensions, Cotterell and Rice (1980) showed that crack path stability is controlled by the sole sign of  $T_{xx}^0$ . In three dimensions, the picture is more complex; the conclusions drawn from the present work can be summarized as follows:

- For negative non-singular stresses  $T_{xx}^0$  or  $T_{zz}^0$ , the system is stable versus perturbations of arbitrary wavelength in the direction of the crack front.
- For positive non-singular stresses  $T_{xx}^0$  or  $T_{zz}^0$ , and non-zero non-singular stresses  $T_{xz}^0$ , perturbations with wavelength

smaller than some critical value set by the structural length scale  $\mathcal{L}_{ij} = (K_I^0/T_{ij}^0)^2$  decrease in time, while perturbations with larger wavelength grow unstably.

The study presented in this paper also provides interesting clues on the influence of non-singular stresses on the onset of the mode I+III out-of-plane instability. The contributions of non-singular stresses cannot be neglected in numerical simulations or experiments, as they typically emerge from the finiteness of the crack dimensions with respect to the size of the embedding structure, and from boundary conditions. Yet, our results cannot claim to provide an explanation to the formation of non-coplanar facets at the very low mode mixity ratios often observed in experiments (Lin et al., 2010; Pham and Ravichandar, 2014; Vasudevan, 2018; Kolvin et al., 2018); indeed, the wavelength of the instability modes evidenced in this work is comparable to the structural length scales  $\mathcal{L}_{ij}$  – of the order of some centimeters –, while the formation of facets has been observed experimentally at scales as low as some tens of microns.

#### Online access to data

The Python script used to generate the figures and the associated data are available at <https://doi.org/10.5281/zenodo.5994126>.

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#### CRedit authorship contribution statement

**Mathias Lebihain:** Conceptualization, Formal analysis, Visualization, Software, Writing - Original Draft; **Jean-Baptiste Leblond:** Conceptualization, Methodology, Formal analysis, Writing - Review & Editing **Laurent Ponson:** Conceptualization, Writing - Review & Editing;

573 **Appendix A. Expressions of the perturbed SIF  $\delta K_p$  in presence of mixed-mode I+III and non-singular stresses**

574 In this Appendix, we provide the expressions of the perturbations of the SIFs arising from those parts of the initial (unperturbed)  
 575 stress field tied to (i) the unperturbed SIFs  $K_I^0, K_{III}^0$ ; (ii) the unperturbed non-singular stresses  $T_{xx}^0, T_{zz}^0, T_{xz}^0$ . These expressions are  
 576 given at first order in the pair of perturbations  $(\phi_x, \phi_y)$  of the crack front and surface. The expressions of the terms pertaining to the  
 577 SIFs  $K_I^0, K_{III}^0$  were given for arbitrary perturbations  $\phi_x, \phi_y$  by Leblond et al. (2011), building on the work of Gao and Rice (1986)  
 578 and Movchan et al. (1998); and we derive here the expressions of the terms pertaining to the non-singular stresses  $T_{xx}^0, T_{zz}^0, T_{xz}^0$ ,  
 579 with a comparison of the partial results of Movchan et al. (1998).

580 *Appendix A.1. First-order perturbations of the stress intensity factors due to the initial stress intensity factors*

At first order in the pair  $(\phi_x, \phi_y)$ , the perturbations  $\delta_K K_p$  of the SIFs tied to the unperturbed SIFs  $K_I^0, K_{III}^0$ , consist of one contribution arising from the in-plane perturbation  $\phi_x$  of the crack front, plus another one arising from the out-of-plane perturbation  $\phi_y$  of the crack surface. According to the work of Gao and Rice (1986) for the in-plane perturbation, and those of Movchan et al. (1998) and Leblond et al. (2011) for the out-of-plane perturbation, the expressions of these perturbations read:

$$\begin{cases} \delta_K K_I(x, z) = \frac{K_I^0}{2\pi} \text{PV} \int_{-\infty}^{+\infty} \frac{\phi_x(x, z') - \phi_x(x, z)}{(z - z')^2} dz' - 2K_{III}^0 \phi_{y,z}(x, z) + \delta K_I^{\text{skew}}(x, z) \\ \delta_K K_{II}(x, z) = -\frac{2}{2 - \nu} K_{III}^0 \phi_{x,z}(x, z) + \frac{K_I^0}{2} \phi_{y,x}(x, z) - \frac{2 - 3\nu}{2 - \nu} \frac{K_I^0}{2\pi} \text{PV} \int_{-\infty}^{+\infty} \frac{\phi_y(x, z') - \phi_y(x, z)}{(z - z')^2} dz' \\ \delta_K K_{III}(x, z) = \frac{2 + \nu}{2 - \nu} \frac{K_{III}^0}{2\pi} \text{PV} \int_{-\infty}^{+\infty} \frac{\phi_x(x, z') - \phi_x(x, z)}{(z - z')^2} dz' + \frac{2(1 - \nu)^2}{2 - \nu} K_I^0 \phi_{y,z}(x, z) \end{cases} \quad (\text{A.1})$$

where the symbol PV denotes a Cauchy principal value. The term  $\delta K_I^{\text{skew}}$  in  $\delta_K K_I$  was calculated for an out-of-plane perturbation  $\phi_y$  independent of  $x$  by Movchan et al. (1998), and in the general case by Leblond et al. (2011), with the following result:

$$\delta K_I^{\text{skew}}(x, z) = \frac{\sqrt{2}}{4\pi} \frac{1 - 2\nu}{1 - \nu} K_{III}^0 \text{Re} \left\{ \int_{-\infty}^x dx' \int_{-\infty}^{+\infty} \frac{\phi_{y,x}(x', z')}{(x - x')^{1/2} [(x - x') + i(z - z')]^{3/2}} dz' \right\} \quad (\text{A.2})$$

581 where the cut of the complex power function is along the half-line of non-positive real numbers. As explained in Section 2,  
 582 expressions (A.1) and (A.2) disregard the influence of the spatial variations of the loading on the variations of the SIFs (because the  
 583 lengths involved, of the order of  $|(\partial K_p^0 / \partial x) / K_p^0|$ , are assumed to be much larger than the typical length over which the perturbations  
 584  $\phi_x, \phi_y$  vary significantly).

585 *Appendix A.2. First-order perturbations of the stress intensity factors due to initial non-singular stresses*

586 The calculation of the variations  $\delta_T K_p$  of the SIFs tied to the non-singular stresses  $T_{xx}^0, T_{zz}^0, T_{xz}^0$  was presented in the work of  
 587 Movchan et al. (1998). These variations were shown to depend only on the out-of-plane perturbation  $\phi_y$  of the crack. They were  
 588 calculated only for a perturbation  $\phi_y$  independent of  $x$  and sinusoidal in  $z$ . Their expressions are derived here for an arbitrary  $\phi_y$ .  
 589 Some details on the derivation are given because of some discrepancies with the results of Movchan et al. (1998), see Appendix B  
 590 below.

591 We first recall that according to Movchan et al. (1998)'s results, the variations of SIFs arising from the out-of-plane perturbation  
 592  $\phi_y$  of the crack surface involve three kinds of terms: (i) *local* terms that relate to the local displacement of the front and orientation of  
 593 the crack surface, characterized by the quantities  $\phi_y, \phi_{y,x}, \phi_{y,z}$ ; (ii) *semi-local* terms that involve integrals over the whole crack front  
 594 of its displacement  $\phi_y$ ; and (iii) *non-local* terms in the form of integrals of  $\phi_y$  extended over the entire crack surface. Furthermore  
 595 we observe from the results derived in Section 3.5 of Movchan et al. (1998), that the variations  $\delta_T K_p$  involve only "morphology  
 596 terms" (in the authors' terminology) which are of type (iii). It is thus possible to retrieve the expressions of these variations by  
 597 considering a perturbation  $\phi_y$  that leaves unchanged both the local position of the crack front and the local orientation of the crack  
 598 surface at all points of the front ( $\phi_y = 0, \phi_{y,x} = \phi_{y,z} = 0$  along the front): under such conditions other (local or semi-local) terms in  
 599 the expressions of the variations of SIFs arising from the out-of-plane perturbation  $\phi_y$  will vanish.

