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# 1 Quasi-static crack front deformations in cohesive materials

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## 7 Abstract

When a crack interacts with material heterogeneities, its front distorts and adopts complex tortuous configurations that are reminiscent of the energy barriers encountered during crack propagation. As such, the study of crack front deformations is key to rationalize the effective failure properties of micro-structured solids and interfaces. Yet, the impact of a localized dissipation in a finite region behind the crack front, called the *process zone*, has often been overlooked. In this work, we derive the equation ruling 3D coplanar crack propagation in heterogeneous *cohesive* materials where the opening of the crack is resisted by some traction in its wake. We show that the presence of a process zone results in two *competing* effects on the deformation of crack fronts: (i) it makes the front *more compliant* to small-wavelength perturbations, and (ii) it *smooths out* local fluctuations of strength and process zone size, from which emerge heterogeneities of fracture energy. Their respective influence on front deformations is shown to strongly impact the stability of perturbed crack fronts, as well as their stationary shapes when interacting with arrays of tough obstacles. Overall, our theory provides a unified framework to predict the variety of front profiles observed in experiments, even when the small-scale yielding hypothesis of linear elastic fracture mechanics breaks down.

8 *Keywords:* Brittle fracture, cohesive zone models, crack front deformation, front stability,  
9 heterogeneous materials

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## 10 1. Introduction

11 Significant efforts have been made in the past decades to unravel the influence of material  
12 heterogeneities on the failure behavior of composites. Understanding how crack fronts deform  
13 may seem a rather anecdotal subject matter in this regard. Yet, the front deformations are remi-  
14 niscent of the interaction between a crack and material disorder, and therefore conceal a wealth  
15 of information on the disorder intensity and its structure. Their study has provided invaluable  
16 insights on the spatio-temporal dynamics of propagating cracks, and on the effective toughness  
17 of composite brittle materials (see Lazarus (2011), Bonamy and Bouchaud (2011) and references  
18 therein).

19 During its interaction with heterogeneities of material properties, the crack front distorts and  
20 adopts complex tortuous configurations. Predicting its propagation path requires considering

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21 all possible geometric configurations, as well as a suitable criterion that selects the “most fa-  
22 vorable” one. This lead to a predominance of perturbative approaches in fracture mechanics,  
23 where quantities of interest such as the stress intensity factors can be linked to perturbations of  
24 the crack front geometry (Gao and Rice, 1986) and that of its surface (Movchan et al., 1998).  
25 In particular, the first-order theory of Rice (1985) based on Bueckner (1987)’s weight functions  
26 has been quite successful in capturing experimental observations of coplanar crack propagation  
27 along weak interfaces. It provides a convincing framework to describe quantitatively the deforma-  
28 tions of a crack front interacting with a single tough defect (Chopin et al., 2011; Patinet et al.,  
29 2013a; Vasoya et al., 2016b) of sometimes complex shape (Xia et al., 2012), periodic arrays  
30 of obstacles (Dalmas et al., 2009), or disorderly-placed heterogeneities (Delaplace et al., 1999).  
31 It also helped in rationalizing the micro-dynamics of depinning of a crack exiting an obstacle  
32 (Chopin et al., 2018), or the intermittent macro-dynamics of crack propagation in disordered me-  
33 dia (Måløy et al., 2006; Ponsón and Bonamy, 2010; Barés et al., 2018). The same theory has  
34 been extensively applied to predict the effective toughness arising from periodic (Gao and Rice,  
35 1989), asymmetric (Xia et al., 2015), or even disordered (Patinet et al., 2013b; Démercy et al.,  
36 2014; Lebihain, 2021) distributions of fracture properties. Higher-order theories also provided  
37 valuable insights on bridging mechanisms (Bower and Ortiz, 1991), on fingering instabilities  
38 (Vasoya et al., 2016a), or on the microbranching transition at higher crack velocities (Kolvin  
39 et al., 2017).

40 A major pitfall of these models is that they arguably overlook the influence of the spatially  
41 localized weakening dynamics near the crack front. Indeed, the LEFM theory is based on the  
42 so-called *small-scale yielding* assumption, which states that all the dissipation occurs in an in-  
43 finitesimally small region in the vicinity of the crack front. As such, LEFM does not provide  
44 any meaningful dissipation length scale, and treats all asperity scales indifferently. Yet, as we  
45 zoom in on the front of a propagating crack, one ultimately finds a region of finite size where the  
46 material behaves inelastically and the validity of the LEFM framework breaks down. It is crucial  
47 to understand how heterogeneities smaller or bigger than this dissipative region may influence  
48 the overall failure behavior of a composite. Cohesive-zone models (Dugdale, 1960; Barenblatt,  
49 1962) provide a way to do that, as they assume that the material does not weaken instantly but  
50 in a region of finite size – called “*process zone*” – located at the crack front where the crack  
51 opening is resisted by a distribution of *cohesive* stresses. In these models, the fracture properties  
52 are not solely characterized by the fracture energy  $G_c$ , but rather by (i) the *strength*  $\sigma_c$  of the  
53 material, and (ii) its *process zone size*  $\omega$ , two quantities from which emerge the fracture energy  
54 (Palmer et al., 1973). The development of a unified framework that describes the failure behavior  
55 of *cohesive* and *heterogeneous* materials may only take place through a preliminary study of the  
56 front deformations *in presence of a finite process zone*.

57  
58 In this work, we extend the LEFM perturbative theory of Rice (1985) to mode I coplanar  
59 crack propagation in *cohesive* materials. This is performed by deriving first Bueckner (1987)’s  
60 crack face weight function for a semi-infinite coplanar crack perturbed within its plane. As a  
61 result, we can explore for the first time the influence of a finite process zone size on the front  
62 deformations of a crack propagating in tensile mode I. We show that a cohesive crack accommo-  
63 dates perturbations differently depending on the size of the perturbation wavelength with respect  
64 to that of the process zone. We also emphasize that the presence of cohesive stress behind the  
65 crack front may strongly modify the fluctuations of fracture energy the crack actually perceives  
66 during its propagation. This theory successfully predicts the front profiles observed in the peel-  
67 ing experiments of Chopin et al. (2011), for which the small-scale yielding hypothesis of LEFM

68 is suspected to break down.

69 The paper is organized as follows: in Section 2, we derive the expression of the crack face  
70 weight function perturbations at first-order in the front deformation, following a method pro-  
71 posed by Rice (1989). It allows us to calculate the stress intensity factor arising from cohesive  
72 stresses acting behind the front for materials translationally invariant in the propagation direc-  
73 tion. Building on this model, we revisit in Section 3 the stability problem of a perturbed crack  
74 front treated by Rice (1985) for perfectly brittle materials. We highlight here the influence of the  
75 finite process zone size on the relative “stiffness” of the crack front to some modal perturbations.  
76 Next, we investigate in Section 4 the stationary shape of a crack front interacting with periodic  
77 arrays of tough obstacles. We show here that the presence of cohesive stresses behind the crack  
78 front does not only affect the stiffness of the crack front, but also influences the energy landscape  
79 the crack actually experiences during its propagation. The front profiles predicted by our theory  
80 are successfully compared to those obtained by Chopin et al. (2011) during adhesive peeling  
81 experiments of an elastomer block from a patterned glass substrate.

## 82 2. First-order variations of the mode I stress intensity factor for a semi-infinite coplanar 83 cohesive crack

### 84 2.1. Cohesive approach to three-dimensional coplanar crack propagation

85 We consider a semi-infinite crack embedded in an infinite body made of an isotropic linear  
86 elastic material. In the initial reference configuration  $\Gamma$ , the crack is planar and its front is straight  
87 (see Fig. 1a). We adopt the usual convention of LEFM, and thus use a Cartesian frame  $Oxyz$  with  
88 origin  $O$  chosen arbitrarily within the crack plane, axis  $(Ox)$  parallel to the direction of crack  
89 propagation,  $(Oy)$  parallel to the direction orthogonal to the crack plane, and  $(Oz)$  parallel to the  
90 crack front. The associated unit vectors are denoted  $(e_x, e_y, e_z)$ . The crack  $\Gamma$  is loaded in pure  
91 mode I through some system of forces independent of the coordinate  $z$ , so that the unperturbed  
92 stress intensity factor (SIF)  $K_I^0$  is independent of the point of observation along the crack front.  
93 The influence of the loading conditions and the specimen geometry may be accounted for in the  
94 evolution of  $K_I^0$  with crack advance.

95  
96 Now, let the front  $\mathcal{F}$  undergo some infinitesimal coplanar perturbation, and note  $\delta a$  the lo-  
97 cal orthogonal distance between the perturbed front  $\mathcal{F}^*$  with respect to the reference one (see  
98 Fig. 1b). The LEFM perturbed stress intensity factor  $K_{\text{Iefm}}$  along the front reads (Rice, 1985):

$$K_{\text{Iefm}}(z) = K_I^0 + \delta K_{\text{Iefm}}(z) = K_I^0 \left[ 1 + \frac{1}{K_I^0} \frac{\partial K_I^0}{\partial a} \delta a(z) - \frac{1}{2\pi} \text{PV} \int_{-\infty}^{+\infty} \frac{\delta a(z) - \delta a(z')}{(z - z')^2} dz' \right] \quad (1)$$

99 where  $\delta K_{\text{Iefm}}$  corresponds to the first-order variations of  $K_{\text{Iefm}}$  due to the front deformations  $\delta a$ ,  
100 and the symbol PV denotes a Cauchy principal value. Equation (1) is non-local, since the rupture  
101 behavior at a given point is affected by the position of the other points due to long-range elastic  
102 interactions.

103  
104 In the following, use will be made of Fourier transforms in the direction  $(Oz)$  of the crack  
105 front. The definition adopted here for the Fourier transform  $\hat{\chi}(k)$  of an arbitrary function  $\chi(z)$  is:

$$\hat{\chi}(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(z) e^{-ikz} dz \Leftrightarrow \chi(z) = \int_{-\infty}^{+\infty} \hat{\chi}(k) e^{ikz} dk \quad (2)$$

106 Then Eq. (1) reads in the Fourier space:

$$\frac{\widehat{\delta K_{\text{Iefm}}}(k)}{K_{\text{I}}^0} = \left( \frac{1}{K_{\text{I}}^0} \frac{\partial K_{\text{I}}^0}{\partial a} - \frac{|k|}{2} \right) \widehat{\delta a}(k) \quad (3)$$

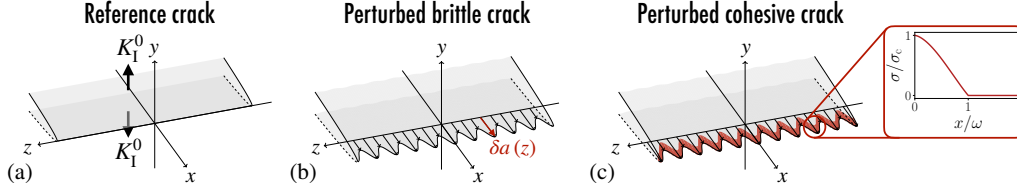


Figure 1: (a) A semi-infinite planar crack  $\Gamma$  with a straight crack front  $\mathcal{F}$  is loaded under pure mode I through some system of forces, giving rise to a stress intensity factor  $K_{\text{I}}^0$ . (b) Due to the presence of microscopic heterogeneities, the crack front  $\mathcal{F}^*$  of the perturbed crack  $\Gamma^*$  distorts by a quantity  $\delta a(z)$  along the direction of propagation ( $Ox$ ). (c) The material does not weaken instantly in the crack wake, but progressively in a closed region behind the rupture front (in red) called *process zone*. Inset: In this region, the cohesive stress  $\sigma$  decay from its peak value  $\sigma_c$ , referred to as the *strength* of the material, down to zero along a typical distance  $\omega$ .

107 The LEFM approach leads to a “non-physical” situation where the stress is found singular at  
 108 the crack front. Cohesive zone models (CZMs) address this issue by assuming that the opening  
 109 of the crack is *resisted* in its wake by a tensile stress  $\sigma(z, x)$ , called *cohesive stress*, acting behind  
 110 the crack front (see Fig. 1c). It typically evolves from a peak value  $\sigma_c$ , called *strength* of the  
 111 material, down to negligible values along a characteristic distance  $\omega$ , called *process zone size*  
 112 (see inset of Fig. 1c). The evolution of  $\sigma$  behind the crack front usually results from a traction-  
 113 separation law  $\sigma = f(\delta)$  describing material degradation with the local opening displacement  $\delta$   
 114 (Dugdale, 1960; Barenblatt, 1962). This cohesive stress  $\sigma$  gives rise to a *negative* stress intensity  
 115 factor  $-K_{\text{czm}}(z)$  that balances  $K_{\text{Iefm}}(z)$ , ensuring then that the stress is non-singular at each point  
 116 along the crack front:

$$K_{\text{total}}(z) = K_{\text{Iefm}}(z) - K_{\text{czm}}(z) = 0 \quad (4)$$

117 Building on Bueckner-Rice’s weight function theory (Bueckner, 1987; Rice, 1989), one may  
 118 express the cohesive stress intensity factor  $K_{\text{czm}}$  as:

$$K_{\text{czm}}(z) = \int_{\Gamma^*} k^*(\Gamma^*; z, z', x) \sigma(\mathbf{x}) dz' dx \quad (5)$$

119 where  $k^*(\Gamma^*; z, z', x)$  is the mode I crack face weight function (CFWF). It corresponds to the  
 120 stress intensity factor generated at point  $z$  by a pair of unitary tensile forces applied at a point  
 121  $(z', x)$  located on the faces of  $\Gamma^*$ .

## 122 2.2. First-order variations of the mode I crack face weight function

123 In order to compute the cohesive stress intensity factor of Eq. (5), we need a closed form for  
 124 the crack face weight function  $k^*$  of the perturbed front  $\mathcal{F}^*$ . We follow here the ideas developed  
 125 by Rice (1989) and Favier et al. (2006), upon which Leblond et al. (2012) built to derive the  
 126 first-order variations of the fundamental kernel  $Z(\Gamma^*; z; z') = \sqrt{\pi}/8 \lim_{x \rightarrow 0} k(\Gamma^*; z; z', x) / \sqrt{x}$ . We  
 127 build upon them here to derive the *variations of the mode I crack face weight function* for a semi-  
 128 infinite coplanar crack at *first order* in the perturbation  $\delta a$ . Our results generalize the calculations

129 of Leblond et al. (2012) that only looked at the asymptotic behavior of  $k^*$  near the crack front  
 130 ( $x \rightarrow 0$ ).

131

132 The evolution of the opening displacement and the stress in the vicinity of the perturbed crack  
 133 front are naturally expressed in the local basis of vectors  $(e_z^*(z), e_x^*(z), e_y)$ , where  $e_z^*(z)$  is tangent  
 134 to the crack front  $\mathcal{F}^*$  at position  $z$ , and  $e_x^*(z)$  is perpendicular to it within the plane  $(zOx)$  and  
 135 oriented in the direction of propagation (see Fig. 2). Following Favier et al. (2006), we need to  
 136 define two distinct crack face weight functions:

- 137 •  $k^*(\mathcal{F}^*; z_0; z_1, x^*)$ , which corresponds to the stress intensity factor generated at  $z = z_0$  by a  
 138 pair of unitary forces applied along  $e_y$  at a distance  $x^*$  behind the point  $z_1$  of the perturbed  
 139 crack front  $\mathcal{F}^*$  in the direction of the vector  $e_x^*(z_1)$ ;
- 140 •  $k(\mathcal{F}^*; z_0; z_1, x)$ , which corresponds to the stress intensity factor generated at  $z = z_0$  by a  
 141 pair of unitary forces applied along  $e_y$  at a distance  $x$  behind the point  $z_1$  of the perturbed  
 142 crack front  $\mathcal{F}^*$  in the direction  $e_x$ .

143 At first order in the perturbation, one has:

$$k^*(\mathcal{F}^*; z_0; z_1, x^*) = k(\mathcal{F}^*; z_0; z_1, x) \quad (6)$$

144 as the error introduced on the position of the point of application of the forces is of second order  
 145 in  $\delta a$  (Favier et al., 2006). Moreover, Rice (1989) showed that provided the crack advance  $\delta a(z)$   
 146 satisfies the condition:

$$\delta a(z_0) = 0 \text{ and } \delta a(z_1) = 0 \quad (7)$$

147  $k(\mathcal{F}^*; z_0; z_1, x)$  can be expressed from the CFWF  $k(\mathcal{F}; z_0; z_1, x)$  of the reference straight front  
 148 following:

$$k(\mathcal{F}^*; z_0; z_1, x) = k(\mathcal{F}; z_0; z_1, x) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} k(\mathcal{F}; z; z_1, x) \frac{\delta a(z)}{(z - z_0)^2} dz \quad (8)$$

149 where  $k(\mathcal{F}; z_0; z_1, x)$  is known analytically for the semi-infinite coplanar crack with a straight  
 150 crack front  $\Gamma$ :

$$k(\mathcal{F}; z_0; z_1, x) = \frac{\sqrt{2}}{\pi^{3/2}} \frac{\sqrt{x}}{(z_0 - z_1)^2 + x^2} \quad (9)$$

151 Yet, the condition (7) is not satisfied for an arbitrary perturbation  $\delta a$ . To circumvent this diffi-  
 152 culty, we follow the ideas of Leblond et al. (2012), and compute the perturbed CFWF associated  
 153 to  $\Gamma^*$  from those of a reference crack  $\Gamma^{**}$  that results from the combination of translatory motion  
 154 and a rotation  $\delta a^{**}$  making  $\delta a^*(z) = \delta a(z) - \delta a^{**}(z)$  vanish in  $z_0$  and  $z_1$ :

$$\delta a^{**}(z) = \delta a(z_0) + \frac{\delta a(z_1) - \delta a(z_0)}{z_1 - z_0} (z - z_0) = \delta a(z_1) + \frac{\delta a(z_1) - \delta a(z_0)}{z_1 - z_0} (z - z_1) \quad (10)$$

155 In the following,  $(e_z^{**}(z_0, z_1), e_x^{**}(z_0, z_1), e_y)$  denotes the natural basis of vectors associated to the  
 156 straight crack front  $\mathcal{F}^{**}$  of  $\Gamma^{**}$ , and  $(z^{**}, x^{**}, y)$  the point coordinates in this basis (see Fig. 2).  
 157 One may then define:

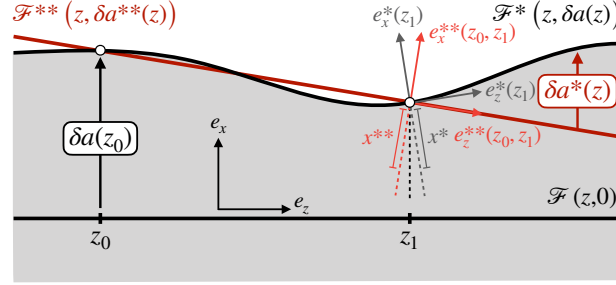


Figure 2: The mode I stress intensity factor  $k(\mathcal{F}^*; z_0; z_1, x^*)$  generated at  $z = z_0$  by a pair of unitary forces applied along  $e_y$  at a distance  $x^*$  behind the point  $z_1$  of the perturbed crack front  $\mathcal{F}^*$  (in black) in the direction of the vector  $e_x^*(z_1)$  can be inferred from that generated by a pair of unitary forces applied along  $e_y$  at a distance  $x^{**}$  behind the point  $z_1$  of the auxiliary front  $\mathcal{F}^{**}$  (in red) in the direction of the vector  $e_x^*(z_1)$ .

- $k^{**}(\Gamma^*; z_0^{**}, z_1^{**}, x^{**})$ , which corresponds to the stress intensity factor generated at  $z^{**} = z_0^{**}$  by a pair of unitary forces applied along  $e_y$  at a distance  $x^{**}$  behind the point  $z_1^{**}$  of the perturbed crack front  $\mathcal{F}^{**}$  in the direction  $e_x^*(z_0, z_1)$ . Since  $\Gamma^{**}$  is a semi-infinite coplanar crack with a straight crack front,  $k^{**}(\Gamma^*; z_0^{**}, z_1^{**}, x^{**})$  reads:

$$k^{**}(\mathcal{F}; z_0^{**}; z_1^{**}, x^{**}) = \frac{\sqrt{2}}{\pi^{3/2}} \frac{\sqrt{x^{**}}}{(z_0^{**} - z_1^{**})^2 + x^{**2}} \quad (11)$$

- $k^*(\Gamma^*; z_0^{**}, z_1^{**}, x^{**})$ , which corresponds to the stress intensity factor generated at  $z^{**} = z_0^{**}$  by a pair of unitary forces applied along  $e_y$  at a distance  $x^{**}$  behind the point  $z_1^{**}$  of the perturbed crack front  $\mathcal{F}^*$  in the direction  $e_x^*(z_0, z_1)$ . Applying Eq. (8) to  $\Gamma^{**}$  and  $\Gamma^*$  yields:

$$k^{**}(\Gamma^*; z_0^{**}, z_1^{**}, x^{**}) = k^{**}(\Gamma^{**}; z_0^{**}, z_1^{**}, x^{**}) + \frac{1}{2\pi} \text{PV} \int_{-\infty}^{+\infty} k^{**}(\mathcal{F}^{**}; z^{**}; z_1^{**}, x^{**}) \frac{\delta a(z^{**}) - \delta a^{**}(z^{**})}{(z^{**} - z_0^{**})^2} dz^{**} \quad (12)$$

Again, the error on the position  $(z^{**}, x^{**})$  with respect to  $(z, x)$  is of second order in  $\delta a$ . In the  $(e_z, e_x, e_y)$  basis, Eq. (12) writes as:

$$k(\Gamma^*; z_0; z_1, x) = \frac{\sqrt{2}}{\pi^{3/2}} \frac{\sqrt{x}}{(z_0 - z_1)^2 + x^2} + \frac{1}{2\pi} \text{PV} \int_{-\infty}^{+\infty} \frac{\sqrt{2}}{\pi^{3/2}} \frac{\sqrt{x}}{(z - z_1)^2 + x^2} \frac{\delta a(z) - \delta a^{**}(z)}{(z - z_0)^2} dz \quad (13)$$

Combined with Eq. (10), Eq. (13) provides a direct way to evaluate Eq. (5) numerically. Yet, as noted by Leblond et al. (2012), the presence of the rational function  $1/((z - z_1)^2 + x^2)(z - z_0)^2$  in the integrand makes it unfit for any analytical calculations. One may then decompose it as:

$$\frac{1}{((z - z_1)^2 + x^2)(z - z_0)^2} = \frac{1}{((z_0 - z_1)^2 + x^2)^2} \left[ \frac{(z_0 - z_1)^2 + x^2}{(z - z_0)^2} - \frac{2(z_0 - z_1)}{(z - z_0)} + \frac{2(z_0 - z_1)(z - z_1)}{(z - z_1)^2 + x^2} + \frac{(z_0 - z_1)^2 - x^2}{(z - z_1)^2 + x^2} \right] \quad (14)$$

170 From Eqs. (10), (13), and (14), one finally gets for the expression of  $k(\Gamma^*; z_0; z_1, x)$ :

$$\begin{aligned}
k(\Gamma^*; z_0; z_1, x) &= k(\Gamma; z_0; z_1, x) + \delta k(\Gamma^*; z_0; z_1, x) \\
&= \frac{\sqrt{2}}{\pi^{3/2}} \frac{\sqrt{x}}{(z_0 - z_1)^2 + x^2} \left[ 1 + \frac{1}{2\pi} \text{PV} \int_{-\infty}^{+\infty} \frac{\delta a(z) - \delta a(z_0)}{(z - z_0)^2} dz \right. \\
&\quad + \frac{2(z_0 - z_1)}{(z_0 - z_1)^2 + x^2} \frac{1}{2\pi} \text{PV} \int_{-\infty}^{+\infty} \left( -\frac{1}{z - z_0} + \frac{(z - z_1)}{(z - z_1)^2 + x^2} \right) \delta a(z) dz \\
&\quad \left. + \frac{(z_0 - z_1)^2 - x^2}{(z_0 - z_1)^2 + x^2} \frac{1}{2\pi} \text{PV} \int_{-\infty}^{+\infty} \frac{\delta a(z) - \delta a(z_1)}{(z - z_1)^2 + x^2} dz + \frac{x}{(z_0 - z_1)^2 + x^2} (\delta a(z_0) - \delta a(z_1)) \right]
\end{aligned} \tag{15}$$

171 We derived here for the first time the analytical expression for the crack face weight function  
172  $k(\Gamma^*; z_0; z_1, x)$  of a perturbed crack at first-order in the perturbation  $\delta a$ . We observe that one re-  
173 trieves, taking the limit  $x \rightarrow 0$  in our Eq. (15), the Eq. (10) of Leblond et al. (2012) that describes  
174 the first-order variations of the fundamental kernel  $Z(\Gamma^*; z; z') = \sqrt{\pi}/8 \lim_{x \rightarrow 0} k(\Gamma^*; z; z', x) / \sqrt{x}$ .  
175 Our equation provides the fundamental ingredients to extend the model of Rice (1985) to cohe-  
176 sive materials. Technical details on the derivation of Eq. (15) are given in Appendix A.

### 177 2.3. First-order variations of the mode I stress intensity factor

178 Using Eq. (15), it is now possible to compute at first order in  $\delta a$  the stress intensity factor  
179  $K_{\text{czm}}$  of Eq. (5) generated by the cohesive stress acting in the wake of the perturbed crack front  
180  $\mathcal{F}^*$ . We perform this in the reduced case where the material is *translationally invariant* in the  
181 propagation direction ( $Ox$ ). In this case, the cohesive stress might be expressed as:

$$\sigma(z, x) = \sigma_c(z) f_w(x/\omega(z)) \tag{16}$$

182 where  $\sigma_c(z)$  and  $\omega(z)$  are the local *strength* and *process zone size* at position  $z$ ,  $x$  is the distance  
183 to the crack tip located at  $(z, \delta a(z))$ , and  $f_w$  is a shape function that relates to the nature of weak-  
184 ening. An example of  $f_w$  is given in the inset of Fig. 1c for the generic linear traction-separation  
185 cohesive law (see Appendix C.4 for more details).

186  
187 The particular choice of cohesive stress evolution in Eq. (16), often referred to as *distance-*  
188 *weakening*, is rather limiting as it does not provide a comprehensive framework to investigate the  
189 influence of spatially distributed material heterogeneities on crack propagation. Yet, it provides  
190 ways to account for the influence of a finite-size dissipation in the crack wake onto the rupture  
191 behavior in a fully analytical manner; see for example (Palmer et al., 1973) for an estimate of the  
192 process zone size in cohesive materials, and (Poliakov et al., 2002) for the displacement/strain  
193 field ahead of a dynamic cohesive rupture tip.