To determine the variations  $\delta_T K_p$  under such conditions, let us expand the solution  $(\mathbf{u}, \boldsymbol{\sigma})$  of the elasticity problem at first order in the perturbation:

$$\begin{cases} \mathbf{u}(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}) + \mathbf{u}^1(\mathbf{x}) \\ \boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}^0(\mathbf{x}) + \boldsymbol{\sigma}^1(\mathbf{x}) \end{cases} \quad (\text{A.3})$$

where the displacement and stress fields  $\mathbf{u}^0$  and  $\boldsymbol{\sigma}^0$  are of order 0, and the fields  $\mathbf{u}^1$  and  $\boldsymbol{\sigma}^1$  of order 1. *All four fields are defined on the unperturbed cracked body*: this is obvious for the fields  $\mathbf{u}^0$  and  $\boldsymbol{\sigma}^0$ , and for the fields  $\mathbf{u}^1$  and  $\boldsymbol{\sigma}^1$  it stems from the fact that they correspond to the derivatives, on the unperturbed configuration, of the displacement and stress fields with respect to the amplitude

of the perturbation. We shall also need to expand the exterior normal vector  $\mathbf{n}(x, y = \phi_y(x, z), z)$  to the perturbed crack surface in a similar way:

$$\mathbf{n}(x, \phi_y(x, z), z) = \mathbf{n}^0 + \mathbf{n}^1(x, z). \quad (\text{A.4})$$

Let us now expand the condition of zero traction on the perturbed crack surface:

$$\begin{aligned} \mathbf{0} &= \boldsymbol{\sigma}(x, \phi_y(x, z), z) \cdot \mathbf{n}(x, \phi_y(x, z), z) = [\boldsymbol{\sigma}^0(x, \phi_y(x, z), z) + \boldsymbol{\sigma}^1(x, \phi_y(x, z), z)] \cdot [\mathbf{n}^0 + \mathbf{n}^1(x, z)] \\ &= \boldsymbol{\sigma}^0(x, 0, z) \cdot \mathbf{n}^0 + \boldsymbol{\sigma}_{,y}^0(x, 0, z) \cdot \mathbf{n}^0 \phi_y(x, z) + \boldsymbol{\sigma}^0(x, 0, z) \cdot \mathbf{n}^1(x, z) + \boldsymbol{\sigma}^1(x, 0, z) \cdot \mathbf{n}^0. \end{aligned}$$

Identifying terms of order 1 in the perturbation in this equation, we get the boundary condition satisfied by the first-order fields on the unperturbed crack:

$$\boldsymbol{\sigma}^1(x, 0, z) \cdot \mathbf{n}^0 = -\boldsymbol{\sigma}_{,y}^0(x, 0, z) \cdot \mathbf{n}^0 \phi_y(x, z) - \boldsymbol{\sigma}^0(x, 0, z) \cdot \mathbf{n}^1(x, z) \equiv \mathbf{t}^{\text{eff}}(x, z). \quad (\text{A.5})$$

We now develop this boundary condition in a more explicit form for those terms, in the unperturbed stress field  $\boldsymbol{\sigma}^0$ , which are connected to the non-singular stresses  $T_{xx}^0, T_{zz}^0, T_{xz}^0$ . First we note that for these terms, the gradient of  $\boldsymbol{\sigma}^0$  is zero so that the term  $-\boldsymbol{\sigma}_{,y}^0(x, 0, z) \cdot \mathbf{n}^0 \phi_y(x, z)$  in Eq. (A.5) vanishes. Second, we remark that the vectors  $\mathbf{n}^0$  and  $\mathbf{n}^1$  read, on the upper (+) and lower (-) unperturbed crack surfaces:

$$\begin{cases} \mathbf{n}^{0\pm} &= \mp \mathbf{e}_y \\ \mathbf{n}^{1\pm}(x, z) &= \pm (\phi_{y,x}(x, z) \mathbf{e}_x + \phi_{y,z}(x, z) \mathbf{e}_z). \end{cases} \quad (\text{A.6})$$