194  
195 A more standard formulation of cohesive zone models is to describe material degradation  
196 with the local crack opening displacement  $\delta$ :

$$\sigma(z) = \sigma_c(z) f_\delta(\delta(z)/D_c(z)) \tag{17}$$

197 where  $D_c(z)$  is the local critical crack opening, above which cohesive stress are negligible, and  
198  $f_\delta$  is a shape function describing material weakening. While we chose here to express directly  
199  $\sigma$  in terms of the variables  $(\sigma_c, \omega, f_w)$ , they usually emerge from the knowledge  $(\sigma_c, D_c, f_\delta)$  and  
200 the resolution of the structural problem. In particular, the process zone size  $\omega$  can be expressed  
201 as:

$$\omega = \alpha \frac{\mu}{\sigma_c} D_c \tag{18}$$



202 where  $\mu$  is the shear modulus, and  $\alpha$  is a proportionality constant that relates to  $f_\delta$ . It can be  
 203 either estimated analytically (Barenblatt, 1962) or computed numerically (Viesca and Garagash,  
 204 2018). The formulation in terms of  $(\sigma_c, \omega, f_w)$  is strictly equivalent to that in  $(\sigma_c, D_c, f_\delta)$  for ma-  
 205 terials that are translationally invariant in the propagation direction ( $Ox$ ). The latter formulation  
 206 would provide a more comprehensive framework to deal with cases where the heterogeneities  
 207 are random.

208  
 209 In the following, perturbations  $\delta a$  of the crack front may arise from the spatial variations of  
 210 strength  $\sigma_c$  and process zone size  $\omega$  along the front, the latter being associated with variations of  
 211 both strength  $\sigma_c$  and critical crack opening  $D_c$ . We decompose  $\sigma_c$  and  $\omega$  in uniform contributions  
 212  $\sigma_c^0$  and  $\omega_0$  associated to a reference homogeneous material, and spatial fluctuations  $\delta\sigma_c$  and  $\delta\omega$ :

$$\begin{cases} \sigma_c(z) = \sigma_c^0 + \delta\sigma_c(z) \\ \omega(z) = \omega_0 + \delta\omega(z) \end{cases} \quad (19)$$

213 where  $\sigma_c^0$  and  $\omega_0$  correspond to the spatial averages of  $\sigma_c$  and  $\omega$  respectively. We can now insert  
 214 Eqs. (6), (15), (16), and (19) into Eq. (5) that gives the cohesive stress intensity factor  $K_{czm}(z)$   
 215 acting along the perturbed crack front  $\mathcal{F}^*$ . It yields:

$$\begin{aligned} K_{czm}(z) &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \sigma(z', x) k^*(\Gamma^*; z; z', x) dz' dx \\ &= \int_0^{+\infty} \int_{-\infty}^{+\infty} \sigma_c(z') f_w(x/\omega(z')) [k(\Gamma; z; z', x) + \delta k(\Gamma^*; z; z', x)] dz' dx \end{aligned} \quad (20)$$

216 The cohesive SIF  $K_{czm}(z)$  can be expressed as the sum of a zero-order term  $K_{czm}^0$ , and first-order  
 217 variations  $\delta K_{czm}(z)$  that relates to the perturbations  $\delta a$ ,  $\delta\sigma_c$  and  $\delta\omega$ . Following Irwin (1958)'s  
 218 criterion,  $K_{czm}^0$  corresponds to the mode I toughness  $K_{Ic}^0$  of the reference material when the crack  
 219 propagates. We show in Appendix B that it writes as:

$$K_{czm}^0 = K_{Ic}^0 = C_w \sqrt{\frac{2}{\pi}} \sigma_c^0 \omega_0^{1/2} \quad (21)$$

220 where  $C_w = \int_0^{+\infty} f_w(u) u^{-1/2} du$  is a pre-factor that relates to the nature of the weakening. We re-  
 221 trieve here the results of Palmer et al. (1973) that derived the expression of the mode I toughness  
 222  $K_{Ic}^0$  of a cohesive crack in 2D. This was expected, as three-dimensional crack propagation in a  
 223 spatially homogeneous reference material can be reduced to a two-dimensional problem.

224

The first-order variations of cohesive stress intensity factor  $\delta K_{czm}(z)$  can be expressed as:

$$\begin{aligned} \frac{\delta K_{czm}}{K_{Ic}^0}(z) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ -\frac{|k|\omega_0}{2} + \frac{1}{C_w} \int_0^{+\infty} -\frac{f'_w(u)}{u^{1/2}} (1 - e^{-|k|\omega_0 u}) du \right] \frac{\widehat{\delta a}(k)}{\omega_0} e^{ikz} dk \\ &+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ \frac{1}{C_w} \int_0^{+\infty} \frac{f_w(u)}{u^{1/2}} e^{-|k|\omega_0 u} du \right] \frac{\widehat{\delta\sigma_c}(k)}{\sigma_c^0} e^{ikz} dk \\ &+ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ \frac{1}{C_w} \int_0^{+\infty} -f'_w(u) u^{1/2} e^{-|k|\omega_0 u} du \right] \frac{\widehat{\delta\omega}(k)}{\omega_0} e^{ikz} dk \end{aligned} \quad (22)$$

225 Details on the derivation of Eq. (22) are given in Appendix B. We further observe that Eq. (22)  
 226 takes a much simpler expression in the Fourier space that one can easily build upon to understand  
 227 the physical implications of a finite-size dissipation on the fracture process. Eq. (22) reads in the  
 228 Fourier space:

$$\frac{\widehat{\delta K_{\text{czm}}}(k)}{K_{\text{Ic}}^0} = \left( \frac{\hat{\mathcal{A}}(|k|\omega_0)}{\omega_0} - \frac{|k|}{2} \right) \widehat{\delta a}(k) + \hat{\Sigma}(|k|\omega_0) \frac{\widehat{\delta \sigma_c}(k)}{\sigma_c^0} + \hat{\Omega}(|k|\omega_0) \frac{\widehat{\delta \omega}(k)}{2\omega_0} \quad (23)$$

229 where:

$$\begin{cases} \hat{\mathcal{A}}(|k|\omega_0) &= -\frac{1}{C_w} \int_0^{+\infty} \frac{f'_w(u)}{u^{1/2}} (1 - e^{-|k|\omega_0 u}) du \\ \hat{\Sigma}(|k|\omega_0) &= \frac{1}{C_w} \int_0^{+\infty} \frac{f_w(u)}{u^{1/2}} e^{-|k|\omega_0 u} du \\ \hat{\Omega}(|k|\omega_0) &= -\frac{2}{C_w} \int_0^{+\infty} f'_w(u) u^{1/2} e^{-|k|\omega_0 u} du \end{cases} \quad (24)$$

230 Combining Eqs. (3) and (23), one finds the equation ruling crack propagation in heterogeneous  
 231 cohesive materials:

$$K_{\text{I}}^0 \left[ 1 + \left( \frac{1}{K_{\text{I}}^0} \frac{\partial K_{\text{I}}^0}{\partial a} - \frac{|k|}{2} \right) \widehat{\delta a}(k) \right] = K_{\text{Ic}}^0 \left[ 1 + \left( \frac{\hat{\mathcal{A}}(|k|\omega_0)}{\omega_0} - \frac{|k|}{2} \right) \widehat{\delta a}(k) + \hat{\Sigma}(|k|\omega_0) \frac{\widehat{\delta \sigma_c}(k)}{\sigma_c^0} + \hat{\Omega}(|k|\omega_0) \frac{\widehat{\delta \omega}(k)}{2\omega_0} \right] \quad (25)$$

232 Equation (25) unveils rich physics about the influence of a finite process zone size on the  
 233 front deformations. It can be reformulated in terms of variations of strength and critical crack  
 234 opening, building on Eq. (18) (see Eq. (B.18) in Appendix B for more details). One can also  
 235 make use of efficient Fast Fourier Transform (FFT) algorithms to solve it efficiently, at a much  
 236 lower computational expense than more standard simulation methods (Geubelle and Rice, 1995).  
 237 As such, our model shows potential to investigate front deformations induced by heterogeneities  
 238 at multiple scales. We focus here on two problems: the configurational stability of a crack  
 239 propagating in a homogeneous yet cohesive material in Section 3, and the influence of material  
 240 heterogeneities on the front deformation in Section 4. Before doing so, several comments are in  
 241 order:

- 242 • First, one expects to find back the results of Rice (1985) and Gao and Rice (1989) in the  
 243 limit  $\omega_0 \rightarrow 0$ . The study of the asymptotic behavior of the cohesive pre-factors  $\mathcal{A}$ ,  $\Sigma$ , and  
 244  $\Omega$  yields in this limit:

$$\hat{\mathcal{A}}(|k|\omega_0) \underset{|k|\omega_0 \rightarrow 0}{\sim} \frac{|k|\omega_0}{2}; \quad \hat{\Sigma}(|k|\omega_0) \underset{|k|\omega_0 \rightarrow 0}{\longrightarrow} 1 \quad \text{and} \quad \hat{\Omega}(|k|\omega_0) \underset{|k|\omega_0 \rightarrow 0}{\longrightarrow} 1 \quad (26)$$

245 So that, in the limit of a perfectly brittle material, the contribution of the front deformations  
 246  $\delta a$  in  $K_{\text{czm}}$  goes to zero. Eq. (4) yields at first-order in  $\delta a$ ,  $\delta \sigma_c$ , and  $\delta \omega$ :

$$K_{\text{Iefm}}(z) = K_{\text{Ic}}^0 \left[ 1 + \frac{\delta \sigma_c(z)}{\sigma_c^0} + \frac{\delta \omega(z)}{2\omega_0} \right] = K_{\text{Ic}}(z) \quad (27)$$

247 where we used Eq. (21) to link the fluctuations of cohesive properties  $\delta \sigma_c$  and  $\delta \omega$  to the  
 248 material toughness  $K_{\text{Ic}}$ . We find back Irwin (1958)'s criterion that describes crack propa-  
 249 gation in perfectly brittle materials.

- Second, the amplitude of the front deformations  $\delta$  are multiplied by a cohesive pre-factor  $\hat{\mathcal{A}}$  that depends on the product of the wavenumber  $k$  and the process zone size  $\omega_0$ . It means that the crack front accommodates a given perturbation length scale differently depending on its size and that of the process zone.
- Third, we observe from Eq. (23) that the process zone acts as a high-frequency filter for the variations of strength  $\delta\sigma_c$  and that of process zone size  $\delta\omega$ . One may thus expect that the influence of heterogeneities on rupture propagation can be averaged at scales below the average process zone size  $\omega_0$ , while the influence of asperities at a scale considerably larger than  $\omega_0$  can be assessed quantitatively within the perturbative framework of LEFM.
- Fourth, the cohesive pre-factors  $\hat{\mathcal{A}}$ ,  $\hat{\Sigma}$ , and  $\hat{\Omega}$  relate to the spatial distribution of weakening characterized by  $f_w$ . As such, distinct behaviors can be expected depending on how the material weakens. The values of  $\hat{\mathcal{A}}$ ,  $\hat{\Sigma}$ , and  $\hat{\Omega}$  and their asymptotic behavior are either given analytically or computed numerically in Appendix C for several types of weakening.

### 3. Stability analysis

As a first application of the newly derived cohesive “line-tension” model of Section 2, we revisit the stability analysis of Rice (1985). It allows us to focus on the impact of a finite-size dissipation on the front deformations only, considering *homogeneous* yet *cohesive* materials.

#### 3.1. Crack front stability to sinusoidal perturbations for perfectly brittle materials

The stability of a semi-infinite crack to some front perturbations has been investigated first by Rice (1985) for perfectly brittle materials. They considered sinusoidal perturbations  $\delta a$  that are characterized by their wavenumber  $k$ , or equivalently by their wavelength  $\lambda = 2\pi/k$ , and their amplitude  $A$  (see Fig. 3a):

$$\delta a(z) = A \cos(kz) \quad (28)$$

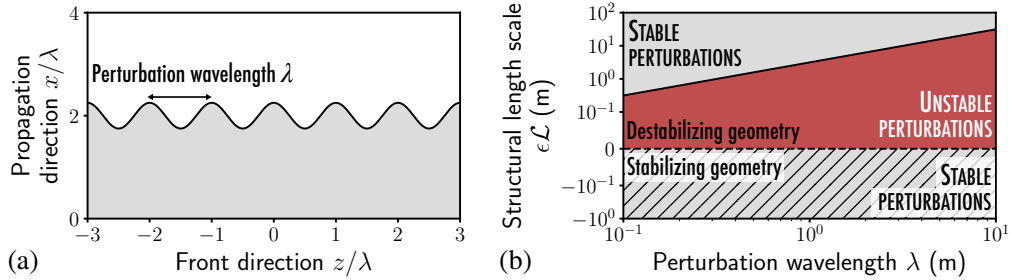


Figure 3: (a) Stability analysis of a sinusoidal crack front (in black solid line) of wavelength  $\lambda$  for perfectly brittle materials; (b) the stability of the crack front is controlled by the sign  $\epsilon$  and the characteristic distance  $\mathcal{L}$  along which the macroscopic  $K_I^0$  varies. For stabilizing geometries  $\epsilon \leq 0$  (hatched surface), the perturbation is stable (in gray) no matter its wavelength. For destabilizing geometries  $\epsilon > 0$  (non-hatched surface), the perturbation is unstable (in red) if its wavelength is larger than a critical value  $\lambda_c$ , and stable (in gray) otherwise. The critical wavelength  $\lambda_c$  is predicted by Eq. (31).

Following Rice (1985), the stability of the perturbed crack is assessed by looking at the value of mode I SIF at the most advanced points of the crack front  $z = n\pi/\lambda$ . If this value

274 is found larger than the applied SIF  $K_1^0$ , crack propagation is considered unstable, and stable  
 275 otherwise. Rice (1985) showed that the configurational stability is partially controlled by (i) the  
 276 sign  $\epsilon$  of the variations  $\partial K_1^0/\partial a$  of loading with crack advance, and (ii) a structural length scale  
 277  $\mathcal{L}$  characterizing their intensity. These two quantities are defined by the following equations:

$$\mathcal{L} = |K_1^0/(\partial K_1^0/\partial a)| \text{ and } \epsilon = \text{sgn}(\partial K_1^0/\partial a) \quad (29)$$

278 The expression of the mode I SIF of Eq. (1) rewrites as:

$$K_1(z) = K_1^0 \left[ 1 + \left( \frac{1}{K_1^0} \frac{\partial K_1^0}{\partial a} - \frac{\pi}{\lambda} \right) A \cos(kz) \right] = K_1^0 \left[ 1 + \left( \frac{\epsilon}{\mathcal{L}} - \frac{\pi}{\lambda} \right) A \cos(kz) \right] \quad (30)$$

279 From Eq. (30), one observes that the crack is *unconditionally stable* for stabilizing geometries  
 280  $\epsilon \leq 0$  for which  $K_1^0$  decreases with crack advance. When  $K_1^0$  increases with crack growth  $\epsilon > 0$   
 281 (destabilizing geometry), the system is *conditionally stable*: small-wavelength perturbations are  
 282 stable, while large wavelengths are unstable. This shift in stability occurs for a critical wave-  
 283 length  $\lambda_c$  that reads:

$$\lambda_c = [\epsilon\pi\mathcal{L}]^+ \quad (31)$$

284 where  $[x]^+$  denotes the positive part of the real  $x$ .

### 285 3.2. Influence of a finite-size dissipation

286 One may then wonder how the presence of a finite-size dissipation influences the stability  
 287 of a crack to sinusoidal perturbations. For homogeneous materials ( $\Delta\sigma_c = \Delta\omega = 0$ ), Eq. (25)  
 288 rewrites as an Irwin-like criterion  $K_1^{\text{eff}} = K_{1c}^0$ , with an effective SIF  $K_1^{\text{eff}}$  that encompasses the  
 289 influence of finite cohesive stresses behind the crack front.  $K_1^{\text{eff}}$  writes as:

$$K_1^{\text{eff}}(z) = K_1^0 \left[ 1 + \left( \frac{\epsilon}{\mathcal{L}} - \frac{\hat{\mathcal{A}}(|k|\omega_0)}{\omega_0} \right) A \cos(kz) \right] \quad (32)$$

290 In the remaining of the manuscript, we assume that the material continuously weakens behind  
 291 the crack front  $f'_w \leq 0$ . From the definition of  $\hat{\mathcal{A}}$  in Eq. (24), one may easily show that  $\hat{\mathcal{A}}$  is  
 292 a (i) *positive* and (ii) *increasing* function of  $|k|\omega_0$ . Moreover, in the limit  $|k|\omega_0 \rightarrow +\infty$ , (iii)  $\hat{\mathcal{A}}$   
 293 *saturates* to a value  $\mathcal{A}_w^\infty$  (see Fig. C.10.a) that reads:

$$\mathcal{A}_w^\infty = \frac{1}{C_w} \int_0^{+\infty} -\frac{f'_w(u)}{u^{1/2}} du \quad (33)$$

294 One then finds:

$$K_1^{\text{eff}}(z) \geq K_1^0 \left[ 1 + \left( \frac{\epsilon}{\mathcal{L}} - \frac{\mathcal{A}_w^\infty}{\omega_0} \right) A \cos(kz) \right] \quad (34)$$

295 From Eqs. (32) and (34), one can distinguish three different instability regimes:

- 296 • **Regime I:** When  $\epsilon = -1$  or  $0$ , the system remains *unconditionally stable* to sinusoidal  
 297 perturbations due to the positiveness of  $\hat{\mathcal{A}}$ .
- 298 • **Regime II:** When  $\epsilon = 1$ , the system is *unconditionally unstable* to sinusoidal perturbations  
 299 of any wavelength if the structural length scale  $\mathcal{L}$  is smaller than a critical value  $\mathcal{L}_c$ , which  
 300 reads:

$$\mathcal{L}_c = \frac{\omega_0}{\mathcal{A}_w^\infty} \quad (35)$$

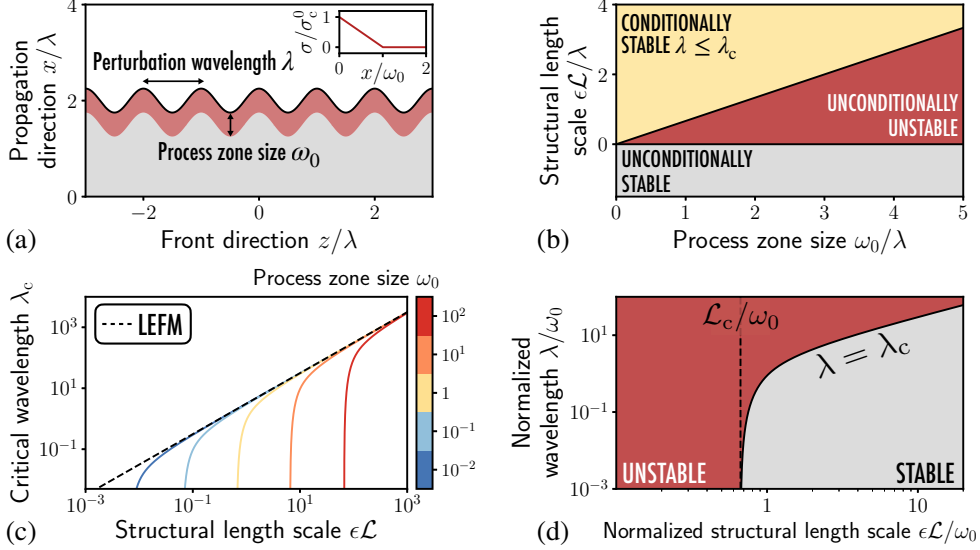


Figure 4: (a) Stability analysis of a sinusoidal crack front (in black solid line) of wavelength  $\lambda$  in presence of a finite process zone size  $\omega_0$  (in red); (b) the stability of the crack front is controlled by the structural length scale  $\mathcal{L} = K_I^0/|\partial K_I^0/\partial a|$  characteristic of the loading variations and their sign  $\epsilon$ . For stabilizing geometries  $\epsilon \leq 0$ , the perturbation is stable no matter its wavelength (in gray; see Regime I in the main text). For destabilizing geometries  $\epsilon > 0$ , the perturbation is stable if its wavelength  $\lambda$  is smaller than a critical value  $\lambda_c$  and unstable otherwise (in yellow; see Regime III in the main text). For a large enough process zone size, the critical wavelength  $\lambda_c$  goes to 0 (in red; see Regime II in the main text), and perturbations of any wavelength are unstable. (c)  $\lambda_c$  depends on both the structural length  $\mathcal{L}$  and the process zone size  $\omega_0$ . (d) When normalized by  $\omega_0$  the values of  $\lambda_c$  all collapse to a master curve (in black solid line) predicted by Eq. (36). Note that, below a critical value  $\mathcal{L}_c/\omega_0$  of the structural length scale (marked by a vertical dashed black line), perturbations of any wavelength  $\lambda$  are unstable (Regime II).

- **Regime III:** When  $\epsilon = 1$  and  $\mathcal{L} \geq \mathcal{L}_c$ , the system is *conditionally stable*: perturbations of wavelength smaller than a critical value  $\lambda_c$  are stable, while larger ones are unstable. For cohesive materials,  $\lambda_c$  reads:

$$\hat{\mathcal{A}}\left(\frac{2\pi\omega_0}{\lambda_c}\right) = \frac{\omega_0}{\mathcal{L}} \quad (36)$$

In contrast with the perfectly brittle case of Eq. (31) where  $\lambda_c$  only depends on the structural length scale  $\mathcal{L}$ , it is also influenced here by the process zone size  $\omega_0$ . We further observe in Fig. 4d that  $\lambda_c$  is smaller than its LEFM value, as  $\hat{\mathcal{A}}(|k|\omega_0)/\omega_0 < |k|/2$ . As such, the introduction of a finite-size region of dissipation can be associated with a *loss of stiffness* of the crack front. This stiffness loss gets stronger as the wavelength  $\lambda$  approaches the process zone size  $\omega_0$  (see Fig. 4d and Fig. C.10 in Appendix C).

The different regimes and the associated stability diagram are summarized in Fig. 4b-c. As can be seen from the conditions of Eqs. (35) and (36), the stability of a perturbed crack strongly depends on the evolution of  $\hat{\mathcal{A}}$  and its asymptotic behavior as  $|k|\omega_0 \rightarrow +\infty$ . It naturally relates to the spatial distributions of cohesive stress behind the crack front, and so to the function  $f_w$ . The influence of the precise shape of the cohesive law is discussed in Appendix E.1.

316 We conclude this section with some potential implications of our results on the stability of  
317 perturbed cracks in a *dynamic* setting. We showed that the stability of a perturbed crack front  
318 of fixed wavelength  $\lambda$  embedded in a destabilizing structure ( $\epsilon > 0$ ) is controlled by the size  
319  $\omega_0$  of the process zone. A crack in such a configuration could be stable when embedded in a  
320 rather brittle material (small  $\omega_0/\lambda$ ) and unstable for a more ductile one (large  $\omega_0/\lambda$ ). Rice (1980)  
321 showed that the process zone size  $\omega_0$  *contracts* as the crack accelerates. One may then imagine  
322 situations where a crack oscillates between a *stable* configuration and an *unstable* one: in a  
323 first stage, the arrested crack becomes unstable because of some large wavelength perturbations  
324 (Regime III). The subsequent decrease in process zone size during the instability could stabilize  
325 the system (Regime III to Regime II). As the crack decelerates, the process zone gets larger,  
326 and the crack becomes unstable again (Regime II to Regime III). The Lorentz contraction of the  
327 process zone with crack velocity may also interact with the dynamic stiffening of the crack front  
328 and crack front waves observed by Morrissey and Rice (2000). Further investigations are needed  
329 to support these ideas. One could build on the spectral method of Geubelle and Rice (1995) to  
330 perform efficient numerical simulations of front stability during quasi-static - to - dynamic and  
331 dynamic - to - quasi-static transients.

#### 332 4. Deformations of a crack front encountering an obstacle

333 We saw that the presence of finite cohesive stress in the crack wake makes the front *more*  
334 *compliant*, especially when the perturbation wavelength is smaller than the process zone size. It  
335 is tempting to think that this stiffness loss consequently increases the deformations of a crack  
336 front induced by some heterogeneities of fracture energy. Yet, one has to bear in mind that the  
337 presence of a finite cohesive zone size also influences the variations of strength and process zone  
338 size, from which emerge the fluctuations of fracture energy. To investigate the ultimate impact  
339 of these two potentially competing mechanisms, we study the interaction of a crack front with  
340 periodic arrays of tough inclusions of increased fracture energy. We recall in Section 4.1 the  
341 results obtained by Gao and Rice (1989) for perfectly brittle materials. We then investigate the  
342 influence of a finite process zone size, first in Section 4.2 where heterogeneities of fracture energy  
343  $G_c$  are associated with fluctuations of strength  $\sigma_c$ , and second in Section 4.3 where they emerge  
344 from variations of process zone size  $\omega$ . We discuss in Section 4.4 the experimental implications  
345 of our findings, and show in Section 4.5 that our theory may explain the deformation profiles  
346 observed in the peeling experiments of Chopin et al. (2011), for which the small-scale yielding  
347 hypothesis is suspected to break down.

##### 348 4.1. Front deformation by a periodic array of obstacles in perfectly brittle materials

349 In the general case, the spatial distribution of fracture energy  $G_c$  can be decomposed in the  
350 sum of a zero order term  $G_c^0$  corresponding to its spatial average, and a first-order term of fluctu-  
351 ations  $\delta G_c$ :

$$G_c(z) = G_c^0 + \delta G_c(z) \quad (37)$$

352 As the crack interacts with the inclusions, front deformations  $\delta a$  arise from the spatial variations  
353  $\delta G_c$  of fracture energy. Building on Griffith (1921)'s criterion and Irwin (1958)'s formula, Eq. (3)  
354 yields at zero order in the perturbation  $\delta a$ :

$$G^0 = \frac{1 - \nu^2}{E} K_I^{02} = G_c^0 \quad (38)$$

355 We find back here the results of Roux et al. (2003), who showed that the effective fracture energy  
 356  $G^0$  of a heterogeneous brittle material corresponds to its spatial average  $G_c^0$  in the *weak pinning*  
 357 limit of infinitely long defects. At first order, one finds:

$$\widehat{\delta a}(k) = -\frac{1}{|k|} \frac{\widehat{\delta G_c}(k)}{G_c^0} \quad (39)$$

358 Eq. (39) links the stationary shape  $\delta a$  of the front deformations to the local fluctuations of  
 359 fracture energy  $\delta G_c$ . Gao and Rice (1989) investigated the deformation of the crack front re-  
 360 sulting from its interaction with a periodic array of tough obstacles of width  $d$  separated by a  
 361 distance  $\Delta L$  (see Fig. 5). The spatial distribution of fracture energy writes as:

$$G_c(z) = \begin{cases} G_c^{\text{obs}} & \text{if } x \in \left[ nL_z - \frac{d}{2}, (n+1)L_z + \frac{d}{2} \right) \\ G_c^{\text{mat}} & \text{otherwise} \end{cases} \quad (40)$$

362 where the fracture energy  $G_c^{\text{obs}}$  of the obstacles is larger than that  $G_c^{\text{mat}}$  of the embedding matrix,  
 363 and  $L_z = \Delta L + d$  corresponds to the spatial period of the distribution. One may additionally  
 364 characterize the fracture energy field through the two quantities:

$$\begin{cases} G_c^0 = (\Delta L/L_z) G_c^{\text{mat}} + (d/L_z) G_c^{\text{obs}} \\ \Delta G_c = G_c^{\text{obs}} - G_c^{\text{mat}} \end{cases} \quad (41)$$

365 where  $\Delta G_c$ , the fracture energy contrast, is a parameter that is assumed small in the following.