Finally we remark that the terms pertaining to the non-singular stresses in the unperturbed stress field  $\boldsymbol{\sigma}^0$  simply read

$$T_{xx}^0 \mathbf{e}_x \otimes \mathbf{e}_x + T_{zz}^0 \mathbf{e}_z \otimes \mathbf{e}_z + T_{xz}^0 (\mathbf{e}_x \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_x).$$

Using these elements, we get the ‘‘effective tractions’’  $\mathbf{t}^{\text{eff}}(x, z)$  appearing in the boundary condition (A.5) for the first-order solution on the upper and lower unperturbed crack surfaces:

$$\begin{aligned} \mathbf{t}^{\text{eff}\pm}(x, z) &= \boldsymbol{\sigma}^1(x, 0, z) \cdot \mathbf{n}^{0\pm} \\ &= -\boldsymbol{\sigma}^0(x, 0, z) \cdot \mathbf{n}^{1\pm}(x, z) \\ &= \mp \left[ T_{xx}^0 \mathbf{e}_x \otimes \mathbf{e}_x + T_{zz}^0 \mathbf{e}_z \otimes \mathbf{e}_z + T_{xz}^0 (\mathbf{e}_x \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_x) \right] \cdot (\phi_{y,x}(x, z) \mathbf{e}_x + \phi_{y,z}(x, z) \mathbf{e}_z) \\ &= \mp \left( T_{xx}^0 \phi_{y,x}(x, z) + T_{xz}^0 \phi_{y,z}(x, z) \right) \mathbf{e}_x \mp \left( T_{xz}^0 \phi_{y,x}(x, z) + T_{zz}^0 \phi_{y,z}(x, z) \right) \mathbf{e}_z. \end{aligned} \quad (\text{A.7})$$

Now since the perturbation of the crack envisaged leaves unchanged the position of the crack front, the solution  $(\mathbf{u}^1, \boldsymbol{\sigma}^1)$  possesses the usual  $r^{-1/2}$  stress singularity near the front, which implies that it is a ‘‘classical’’ solution of bounded total energy, to which standard theorems of elasticity theory apply.<sup>2</sup> It follows that the p-th SIF at the point  $(x, z)$  of the unperturbed crack front, corresponding to this solution, is given in terms of Bueckner (1987) weight functions and the effective tractions by the integral

$$\int_{-\infty}^x dx' \int_{-\infty}^{+\infty} h_{pi}(x, z; x', z') t_i^{\text{eff}\pm}(x', z') dz' \quad (\text{A.8})$$

where  $h_{pi}(x, z; x', z')$ , a *crack-face weight function*, is the p-th SIF generated at the point  $(x, z)$  of the front by a pair of opposite (equilibrated) unit forces applied in the directions  $\pm \mathbf{e}_i$  at the points  $(x', z')$  of the unperturbed upper (+) and lower (-) crack surfaces. But since the perturbation envisaged also leaves unchanged the orientation of the crack surface, these SIFs on the unperturbed crack front simply coincide with those,  $\delta_T K_p(x, z)$ , arising from the unperturbed non-singular stresses in the perturbed solution.<sup>3</sup> It follows, using Eqs. (A.7) and (A.8) and the expressions of the crack-face weight functions for a semi-infinite crack in an infinite body, provided for instance in (Gao and Rice, 1986), that:

$$\left\{ \begin{aligned} \delta_T K_I(x, z) &= 0 \\ \delta_T K_{II}(x, z) &= -\frac{\sqrt{2}}{\pi^{3/2}} \int_{-\infty}^x dx' \int_{-\infty}^{+\infty} \frac{\sqrt{x-x'}}{(x-x')^2 + (z-z')^2} \left[ 1 + \frac{2\nu}{2-\nu} \frac{(x-x')^2 - (z-z')^2}{(x-x')^2 + (z-z')^2} \right] \left[ T_{xx}^0 \phi_{y,x}(x', z') + T_{xz}^0 \phi_{y,z}(x', z') \right] dz' \\ &\quad - \frac{\sqrt{2}}{\pi^{3/2}} \int_{-\infty}^x dx' \int_{-\infty}^{+\infty} \frac{\sqrt{x-x'}}{(x-x')^2 + (z-z')^2} \frac{4\nu}{2-\nu} \frac{(x-x')(z-z')}{(x-x')^2 + (z-z')^2} \left[ T_{xz}^0 \phi_{y,x}(x', z') + T_{zz}^0 \phi_{y,z}(x', z') \right] dz' \\ \delta_T K_{III}(x, z) &= -\frac{\sqrt{2}}{\pi^{3/2}} \int_{-\infty}^x dx' \int_{-\infty}^{+\infty} \frac{\sqrt{x-x'}}{(x-x')^2 + (z-z')^2} \frac{4\nu}{2-\nu} \frac{(x-x')(z-z')}{(x-x')^2 + (z-z')^2} \left[ T_{xx}^0 \phi_{y,x}(x', z') + T_{xz}^0 \phi_{y,z}(x', z') \right] dz' \\ &\quad - \frac{\sqrt{2}}{\pi^{3/2}} \int_{-\infty}^x dx' \int_{-\infty}^{+\infty} \frac{\sqrt{x-x'}}{(x-x')^2 + (z-z')^2} \left[ 1 - \frac{2\nu}{2-\nu} \frac{(x-x')^2 - (z-z')^2}{(x-x')^2 + (z-z')^2} \right] \left[ T_{xz}^0 \phi_{y,x}(x', z') + T_{zz}^0 \phi_{y,z}(x', z') \right] dz'. \end{aligned} \right. \quad (\text{A.9})$$

<sup>2</sup>If the perturbation implied a displacement of the front, the stress  $\boldsymbol{\sigma}^1$  would behave near it like  $r^{-3/2}$  so that the corresponding total energy would be infinite, implying a non-classical solution to which usual theorems would *not* apply.

<sup>3</sup>If the orientation of the surface was modified in the perturbation, the relation between these two sets of SIFs would involve additional terms tied to the small rotations  $\phi_{y,x}$  and  $\phi_{y,z}$  of this surface.

600 **Appendix B. Expressions of the variations of the stress intensity factors using Fourier transforms**

601 In this Appendix, we provide the expressions of the variations  $\delta K_p$  of the SIFs for the perturbations defined in Eq. (5), using the  
602 Fourier transforms  $\hat{\psi}_x, \hat{\psi}_y$  of the functions  $\psi_x, \psi_y$ .

603 It is assumed below that  $\text{Re}(\xi) > -1$  or equivalently  $\text{Re}(\lambda) > -|k|$  (which is true for all values of the wavenumber  $k$  if  $\text{Re}(\lambda) > 0$ );  
604 this condition is indeed required for convergence of the integrals obtained from the general expressions (A.1) of  $\delta_K K_p$  and (A.9) of  
605  $\delta_T K_p$ .

606 *Appendix B.1. Variations of the stress intensity factors tied to  $K_I^0, K_{III}^0$*

For perturbations of the form (5), the expression of the variations  $\delta_K K_p$  of Eq. (A.1) are given by Leblond et al. (2019) as:

$$\begin{cases} \delta_K K_I(x, z) &= \text{Re} \left\{ e^{\lambda x} \int_{-\infty}^{+\infty} \left[ -K_I^0 \frac{|k|}{2} \hat{\psi}_x(k) - iK_{III}^0 \left( 2 - \frac{1-2\nu}{\sqrt{2}(1-\nu)} \frac{\lambda}{(\lambda+|k|)^{1/2}} \right) \hat{\psi}_y(k) \right] e^{ikz} dk \right\} \\ \delta_K K_{II}(x, z) &= \text{Re} \left\{ e^{\lambda x} \int_{-\infty}^{+\infty} \left[ -iK_{III}^0 \frac{2}{2-\nu} k \hat{\psi}_x(k) + K_I^0 \left( \frac{\lambda}{2} + \frac{2-3\nu}{2(2-\nu)} |k| \right) \hat{\psi}_y(k) \right] e^{ikz} dk \right\} \\ \delta_K K_{III}(x, z) &= \text{Re} \left\{ e^{\lambda x} \int_{-\infty}^{+\infty} \left[ -K_{III}^0 \frac{2+\nu}{2(2-\nu)} |k| \hat{\psi}_x(k) + iK_I^0 \frac{2(1-\nu)^2}{2-\nu} k \hat{\psi}_y(k) \right] e^{ikz} dk \right\} \end{cases} \quad (\text{B.1})$$

607 *Appendix B.2. Variations of the stress intensity factors tied to  $T_{xx}^0, T_{zz}^0, T_{xz}^0$*

We provide here the expressions of the variations  $\delta_T K_p$  of Eq. (A.9) for perturbations of the form (5). As an example of the kind  
of calculations required, here are those necessary to evaluate the first integral,  $\delta_T^1 K_{II}$ , appearing in the expression (A.9)<sub>2</sub> of  $\delta_T K_{II}$ :

$$\begin{aligned} \delta_T^1 K_{II}(x, z) &= -\frac{\sqrt{2}}{\pi^{3/2}} \int_{-\infty}^x dx' \int_{-\infty}^{+\infty} \frac{\sqrt{x-x'}}{(x-x')^2 + (z-z')^2} \left[ 1 + \frac{2\nu}{2-\nu} \frac{(x-x')^2 - (z-z')^2}{(x-x')^2 + (z-z')^2} \right] \left[ T_{xx}^0 \phi_{y,x}(x', z') + T_{xz}^0 \phi_{y,z}(x', z') \right] dz' \\ &= -\frac{\sqrt{2}}{\pi^{3/2}} \text{Re} \left\{ \int_{-\infty}^x e^{\lambda x'} dx' \int_{-\infty}^{+\infty} \frac{\sqrt{x-x'}}{(x-x')^2 + (z-z')^2} \left[ 1 + \frac{2\nu}{2-\nu} \frac{(x-x')^2 - (z-z')^2}{(x-x')^2 + (z-z')^2} \right] \left[ \lambda T_{xx}^0 \psi_y(z') + T_{xz}^0 \psi_{y,z}(z') \right] dz' \right\} \\ &= -\frac{\sqrt{2}}{\pi^{3/2}} \text{Re} \left\{ \int_{-\infty}^x e^{\lambda x'} dx' \int_{-\infty}^{+\infty} \frac{\sqrt{x-x'}}{(x-x')^2 + (z-z')^2} \left[ 1 + \frac{2\nu}{2-\nu} \frac{(x-x')^2 - (z-z')^2}{(x-x')^2 + (z-z')^2} \right] dz' \int_{-\infty}^{+\infty} (\lambda T_{xx}^0 + ik T_{xz}^0) \hat{\psi}_y(k) e^{ikz'} dk \right\} \\ &= -\frac{\sqrt{2}}{\pi^{3/2}} \text{Re} \left\{ \int_{-\infty}^x e^{\lambda x'} dx' \int_{-\infty}^{+\infty} \frac{\sqrt{x-x'}}{(x-x')^2 + (z-z')^2} \left[ \frac{2-3\nu}{2-\nu} + \frac{4\nu}{2-\nu} \frac{(x-x')^2}{(x-x')^2 + (z-z')^2} \right] dz' \int_{-\infty}^{+\infty} (\lambda T_{xx}^0 + ik T_{xz}^0) \hat{\psi}_y(k) e^{ikz'} dk \right\} \\ &\xrightarrow{\substack{u = x-x' \\ v = z-z'}} = -\frac{\sqrt{2}}{\pi^{3/2}(2-\nu)} \text{Re} \left\{ e^{\lambda x} \int_{-\infty}^{+\infty} (\lambda T_{xx}^0 + ik T_{xz}^0) \hat{\psi}_y(k) e^{ikz} dk \int_0^{+\infty} \sqrt{u} e^{-\lambda u} du \int_{-\infty}^{+\infty} \left[ \frac{2-3\nu}{u^2+v^2} + 4\nu \frac{u^2}{(u^2+v^2)^2} \right] e^{-ikv} dv \right\} \\ &= -\frac{\sqrt{2}}{\pi^{3/2}(2-\nu)} \text{Re} \left\{ e^{\lambda x} \int_{-\infty}^{+\infty} (\lambda T_{xx}^0 + ik T_{xz}^0) \hat{\psi}_y(k) e^{ikz} dk \int_0^{+\infty} \pi \frac{e^{-(\lambda+|k|)u}}{\sqrt{u}} [2-3\nu + 2\nu(1+|k|u)] du \right\} \\ &= -\sqrt{2} \text{Re} \left\{ e^{\lambda x} \int_{-\infty}^{+\infty} (\lambda T_{xx}^0 + ik T_{xz}^0) \left[ \frac{1}{(\lambda+|k|)^{1/2}} + \frac{\nu}{2-\nu} \frac{|k|}{(\lambda+|k|)^{3/2}} \right] \hat{\psi}_y(k) e^{ikz} dk \right\}. \end{aligned} \quad (\text{B.2})$$