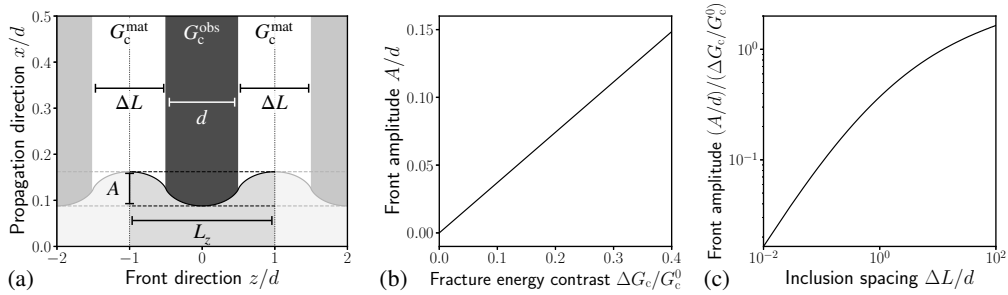


Figure 5: (a) The crack front (in solid black line) is deformed by a periodic array of tough obstacles (in gray) of width  $d$ , separated by a distance  $\Delta L$ . The obstacles have a fracture energy  $G_c^{\text{obs}}$  slightly larger than that of the matrix  $G_c^{\text{mat}}$ . The front deformations are characterized by their amplitude  $A$ , defined as the distance between the most advanced point of the front and the less advanced one. (b) For a *perfectly brittle* material,  $A$  increases *linearly* with the fracture energy contrast  $\Delta G_c/G_c^0$ . (c)  $A$  depends non-linearly on the ratio of the obstacle spacing  $\Delta L$  to their size  $d$ .

366 A quantity of interest is the amplitude  $A$  of the front deformations, defined as the distance  
 367 between the most advanced point of the front and the less advanced one (see Fig. 5a). From  
 368 Eq. (39), one may notice that  $A$  increases linearly with  $\Delta G_c/G_c^0$  (see Fig. 5b). The dependence of  
 369  $A$  with the inclusion spacing  $\Delta L$  is non-linear, and can be computed numerically from Eq. (39)  
 370 for spatial distributions of  $G_c$  described by Eq. (40) (see Fig. 5c).

371 One may then wonder how these observations fare in the framework of cohesive materials,  
 372 where the fracture properties are not solely described by the fracture energy  $G_c$ , but rather by the

373 strength  $\sigma_c$  and the process zone size  $\omega$ . Using Irwin (1958)'s formula combined with Eq. (21),  
 374 one finds:

$$G_c^0 = \frac{1 - \nu^2}{E} K_{Ic}^{0,2} = \frac{1 - \nu^2}{E} \frac{2}{\pi} C_w^2 \sigma_c^{0,2} \omega_0 \quad (42)$$

375 As such, the variations of fracture energy  $\delta G_c$  in cohesive materials can either emerge from the  
 376 variations of strength  $\delta \sigma_c$ , or from that of process zone size  $\delta \omega$ . At first order in those two  
 377 quantities, one has:

$$\frac{\delta G_c(z)}{G_c^0} = 2 \frac{\delta \sigma_c(z)}{\sigma_c^0} + \frac{\delta \omega(z)}{\omega_0} \quad (43)$$

378 Next, we explore two limit cases: that of Section 4.2 where heterogeneities of fracture energy  
 379 solely emerge from local variations of strength, and that of Section 4.3 where they are associated  
 380 with fluctuations of process zone size only.

#### 381 4.2. Heterogeneities of strength

382 We first consider the case where the heterogeneities of fracture energy emerge from variations  
 383 of strength only. The strength of the obstacles  $\sigma_c^{\text{obs}}$  is larger than that of the matrix  $\sigma_c^{\text{mat}}$ , while  
 384 they are equally brittle  $\omega^{\text{obs}} = \omega^{\text{mat}} = \omega_0$ . Note that in the standard formulation of cohesive-zone  
 385 models that deal with the pair  $(\sigma_c, D_c)$  instead of  $(\sigma_c, \omega)$ , this corresponds to a simultaneous  
 386 proportional increase of strength  $\sigma_c$  and critical opening  $D_c$ .

387 We define the average strength  $\sigma_c^0$  and the strength contrast  $\Delta \sigma_c$  as:

$$\begin{cases} \sigma_c^0 = (\Delta L/L_z) \sigma_c^{\text{mat}} + (d/L_z) \sigma_c^{\text{obs}} \\ \Delta \sigma_c = \sigma_c^{\text{obs}} - \sigma_c^{\text{mat}} \end{cases} \quad (44)$$

388 The resulting fluctuations of fracture energy are linked to the variations of strength following:

$$\Delta G_c / G_c^0 = 2 \Delta \sigma_c / \sigma_c^0 \quad (45)$$

389 The propagation criterion of (25) yields at first order in the perturbations  $\delta a$  and  $\delta \sigma_c$ :

$$-\hat{\mathcal{A}}(|k|\omega_0) \frac{\widehat{\delta a}(k)}{\omega_0} = \hat{\Sigma}(|k|\omega_0) \frac{\widehat{\delta \sigma_c}(k)}{\sigma_c^0} = \frac{\delta G_c^{\text{eff}}(k)}{2G_c^0} \quad (46)$$

390 meaning that the front deformation increases linearly with the normalized strength contrast  
 391  $\Delta \sigma_c / \sigma_c^0$ , and so with the normalized fracture energy contrast  $\Delta G_c / G_c^0$ .

392  
 393 The presence of a cohesive stress in the crack wake influences the amplitude of the front  
 394 deformations through two *competing* mechanisms. We saw in Section 3 that (i) it *decreases the*  
 395 *front stiffness* through the cohesive pre-factor  $\hat{\mathcal{A}}$ . We now observe from Eq. 46 that (ii) it also  
 396 *smooths out the fluctuations of strength*  $\delta \sigma_c$  ( $\hat{\Sigma}(|k|\omega_0) \leq 1$ ), leaving its average value  $\sigma_c^0$   
 397 unchanged. The overall decrease in magnitude of the strength fluctuations is mostly prevalent at  
 398 scales lower than the average process zone size  $\omega_0$  (see the evolution of  $\hat{\Sigma}$  in Fig. C.10b). While  
 399 the first effect leads to an increase of the front deformations with respect to the perfectly brittle  
 400 case, the second effect should decrease it as the crack experiences an effectively smoother dis-  
 401 tribution  $\delta G_c^{\text{eff}}$  of fracture energy. Examples of the effective fluctuations of fracture energy  $\delta G_c^{\text{eff}}$   
 402 are shown in Fig. 6c for several values of process zone size to obstacle width ratio  $\omega_0/d$ .

403



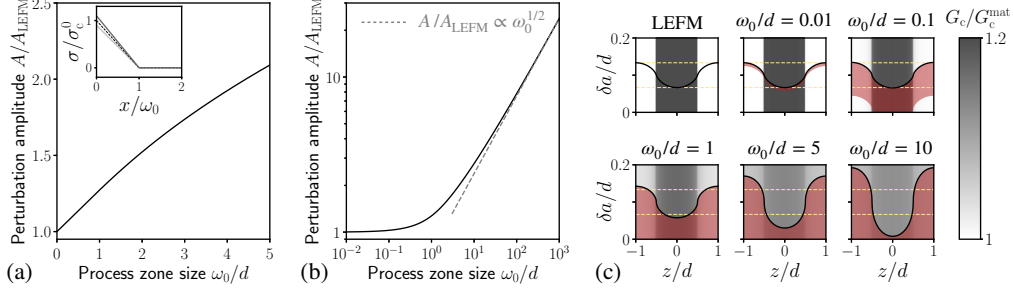


Figure 6: (a) Influence of the average process zone size  $\omega_0$  on the amplitude  $A$  of the front deformations for a periodic array of obstacles of width  $d$ ; Inset: the obstacles (in solid dark gray line) are stronger than the matrix  $\sigma_c^{\text{obs}} = 1.1\sigma_c^{\text{mat}}$  (in solid light gray line), but they are equally brittle  $\omega_c^{\text{obs}} = \omega_c^{\text{mat}}$ , so that  $G_c^{\text{obs}} \simeq 1.2G_c^{\text{mat}}$ . For moderate values of  $\omega_0/d$ , the amplitude  $A$  increases almost linearly with the process zone size  $\omega_0$  from its reference LEFM value. (b) For larger values of  $\omega_0/d$ , it scales as  $\propto (\omega_0/d)^{1/2}$ . (c) The presence of a finite process zone size smooths out the small-scale variations of strength, so that the amplitude of the apparent fracture energy field (in gray scale) is decreased from its material value (LEFM case). Yet, the decrease in apparent fracture energy amplitude for increasing process zone size (in red) is weaker than the associated decrease in front stiffness, so that the deformation amplitude (extremal points of the crack front in solid black line) is ultimately larger than in the LEFM case (in yellow dashed line).

404 We show in Fig. 6a the evolution of the front deformation amplitude  $A$  with the process zone  
 405 size  $\omega_0$ . The competition between the two opposite effects mentioned above results in an overall  
 406 *increase* of the amplitude of the front deformations with the process zone size. This increase in  
 407 front amplitude is observed no matter the inclusion spacing (see Fig. D.14 and Appendix D.1  
 408 for more details).

409 In accordance with the asymptotic behaviors derived in Eq. (26),  $A$  converges to its LEFM  
 410 value  $A_{\text{LEFM}}$  as the ratio between the process zone size  $\omega_0$  and the obstacle width  $d$  goes to zero  
 411 (see Fig. 6a). We further notice that the amplitude of the deformations grows as  $(\omega_0/d)^{1/2}$   
 412 for process zone sizes much larger than the obstacle width (see Fig. 6b). This was expected from the  
 413 asymptotic behavior of  $\hat{\Sigma}$  when  $|k|\omega_0 \rightarrow +\infty$ :

$$\hat{\Sigma}(|k|\omega_0) \underset{|k|\omega_0 \rightarrow +\infty}{\sim} \frac{\Sigma_w^\infty}{(|k|\omega_0)^{1/2}}, \text{ with } \Sigma_w^\infty = \frac{\sqrt{\pi}f_w(0)}{C_w} \quad (47)$$

414 The front deformations are thus amplified by the existence of cohesive stresses, as the gain in  
 415 front compliance overcomes the smoothing of heterogeneities by the process zone. This behavior  
 416 and the associated scaling are retrieved independently of how the material weakens behind the  
 417 crack front (see Fig. E.18 and Appendix E.2 for more details).

418 A large process zone size lead to a localization of the deformation at the edges of the defect  
 419 (see Fig. 6c, and Eq. (D.2) for an analytical expression of the front deformations in the limit  
 420 case of a single defect embedded in an infinite matrix). Similar deformation patterns have been  
 421 previously observed in peeling experiments of a silicon elastomer block from a heterogeneous  
 422 glass substrate (Chopin et al., 2011). They are not ubiquitous, as other peeling experiments lead  
 423 to deformations closer to the LEFM predictions (Patinet et al., 2013a; Vasoya et al., 2016b). Our  
 424 theory may then provide a comprehensive framework to bridge various experimental observa-  
 425 tions where the small-scale yielding condition of LEFM may not always be met. We explore  
 426 further this avenue in the final Section 4.5.

427 **4.3. Heterogeneities of process zone size**

428 We explore next the situation where the heterogeneities of fracture energy solely emerge  
 429 from variations of process zone size. The process zone size of the obstacles  $\omega^{\text{obs}}$  is now larger  
 430 than that of the matrix  $\omega^{\text{mat}}$ , but both materials are equally strong  $\sigma_c^{\text{obs}} = \sigma_c^{\text{mat}} = \sigma_c^0$ . In the  
 431 standard formulation of cohesive-zone models that deal with the pair  $(\sigma_c, D_c)$  instead of  $(\sigma_c, \omega)$ ,  
 432 this corresponds to an increase of critical opening  $D_c$  only.

433 We define the average process zone size  $\omega^0$  and the size contrast  $\Delta\omega$  as:

$$\begin{cases} \omega_0 = (\Delta L/L_z) \omega^{\text{mat}} + (d/L_z) \omega^{\text{obs}} \\ \Delta\omega = \omega^{\text{obs}} - \omega^{\text{mat}} \end{cases} \quad (48)$$

434 The resulting fluctuations of fracture energy are linked to the variations of process zone size  
 435 following:

$$\Delta G_c/G_c^0 = \Delta\omega/\omega_0 \quad (49)$$

436 The propagation criterion of (25) yields at first order in the perturbations  $\delta a$  and  $\delta\omega$ :

$$-\hat{\mathcal{A}}(|k|\omega_0) \frac{\widehat{\delta a}(k)}{\omega_0} = \hat{\Omega}(|k|\omega_0) \frac{\widehat{\delta\omega}(k)}{2\omega_0} = \frac{\delta G_c^{\text{eff}}(k)}{2G_c^0} \quad (50)$$

437 meaning that the front deformations increase linearly with the normalized process zone size con-  
 438 trast  $\Delta\omega/\omega_0$ , and so with the normalized fracture energy contrast  $\Delta G_c/G_c^0$ .

439

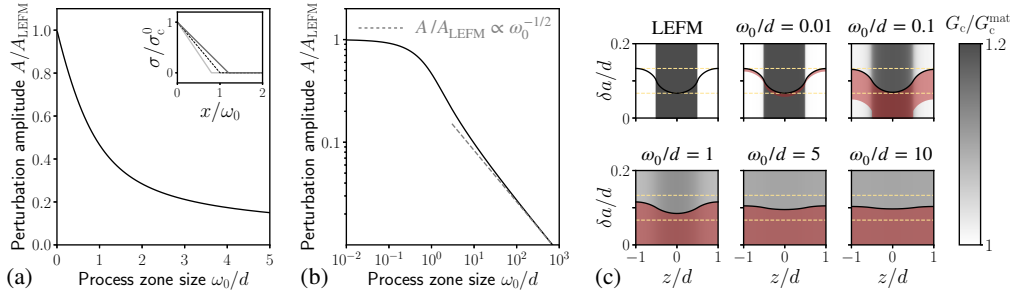


Figure 7: (a) Influence of the average process zone size  $\omega_0$  on the amplitude  $A$  of the front deformations for a periodic array of obstacles of width  $d$ ; Inset: the obstacles (in solid dark gray line) are more ductile than the matrix  $\omega^{\text{obs}} = 1.2\omega^{\text{mat}}$  (in solid light gray line), but they are equally strong  $\sigma_c^{\text{obs}} = \sigma_c^{\text{mat}}$ , so that  $G_c^{\text{obs}} \simeq 1.2G_c^{\text{mat}}$ . For moderate values of  $\omega_0/d$ , the deformation amplitude  $A$  decreases almost linearly with the process zone size  $\omega_0$  from its reference LEFM value. (b) For larger values of  $\omega_0/d$ , it scales as  $\propto (\omega_0/d)^{-1/2}$ . (c) The presence of a finite process zone size smooths out the small-scale variations of process zone size, so that the apparent amplitude of the fracture energy field (in gray scale) is decreased from its material value (LEFM case). Furthermore, the decrease in apparent fracture energy amplitude for increasing process zone size (in red) is stronger than the associated decrease in front stiffness, so that the deformation amplitude (extremal points of the crack front in solid black line) is ultimately smaller than in the LEFM case (in yellow dashed line).

440 Again, the amplitude  $A$  of the front deformations and its evolution with the average process  
 441 zone size  $\omega_0$  result from the competition between an increased compliance of the crack front,  
 442 which promotes front deformations, and the decrease in magnitude of the effective fracture en-  
 443 ergy fluctuations  $\delta G_c^{\text{eff}}$ , which smooths out the front. Surprisingly, we observe in Fig. 7a that

444 the amplitude  $A$  decreases with the average process zone size  $\omega_0$  for heterogeneities of pro-  
 445 cess zone size. This is explained by the asymptotic behavior of the cohesive pre-factor  $\hat{\Omega}$  when  
 446  $|k|\omega_0 \rightarrow +\infty$ :

$$\hat{\Omega}(|k|\omega_0) \underset{|k|\omega_0 \rightarrow +\infty}{\sim} \frac{\Omega_w^\infty}{(|k|\omega_0)^{3/2}}, \text{ with } \Omega_w^\infty = -\frac{\sqrt{\pi}f'_w(0)}{C_w} \quad (51)$$

447 In that case, the increase in front compliance with the average process zone size cannot com-  
 448 pensate for the sharp decrease in magnitude of the process zone size fluctuations. It results in  
 449 front deformations vanishing as  $(\omega_0/d)^{-1/2}$  for average process zone sizes larger than the obstacle  
 450 width (see Fig. 7b). This decrease in front amplitude is observed no matter the inclusion spacing  
 451 (see Fig. D.15 and Appendix D.1 for more details). However, the associated scaling strongly  
 452 depends on how the material weakens behind the crack front (see Fig. E.19). This is explored in  
 453 more details in Appendix E.2.

454  
 455 The deformation patterns showed in Fig. 7 do not correspond to front configurations observed  
 456 experimentally. This is expected as real materials usually display variations of both strength and  
 457 process zone size, and the influence of *amplified* spatial variations of strength should often prevail  
 458 over that *vanishing* of process zone size in the limit  $\omega_0/d \ll 1$  of large process zone size.

459 Overall, our model shed light on the strong influence of the nature of heterogeneities on  
 460 the front deformations. When extended to a dynamic setting, it helps to rationalize the front  
 461 deformations measured in numerical simulations of dynamic rupture where a crack interacts with  
 462 heterogeneities of cohesive properties (Roch et al., 2022). It may also explain changes in front  
 463 roughness observed in quasi-static simulations of coplanar crack propagation based on cohesive  
 464 zone models (Sevillano et al., 2007). In accordance with our findings, the authors show that  
 465 crack fronts interacting with heterogeneities of strength get much rougher than those interacting  
 466 with obstacles of larger process zone size.

#### 467 4.4. Implications for the measurement of fracture energy variations from front deformations

468 We saw that the front deformations are *amplified* by the presence of a finite process zone for  
 469 heterogeneities of strength (see Section 4.2), while it is found *vanishing* for heterogeneities of  
 470 process zone size (see Section 4.3). The nature of the heterogeneities thus strongly influences  
 471 the way cracks distort when interacting with material heterogeneities. Yet, both types of hetero-  
 472 geneities impact similarly the overall value of fracture energy (see Eq. (43)). As such, one may  
 473 wonder if the amplitude of the front deformations constitutes a robust measure of the fracture  
 474 energy contrast.

475  
 476 To tackle this issue, we study the general case where heterogeneities of fracture energy  
 477 emerge from fluctuations of both strength and process zone size. Together with Eqs. (1) and  
 478 (21), the condition (4) of finiteness of the stress at the crack front yields at zero order in the  
 479 perturbations  $\delta a$ ,  $\delta\sigma_c$  and  $\delta\omega$ :

$$G^0 = G_c^0 \quad (52)$$

480 In other words, the presence of a cohesive zone does not influence the effective toughness of a  
 481 heterogeneous material in the *weak pinning* regime.

482 At first order, one finds:

$$\widehat{\delta a}(k) = -\omega_0 \frac{\hat{\Sigma}(|k|\omega_0)}{\hat{\mathcal{A}}(|k|\omega_0)} \frac{\widehat{\delta\sigma_c}(k)}{\sigma_c^0} - \omega_0 \frac{\hat{\Omega}(|k|\omega_0)}{\hat{\mathcal{A}}(|k|\omega_0)} \frac{\widehat{\delta\omega}(k)}{2\omega_0} \quad (53)$$

483 One may then define multiple paths of increasing fracture energy contrast  $\Delta G_c^0/G_c^0$  from a path of  
484 increasing strength (Path 1 of Fig. 8a) to one of increasing process zone size (Path 3 of Fig. 8a).  
485 We observe in Fig. 8b-d that the former constitutes an upper bound of the front deformation am-  
486 plitude  $A$ , while the latter represents a lower bound. Mixed paths (Path 2 of Fig. 8a) are found  
487 in between. When the average process zone size  $\omega_0$  is way smaller than the obstacle width  $d$   
488 (see Fig. 8b), the two bounds converge to the LEFM value and the amplitude of the front deforma-  
489 tion constitutes a good proxy for the contrast  $\Delta G_c/G_c^0$  of the fracture energy (Patinet et al.,  
490 2013a). For larger process zone sizes, the two bounds depart from one another (see yellow area  
491 on Fig. 8b-d). Yet, a lower dispersion on  $\Delta G_c/G_c^0$  can be observed when the obstacle spacing  $\Delta L$   
492 is larger than the process zone size  $\omega_0$  (see Fig. 8e-g). This is explained by the fact that the spec-  
493 trum of the fracture energy fluctuations is then mostly carried by large-wavelengths. As such,  
494 one should rather measure a contrast in fracture energy from experiments of a crack interacting  
495 with a single defect (Patinet et al., 2013a) rather than arrays of close obstacles (Dalmas et al.,  
496 2009).  
497

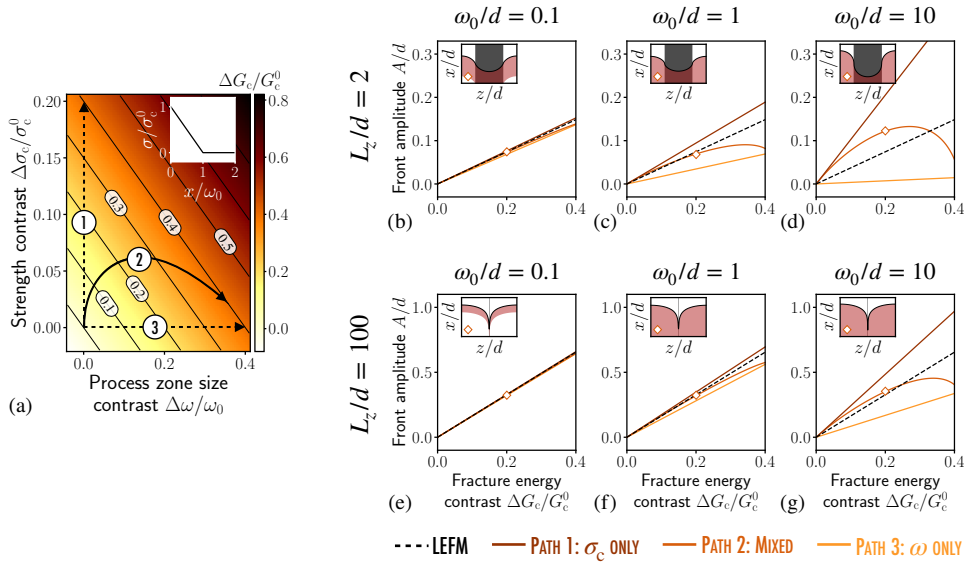


Figure 8: (a) An increase of material fracture energy can be achieved either by increasing its strength at constant brittleness (see dashed vertical line of path 1), by increasing its ductility at constant strength (see dashed horizontal line of path 3), or by any mixed path combining an increase of strength with that of ductility (see curved black line of path 2). (b-g) While the front amplitude increases linearly with the ratio of the fracture energy of the obstacles  $G_c^{\text{obs}}$  to that of the matrix  $G_c^{\text{mat}}$  in the limit cases of LEFM (dashed black line), and path 1 (solid brown line) & 3 (solid light orange line), it usually evolves non-linearly with the fracture energy contrast  $\Delta G_c/G_c^0$  in presence of a finite-size dissipation (see path 2 in solid dark orange line). The amplitude of the front deformations may vary between the two curves associated with paths 1 & 3 (light yellow area). These variations increase with the process zone size  $\omega$  (see b & e, c & f, d & g), but decrease with the obstacle spacing  $L_z$  (see b-d & e-g). Insets: front deformation profiles for  $\Delta G_c/G_c^0 = 1.2$ .

498 It is also worth noticing that a non-linear dependence of the deformation amplitude  $A$  with  
499 the contrast of fracture energy can be observed for cohesive materials (see Path 2 of Fig. 8b-g),  
500 even within the linear theory. This is solely due to the differential impact of heterogeneities of

501 strength or process zone size on the front deformations.

#### 502 4.5. Comparison with peeling experiments along heterogeneous interfaces

503 We conclude this section by comparing the output of our model to deformation profiles ob-  
 504 tained in experiments. The first-order theory of Rice (1985) has been shown to reproduce a wide  
 505 variety of experimental observations of crack front deformations in experiments of peeling or  
 506 fracture of heterogeneous interfaces (Dalmas et al., 2009), even if more refined models taking  
 507 into account the finite thickness of the fracture specimen (Legrand et al., 2011) or higher-order  
 508 terms (Vasoya et al., 2013) yield more accurate results (see (Patinet et al., 2013a) and (Vasoya  
 509 et al., 2016b) respectively). Chopin et al. (2011) noted nonetheless some discrepancies between  
 510 the LEFM predictions and the deformation profile of the crack front in their peeling experiments  
 511 of a silicon elastomer block from a patterned glass substrate, in particular close to the obstacle.  
 512 We show here that they may emerge from a non-negligible process zone size, and that our model  
 513 of Eq. (15) provides a better fit for the experimental profile of crack front deformations in cases  
 514 where the small-scale yielding assumption breaks down.

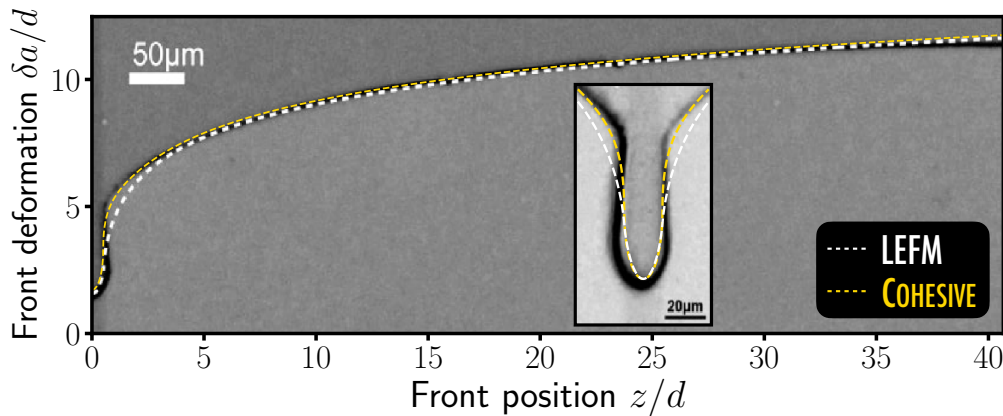


Figure 9: Prediction of the crack front deformations in peeling experiments of a silicon elastomer block from a patterned glass substrate (adapted from (Chopin et al., 2011)). A  $20\mu\text{m}$ -large tough obstacle is placed at the center of the  $22\text{mm}$ -large block, and deforms the crack front. Chopin et al. (2011)'s LEFM fit of the front deformations is plotted in white dashed line for  $\Delta K_{Ic}/K_{Ic}^0 \approx 3$ . Our cohesive fit of Eq. 25 for  $\Delta K_{Ic}/K_{Ic}^0 = 2.9$  ( $\Delta\sigma_c/\sigma_c^0 = 1.65$  and  $\Delta\omega/\omega_0 = 2.5$ ) and  $\omega_0/d = 2.7$  is plotted in yellow dashed line; Inset: Zoom on the front deformations near the obstacle. The large deformations observed at the edges of the defect are a signature of the cohesive nature of the interface.