The last steps of the calculation here used the following results (see formulae (3.723.2), (3.729.1) and (3.944.6) of Gradshteyn and  
Ryzhik (2014)):

$$\begin{cases} \int_{-\infty}^{+\infty} \frac{e^{-ikv}}{u^2+v^2} dv = \frac{\pi}{u} e^{-|k|u} & ; \quad \int_{-\infty}^{+\infty} \frac{e^{-ikv}}{(u^2+v^2)^2} dv = \frac{\pi}{2u^3} (1+|k|u) e^{-|k|u} \\ \int_{-\infty}^{+\infty} \frac{e^{-(\lambda+|k|)u}}{\sqrt{u}} du = \sqrt{\pi} (\lambda+|k|)^{-1/2} & ; \quad \int_{-\infty}^{+\infty} \sqrt{u} e^{-(\lambda+|k|)u} du = \frac{\sqrt{\pi}}{2} (\lambda+|k|)^{-3/2}. \end{cases} \quad (\text{B.3})$$

The three remaining integrals of Eq. (A.9) may be calculated in a similar way, using in addition formula (3.729.2) of Gradshteyn  
and Ryzhik (2014). The final results read as follows:

$$\begin{cases} \delta_T K_I(x, z) &= 0 \\ \delta_T K_{II}(x, z) &= \sqrt{2} \text{Re} \left\{ e^{\lambda x} \int_{-\infty}^{+\infty} \left[ -\left( 1 + \frac{\nu}{2-\nu} \frac{|k|}{\lambda+|k|} \right) \frac{\lambda}{(\lambda+|k|)^{1/2}} T_{xx}^0 - \frac{\nu}{2-\nu} \frac{k^2}{(\lambda+|k|)^{3/2}} T_{zz}^0 - i \left( 1 - \frac{\nu}{2-\nu} \frac{\lambda-|k|}{\lambda+|k|} \right) \frac{k}{(\lambda+|k|)^{1/2}} T_{xz}^0 \right] \hat{\psi}_y(k) e^{ikz} dk \right\} \\ \delta_T K_{III}(x, z) &= \sqrt{2} \text{Re} \left\{ e^{\lambda x} \int_{-\infty}^{+\infty} \left[ i \frac{\nu}{2-\nu} \frac{\lambda k}{(\lambda+|k|)^{3/2}} T_{xx}^0 - i \left( 1 - \frac{\nu}{2-\nu} \frac{|k|}{\lambda+|k|} \right) \frac{k}{(\lambda+|k|)^{1/2}} T_{zz}^0 - \left( \lambda - \frac{\nu}{2-\nu} \frac{\lambda-|k|}{\lambda+|k|} |k| \right) \frac{1}{(\lambda+|k|)^{1/2}} T_{xz}^0 \right] \hat{\psi}_y(k) e^{ikz} dk \right\}. \end{cases} \quad (\text{B.4})$$