515 Chopin et al. (2011) performed peeling experiments of a silicon elastomer block from a pat-  
 516 terned glass substrate. The  $22$  or  $72\text{mm}$ -wide and  $10\text{mm}$ -thick elastomer block is made of black  
 517 cross-linked PDMS (Sylgard170, Dow Corning) of Young's modulus  $E \approx 2\text{MPa}$ . It is peeled  
 518 from a rigid substrate along a heterogeneous interface consisting of a  $20\mu\text{m}$ -wide PDMS-glass  
 519 obstacle embedded in a weak PDMS-chromium layer. The toughness contrast between the two  
 520 types of interfaces has been estimated to  $\Delta K_{Ic}/K_{Ic}^0 = 5 - 9$  from independent peeling tests along  
 521 homogeneous interfaces. During the experiments, the crack front is pinned by the tough obsta-  
 522 cle, and front deformations emerge from the interaction between the crack and the heterogeneous  
 523 toughness field. Chopin et al. (2011) showed that the deformation profile predicted by LEFM  
 524 for a single tough defect (see Eq. (D.1)) corresponds to that observed in the experiments, when

525 looked at far away from the defect (see dashed white line in Fig. 9). Yet, two major discrepan-  
526 cies were observed: (i) the LEFM best-fit of the deformation profile corresponds to a toughness  
527 contrast  $\Delta K_{Ic}/K_{Ic}^0 \simeq 3$ , which is lower than the one estimated from the interfacial properties  
528  $\Delta K_{Ic}/K_{Ic}^0 = 5 - 9$ ; (ii) the experimental crack front displays much sharper variations at the edges  
529 of the obstacle than those predicted by LEFM. The first discrepancy can be attributed to finite-  
530 thickness and second-order effects, as they have been shown to reduce crack front deformations  
531 (Vasoya et al., 2016b). We argue here that the second discrepancy relates to a non-negligible  
532 process zone size in their experiments.

533  
534 Indeed, we saw in Section 4.2 that a finite process zone size leads to an overall increase of  
535 the amplitude of the front deformations. In that case, the front deformations are concentrated at  
536 the obstacles edges (see Fig 6), as observed in the experiments of Chopin et al. (2011). While the  
537 localization of the front deformation is quantitatively grasped by our cohesive framework (see  
538 Eq. (D.2) describing the front deformations for the single defect and large process zone size),  
539 this feature cannot be predicted either by the finite-size effects emerging from the finite thickness  
540 of the PDMS block with respect to the defect size (Legrand et al., 2011) or from higher-order  
541 effects (Vasoya et al., 2016b).

542 The experimental front profile is best fitted by our model (see dashed yellow line of Fig 6)  
543 for a process zone size  $\omega_0 \simeq 2.7d = 54\mu\text{m}$ , a strength contrast  $\Delta\sigma_c/\sigma_c^0 = 1.65$  and a process  
544 zone size contrast  $\Delta\omega/\omega_0 = 2.5$ , which are equivalent to a toughness contrast  $\Delta K_{Ic}/K_{Ic}^0 = 2.9$ .  
545 The size of the process zone  $\omega_0 \simeq 54\mu\text{m}$  is compatible with values of the adhesive length of soft  
546 materials (Creton and Ciccotti, 2016). Moreover, our measurements yield a value of toughness  
547 contrast similar to that of Chopin et al. (2011) based on LEFM. This was somehow expected as,  
548 in the case of the single obstacle, the amplitude of the front deformations is mostly controlled by  
549 the toughness contrast  $\Delta K_{Ic}/K_{Ic}^0$  when  $\omega_0 \simeq 54\mu\text{m} \ll \Delta L \simeq 22\text{mm}$  (see Section 4.4). Promising  
550 directions for a better match of the experimental data would consist in accounting for the influ-  
551 ence of the finite thickness of the PDMS block, and second-order terms. Following the theory  
552 of Legrand et al. (2011), the former effect is expected to be rather small in the experiments of  
553 (Chopin et al., 2011), as the thickness of the PDMS block ( $h \simeq 10\text{mm}$ ) is much larger than the de-  
554 fect width ( $d \simeq 20\mu\text{m}$ ). On the contrary, the influence of second-order terms could be meaningful  
555 as the toughness contrast between the matrix and the obstacle is rather large ( $\Delta K_{Ic}/K_{Ic}^0 = 5 - 9$ ),  
556 and they have been shown to reduce significantly crack front deformations even for contrasts as  
557 small as 1 (Vasoya et al., 2016b). Note that mixed mode effects due to a friction-induced shear  
558 loading can be observed in the experimental setup of Chopin et al. (2011), and may also explain  
559 some discrepancies between the experimental observations and the theoretical predictions.

## 560 5. Conclusion

561 This study provides a theoretical framework based on a perturbative approach of LEFM that  
562 describes the influence of a finite process zone on the deformations of a crack front by hetero-  
563 geneities of fracture properties. Namely, we extended in equation (25) the first-order theory of  
564 Rice (1985) designed for perfectly brittle materials to the broader case of cohesive materials,  
565 where crack advance is resisted by cohesive stress in its wake. Our model allowed us to re-  
566 visit Rice (1985)'s problem of crack front stability to sinusoidal perturbations and Gao and Rice  
567 (1989)'s of crack pinning by an array of tough obstacles, and to stress out the influence of a finite  
568 process zone on these matters.

569

570 We first showed that cohesive cracks accommodate a perturbation differently depending on  
571 the size of its wavelength with respect to that of the process zone. In particular, perturbations  
572 of wavelength smaller than the process zone size are amplified with respect to the LEFM case,  
573 while perturbations of wavelength larger than the process zone size are left unchanged. This was  
574 interpreted as a *stiffness loss of the crack front* due to the presence of cohesive stresses in the  
575 crack wake. As a result, cracks may be unstable to perturbations of any wavelength if embedded  
576 in a *destabilizing* structure, where the mode I SIF strongly increases with crack advance.

577 The loss in front stiffness may then lead to amplified front deformations of a crack interacting  
578 with periodic arrays of tough obstacles. We showed that it was not always the case, as the  
579 process zone also *smooths out the local fluctuations of strength and process zone size* that occur  
580 at a scale lower than the process zone size. Interestingly, we showed that the gain in front  
581 compliance overcomes the smoothing of perturbations in the case of heterogeneities of strength,  
582 while the decrease in amplitude of the fluctuations is found sharper than the stiffness loss in the  
583 case of heterogeneities of process zone size. As a result, the front deformations are *amplified*  
584 by the presence of a process zone when the increased fracture energy is associated with a higher  
585 strength of the obstacle, and *reduced* when it emerges from a larger process zone size.

586 Overall, our theory reconciles the wide variety of front profiles observed in experiments,  
587 when the small-scale yielding hypothesis of linear elastic fracture mechanics breaks down, as in  
588 (Chopin et al., 2011).

589

590 Future works may focus on the extension of our model to disordered microstructures that  
591 are no more invariant in the front direction. A particular attention should then be devoted to the  
592 interaction between the finite process zone size, characteristics of the dissipation, and the Larkin  
593 length that emerges from the disorder (Larkin and Ovchinnikov, 1979). This may ultimately  
594 provide insights on the influence of a finite process zone size on the effective fracture properties  
595 of quasi-brittle materials. Another promising direction is to investigate crack front deformations  
596 in a dynamic setting, where the contraction of the process zone with crack velocity (Rice, 1980)  
597 interacts with the dynamic stiffening of the crack front observed by Morrissey and Rice (2000).  
598 This is done in a follow-up study, in which our framework is extended to dynamic fracture, and  
599 theoretical predictions are compared to the results of numerical simulations (Roch et al., 2022).

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604 work (Chopin et al., 2011).

## 605 **CRedit authorship contribution statement**

606 **Mathias Lebihain:** Conceptualization, Formal analysis, Investigation, Visualization, Software,  
607 Writing - Original Draft; **Thibault Roch:** Conceptualization, Investigation, Writing - Review &  
608 Editing; **Jean-Fraçois Molinari:** Conceptualization, Funding acquisition, Writing - Review &  
609 Editing

610 **Appendix A. Calculation of  $k(\mathcal{F}^*; z_0; z_1, x)$**

611 This appendix aims at establishing Eq. (15) from Eqs. (10), (13), and (14). In particular, we  
612 want to find a more suitable expression for:

$$I(z_0, z_1) = \frac{1}{2\pi} \text{PV} \int_{-\infty}^{+\infty} \frac{\delta a(z) - \delta a^{**}(z)}{((z - z_1)^2 + x^2)(z - z_0)^2} dz \quad (\text{A.1})$$

613 As noted by Leblond et al. (2012), each of the integrals that comprise  $I(z_0, z_1)$  individually  
614 diverges at infinity, so that it is necessary to integrate first between finite bounds  $-L_-$  and  $L_+$   
615 before taking the limit  $L_-, L_+ \rightarrow +\infty$  in the final combination of integrals.

616 **Terms in  $1/(z - z_0)^2$**  – They read with a pre-factor  $1/((z_0 - z_1)^2 + x^2)$ :

$$\text{PV} \int_{-L_-}^{L_+} \frac{\delta a(z) - \delta a^{**}(z)}{(z - z_0)^2} dz = \text{PV} \int_{-L_-}^{L_+} \frac{\delta a(z) - \delta a(z_0)}{(z - z_0)^2} dz - \ln \left[ \frac{L_+ - z_0}{L_- + z_0} \right] \frac{\delta a(z_0) - \delta a(z_1)}{z_0 - z_1} \quad (\text{A.2})$$

617 **Terms in  $1/(z - z_0)$**  – They read with a pre-factor  $-2(z_0 - z_1)/((z_0 - z_1)^2 + x^2)^2$ :

$$\int_{-L_-}^{L_+} \frac{\delta a(z) - \delta a^{**}(z)}{(z - z_0)} dz = \text{PV} \int_{-L_-}^{L_+} \frac{\delta a(z)}{(z - z_0)} dz - \ln \left[ \frac{L_+ - z_0}{L_- + z_0} \right] \delta a(z_0) - (L_+ + L_-) \frac{\delta a(z_0) - \delta a(z_1)}{z_0 - z_1} \quad (\text{A.3})$$

618 **Terms in  $1/((z - z_1)^2 + x^2)^2$**  – They read with a pre-factor  $((z_0 - z_1)^2 - x^2)/((z_0 - z_1)^2 + x^2)^2$ :

$$\text{PV} \int_{-L_-}^{L_+} \frac{\delta a(z) - \delta a^{**}(z)}{((z - z_1)^2 + x^2)^2} dz = \text{PV} \int_{-L_-}^{L_+} \frac{\delta a(z) - \delta a(z_1)}{((z - z_1)^2 + x^2)^2} dz - \ln \left[ \sqrt{\frac{(L_+ - z_1)^2 + x^2}{(L_- + z_1)^2 + x^2}} \right] \frac{\delta a(z_0) - \delta a(z_1)}{z_0 - z_1} \quad (\text{A.4})$$

619 **Terms in  $(z - z_1)/((z - z_1)^2 + x^2)^2$**  – They read with a pre-factor  $2(z_0 - z_1)/((z_0 - z_1)^2 + x^2)^2$ :

$$\int_{-L_-}^{L_+} \frac{(z - z_1)}{((z - z_1)^2 + x^2)^2} [\delta a(z) - \delta a^{**}(z)] dz = \text{PV} \int_{-L_-}^{L_+} \frac{(z - z_1)}{((z - z_1)^2 + x^2)^2} \delta a(z) dz - \ln \left[ \sqrt{\frac{(L_+ - z_1)^2 + x^2}{(L_- + z_1)^2 + x^2}} \right] \delta a(z_1) \\ - (L_+ + L_-) \frac{\delta a(z_0) - \delta a(z_1)}{z_0 - z_1} + x \left[ \arctan \left( \frac{L_+ - z_1}{x} \right) + \arctan \left( \frac{L_- + z_1}{x} \right) \right] \frac{\delta a(z_0) - \delta a(z_1)}{z_0 - z_1} \quad (\text{A.5})$$

620 We observe that the terms proportional to  $(L_+ + L_-)$  of Eqs (A.3) and (A.5) cancel out, we get:

$$\text{PV} \int_{-L_-}^{L_+} \frac{\delta a(z) - \delta a^{**}(z)}{((z - z_1)^2 + x^2)(z - z_0)^2} dz = \left[ \frac{1}{(z_0 - z_1)^2 + x^2} \text{PV} \int_{-L_-}^{L_+} \frac{\delta a(z) - \delta a(z_0)}{(z - z_0)^2} dz - \frac{2(z_0 - z_1)}{((z_0 - z_1)^2 + x^2)^2} \text{PV} \int_{-L_-}^{L_+} \frac{\delta a(z)}{z - z_0} dz \right. \\ \left. + \frac{(z_0 - z_1)^2 - x^2}{((z_0 - z_1)^2 + x^2)^2} \text{PV} \int_{-L_-}^{L_+} \frac{\delta a(z) - \delta a(z_1)}{(z - z_1)^2 + x^2} dz + \frac{2(z_0 - z_1)}{((z_0 - z_1)^2 + x^2)^2} \text{PV} \int_{-L_-}^{L_+} \frac{(z - z_1)}{(z - z_1)^2 + x^2} \delta a(z) dz \right. \\ \left. + \frac{(z_0 - z_1)^2 - x^2}{((z_0 - z_1)^2 + x^2)^2} \left[ \ln \left( \frac{L_+ - z_0}{L_- + z_0} \right) - \ln \left( \sqrt{\frac{(L_+ - z_1)^2 + x^2}{(L_- + z_1)^2 + x^2}} \right) \right] \frac{\delta a(z_0)}{(z_0 - z_1)} \right. \\ \left. + \frac{1}{(z_0 - z_1)^2 + x^2} \left[ \ln \left( \sqrt{\frac{(L_+ - z_1)^2 + x^2}{(L_- + z_1)^2 + x^2}} \right) - \ln \left( \frac{L_+ - z_0}{L_- + z_0} \right) \right] \frac{\delta a(z_1)}{(z_0 - z_1)} \right. \\ \left. + \frac{2x}{((z_0 - z_1)^2 + x^2)^2} \left[ \arctan \left( \frac{L_+ - z_1}{x} \right) + \arctan \left( \frac{L_- + z_1}{x} \right) \right] (\delta a(z_0) - \delta a(z_1)) \right] \quad (\text{A.6})$$