For the sinusoidal perturbation independent of  $x$  considered by Gao (1992) and Movchan et al. (1998),  $\phi_y(z) = A \cos(kz)$ ,  $k > 0$  (that is,  $\lambda = 0$  and  $\hat{\psi}_y(k') = \frac{A}{2} [\delta(k' - k) + \delta(k' + k)]$ ), one thus gets:

$$\begin{cases} \delta_T K_{II}(z) = A \frac{\sqrt{2k}}{2-\nu} \left[ 2T_{xz}^0 \sin(kz) - \nu T_{zz}^0 \cos(kz) \right] \\ \delta_T K_{III}(z) = A \frac{\sqrt{2k}}{2-\nu} \left[ -\nu T_{xz}^0 \cos(kz) + 2(1-\nu) T_{zz}^0 \sin(kz) \right]. \end{cases} \quad (\text{B.5})$$

608 These expressions agree with Eqs (40) and (41) of Gao (1992).<sup>4</sup> When comparing our results to Eqs. (3.44) and (3.45) of Movchan  
609 et al. (1998), however, we note sign discrepancies in the term proportional to  $T_{zz}^0$  in the expression of  $\delta_T K_{II}$ , and in the term  
610 proportional to  $T_{xz}^0$  in the expression of  $\delta_T K_{III}$ . It is unfortunately not possible to check Movchan et al. (1998)'s calculations  
611 because they are not detailed in their paper.

## 612 Appendix C. Procedure of solution of Eq. (18)

The study of general (complex) solutions in  $\xi$  of Eq. (18) is made easier by adopting the change of variable suggested by Leblond et al. (2011):

$$\zeta = \frac{1}{\sqrt{\xi+1}} \quad \Leftrightarrow \quad \xi = \frac{1}{\zeta^2} - 1 \quad (\text{with } \text{Re}(\xi) > -1 \Leftrightarrow |\text{Arg}(\zeta)| < \frac{\pi}{4}). \quad (\text{C.1})$$

Equation (18) reduces, after some manipulations, to the following algebraic equation of the 5th degree in  $\zeta$ :

$$\begin{aligned} & 2\sqrt{2}\nu(\rho_0^2 - (1+\nu)) \left[ q_{xx}^0 - 2isq_{xz}^0 - q_{zz}^0 \right] \zeta^5 + 4\sqrt{2} \left[ -(1-2\nu)\rho_0^2 - ((1+\nu)\rho_0^2 + (1-\nu)^2)q_{xx}^0 + ((3+\nu)\rho_0^2 + (1-\nu)^2)isq_{xz}^0 + 2\rho_0^2q_{zz}^0 \right] \zeta^3 \\ & + \left[ 2(4-\nu)\rho_0^2 + 2\nu(1-\nu) \right] \zeta^2 + 2\sqrt{2} \left[ ((2+\nu)\rho_0^2 + (1-\nu)(2-\nu))q_{xx}^0 - 4\rho_0^2isq_{xz}^0 \right] \zeta - (2+\nu)\rho_0^2 - (1-\nu)(2-\nu) = 0. \end{aligned} \quad (\text{C.2})$$

613 The roots of Eq. (C.2) are calculated numerically using the function roots of the Python library Numpy (Harris et al., 2020). The  
614 method relies on evaluation of the eigenvalues of the companion matrix. Only the roots satisfying the condition  $|\text{Arg}(\zeta)| < \pi/4$  are  
615 of interest, so that the number of retained roots may be less than 5.

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<sup>4</sup>They do not agree with Eq. (44) of (Gao, 1992), but the origin of the discrepancy can easily be traced back to a trivial typographic or algebraic error in the transition from his Eqs. (40) and (41) to his Eq. (44).

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