621 We then notice that the logarithmic terms go to zero when  $L_-, L_+ \rightarrow +\infty$ . In this limit, one finds:

$$\text{PV} \int_{-\infty}^{+\infty} \frac{\delta a(z) - \delta a^{**}(z)}{((z - z_1)^2 + x^2)(z - z_0)^2} dz = \left[ \frac{1}{(z_0 - z_1)^2 + x^2} \text{PV} \int_{-\infty}^{+\infty} \frac{\delta a(z) - \delta a(z_0)}{(z - z_0)^2} dz \right. \\ \left. - \frac{2(z_0 - z_1)}{((z_0 - z_1)^2 + x^2)^2} \text{PV} \int_{-\infty}^{+\infty} \frac{\delta a(z)}{z - z_0} dz + \frac{(z_0 - z_1)^2 - x^2}{((z_0 - z_1)^2 + x^2)^2} \text{PV} \int_{-\infty}^{+\infty} \frac{\delta a(z) - \delta a(z_1)}{(z - z_1)^2 + x^2} dz \right. \\ \left. + \frac{2(z_0 - z_1)}{((z_0 - z_1)^2 + x^2)^2} \text{PV} \int_{-\infty}^{+\infty} \frac{(z - z_1)}{(z - z_1)^2 + x^2} \delta a(z) dz + \frac{2\pi x}{((z_0 - z_1)^2 + x^2)^2} (\delta a(z_0) - \delta a(z_1)) \right] \quad (\text{A.7})$$



622 One finally gets Eq. (15) from Eqs. (13) and (A.7).

### 623 **Appendix B. Calculation of $\delta K_{\text{czm}}(z)$ and its Fourier transform**

624 In this appendix, we derive Eqs. (21) and (22) from Eqs. (15) and (20). To do so, we first  
625 express  $k(\Gamma^*; z_0; z_1, x)$  as:

$$k(\Gamma^*; z_0; z_1, x) = k_0(z_0, z_1, x) [1 + \delta\mathcal{K}(z_0; z_1, x)] \quad (\text{B.1})$$

626 where  $k_0$  corresponds to the CFWF associated to the reference straight front  $\Gamma$ :

$$k_0(z_0, z_1, x) = \frac{\sqrt{2}}{\pi^{3/2}} \frac{\sqrt{x}}{(z_0 - z_1)^2 + x^2} \quad (\text{B.2})$$

627 and  $\delta\mathcal{K}$  can be decomposed into five different first-order terms following Eq. (15):

$$\begin{aligned} \delta\mathcal{K}(z_0, z_1, x) &= \delta\mathcal{K}_1(z_0, z_1, x) + \delta\mathcal{K}_2(z_0, z_1, x) + \delta\mathcal{K}_3(z_0, z_1, x) + \delta\mathcal{K}_4(z_0, z_1, x) + \delta\mathcal{K}_5(z_0, z_1, x) \\ &= \frac{1}{2\pi} \text{PV} \int_z \frac{\delta a(z) - \delta a(z_0)}{(z - z_0)^2} dz - \frac{2(z_0 - z_1)}{(z_0 - z_1)^2 + x^2} \frac{1}{2\pi} \text{PV} \int_z \frac{\delta a(z)}{z - z_0} dz \\ &\quad + \frac{2(z_0 - z_1)}{(z_0 - z_1)^2 + x^2} \frac{1}{2\pi} \text{PV} \int_z \frac{(z - z_1)}{(z - z_1)^2 + x^2} \delta a(z) dz + \frac{(z_0 - z_1)^2 - x^2}{(z_0 - z_1)^2 + x^2} \frac{1}{2\pi} \text{PV} \int_z \frac{\delta a(z) - \delta a(z_1)}{(z - z_1)^2 + x^2} dz \\ &\quad + \frac{x}{(z_0 - z_1)^2 + x^2} (\delta a(z_0) - \delta a(z_1)) \end{aligned} \quad (\text{B.3})$$

628 **Decomposition of the cohesive SIF** – Eq. (20) rewrites as:

$$K_{\text{czm}}(z_0) = \int_u \int_{z_1} \sigma_c(z_1) f_w(u) \omega(z_1) k_0(z_0, z_1, \omega(z_1)u) [1 + \delta\mathcal{K}(z_0, z_1, \omega(z_1)u)] dz_1 du \quad (\text{B.4})$$

629 where  $\omega(z_1)k_0(z_0, z_1, \omega(z_1)u)$  can be rewritten as the sum of a zero-order term and a first-order  
630 term in  $\delta\omega$ :

$$\begin{aligned} \omega(z_1)k_0(z_0, z_1, \omega(z_1)u) &= \frac{\sqrt{2}}{\pi^{3/2}} \frac{\omega(z_1)^{3/2} u^{1/2}}{(z_0 - z_1)^2 + (\omega(z_1)u)^2} \\ &= \omega_0 k_0(z_0, z_1, \omega_0 u) \left[ 1 + \left( \frac{3}{2} - \frac{2\omega_0^2 u^2}{(z_0 - z_1)^2 + (\omega_0 u)^2} \right) \frac{\delta\omega(z_1)}{\omega_0} \right] \end{aligned} \quad (\text{B.5})$$

631 and  $\delta\mathcal{K}(z_0; z_1, \omega(z_1)u) = \delta\mathcal{K}(z_0; z_1, \omega_0 u)$  since it is already composed of first-order terms in  $\delta a$ .

632 **Zero-order terms** – In the end, one finds back from Eq. (B.4) at order 0:  
633

$$\begin{aligned} K_{\text{czm}}^0 &= \int_u \int_{z_1} \sigma_c^0 f_w(u) \omega_0 k_0(z_0, z_1, \omega_0 u) dz_1 du \\ &= \left[ \int_u \frac{f_w(u)}{u^{1/2}} du \right] \sqrt{\frac{2}{\pi}} \sigma_c^0 \omega_0^{1/2} = C_w \sqrt{\frac{2}{\pi}} \sigma_c^0 \omega_0^{1/2} = K_{\text{lc}}^0 \end{aligned} \quad (\text{B.6})$$

634 which corresponds to Eq. (21). We find back the results of Palmer et al. (1973) derived in the 2D  
635 case. This was expected since the zero-order terms relate to the reference semi-infinite crack  $\Gamma$   
636 with a straight front  $\mathcal{F}$  embedded in a homogeneous medium, which is translationally invariant

637 in the ( $Oz$ ) direction.

638

639 **First-order terms** – At order 1 in  $\delta\sigma_c$ ,  $\delta\omega$  and  $\delta a$ , one gets:

$$\begin{aligned} \delta K_{\text{czm}}(z_0) &= \int_u \int_{z_1} \frac{\sqrt{2}}{\pi^{3/2}} \frac{\sigma_c^0 \omega_0^{3/2} u^{1/2}}{(z_0 - z_1)^2 + (\omega_0 u)^2} f_w(u) \frac{\delta\sigma_c(z_1)}{\sigma_c^0} dz_1 du \\ &+ \int_u \int_{z_1} \frac{\sqrt{2}}{\pi^{3/2}} \frac{\sigma_c^0 \omega_0^{3/2} u^{1/2}}{(z_0 - z_1)^2 + (\omega_0 u)^2} f_w(u) \left[ \frac{3}{2} - \frac{2\omega_0^2 u^2}{(z_0 - z_1)^2 + (\omega_0 u)^2} \right] \frac{\delta\omega(z_1)}{\omega_0} dz_1 du \\ &+ \int_u \int_{z_1} \frac{\sqrt{2}}{\pi^{3/2}} \frac{\sigma_c^0 \omega_0^{3/2} u^{1/2}}{(z_0 - z_1)^2 + (\omega_0 u)^2} f_w(u) \left[ \sum_i \delta\mathcal{K}_i(z_0, z_1, \omega_0 u) \right] dz_1 du \end{aligned} \quad (\text{B.7})$$

640 The next step is to calculate each of the 7 integrals of Eq. (B.7). The calculations build upon the  
 641 Fourier representation  $\widehat{\delta\sigma_c}$ ,  $\widehat{\delta\omega}$ , and  $\widehat{\delta a}$  of  $\delta\sigma_c$ ,  $\delta\omega$ , and  $\delta a$ , and the expression of the following  
 642 integrals (see Eqs. (3.723.2), (3.723.3), (3.729.1), (3.729.2) and (3.729.3) of Gradshteyn and  
 643 Ryzhik (2014)):

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{e^{ikv}}{v} dv &= i\pi \text{sgn}(k) & \text{and} & \int_{-\infty}^{+\infty} \frac{e^{ikv}}{v^2 + (\omega_0 u)^2} dv = \frac{\pi}{\omega_0 u} e^{-|k|\omega_0 u} \\ \int_{-\infty}^{+\infty} \frac{ve^{ikv}}{v^2 + (\omega_0 u)^2} dv &= i\pi e^{-|k|\omega_0 u} & \text{and} & \int_{-\infty}^{+\infty} \frac{e^{ikv}}{[v^2 + (\omega_0 u)^2]^2} dv = \frac{\pi}{2} \frac{1 + |k|\omega_0 u}{(\omega_0 u)^3} e^{-|k|\omega_0 u} \\ \int_{-\infty}^{+\infty} \frac{ve^{ikv}}{[v^2 + (\omega_0 u)^2]^2} dv &= i\frac{\pi}{2} \frac{|k|}{\omega_0 u} e^{-|k|\omega_0 u} & \text{and} & \int_{-\infty}^{+\infty} \frac{v^2 - (\omega_0 u)^2}{[v^2 + (\omega_0 u)^2]^2} e^{ikv} dv = -\pi|k| e^{-|k|\omega_0 u} \end{aligned} \quad (\text{B.8})$$

644 where  $\text{sgn}(k)$  denotes the sign of the real number  $k$ .

645

646 **Terms in  $\delta\sigma_c$**  – We denote  $\delta K_{\text{czm}}^\sigma$  the contributions of  $\delta\sigma_c$  to the variations  $\delta K_{\text{czm}}$  of the cohesive  
 647 SIF. From Eq. (B.7), it reads:

$$\begin{aligned} \delta K_{\text{czm}}^\sigma(z_0) &= \int_u \int_{z_1} \frac{\sqrt{2}}{\pi^{3/2}} \frac{\sigma_c^0 \omega_0^{3/2} u^{1/2}}{(z_0 - z_1)^2 + (\omega_0 u)^2} f_w(u) \frac{\delta\sigma_c(z_1)}{\sigma_c^0} dz_1 du \\ &= \int_u \left[ du \frac{\sqrt{2}}{\pi^{3/2}} \sigma_c^0 \omega_0^{3/2} u^{1/2} f_w(u) \frac{1}{2\pi} \int_k dk \frac{\widehat{\delta\sigma_c}(k)}{\sigma_c^0} e^{ikz_0} \left( \int_{z_1} \frac{e^{ik(z_1 - z_0)}}{(z_1 - z_0)^2 + (\omega_0 u)^2} dz_1 \right) \right] \\ &= \sqrt{\frac{2}{\pi}} \sigma_c^0 \omega_0^{1/2} \frac{1}{2\pi} \int_k \left[ \int_u \frac{f_w(u)}{u^{1/2}} e^{-|k|\omega_0 u} du \right] \frac{\widehat{\delta\sigma_c}(k)}{\sigma_c^0} e^{ikz_0} dk \\ &= K_{\text{lc}}^0 \cdot \frac{1}{2\pi} \int_k \left[ \frac{1}{C_w} \int_u \frac{f_w(u)}{u^{1/2}} e^{-|k|\omega_0 u} du \right] \frac{\widehat{\delta\sigma_c}(k)}{\sigma_c^0} e^{ikz_0} dk \end{aligned} \quad (\text{B.9})$$

648 **Terms in  $\delta\omega$**  – We denote  $\delta K_{\text{czm}}^\omega$  the contributions of  $\delta\omega$  to the variations  $\delta K_{\text{czm}}$  of the cohesive

649 SIF. From Eq. (B.7), it reads:

$$\begin{aligned}
\delta K_{\text{czm}}^\omega(z_0) &= \int_u \int_{z_1} \frac{\sqrt{2}}{\pi^{3/2}} \frac{\sigma_c^0 \omega_0^{3/2} u^{1/2}}{(z_0 - z_1)^2 + (\omega_0 u)^2} f_w(u) \left[ \frac{3}{2} - \frac{2\omega_0^2 u^2}{(z_0 - z_1)^2 + (\omega_0 u)^2} \right] \frac{\delta\omega(z_1)}{\omega_0} dz_1 du \\
&= \int_u \left[ du \frac{\sqrt{2}}{\pi^{3/2}} \sigma_c^0 \omega_0^{3/2} u^{1/2} f_w(u) \frac{1}{2\pi} \int_k dk \frac{3}{2} \frac{\widehat{\delta\omega}(k)}{\omega_0} e^{ikz_0} \left( \int_{z_1} \frac{e^{ik(z_1 - z_0)}}{(z_1 - z_0)^2 + (\omega_0 u)^2} dz_1 \right) \right] \\
&\quad - \int_u \left[ du \frac{\sqrt{2}}{\pi^{3/2}} \sigma_c^0 \omega_0^{3/2} u^{1/2} f_w(u) \frac{1}{2\pi} \int_k dk \frac{\widehat{\delta\omega}(k)}{\omega_0} e^{ikz_0} \left( \int_{z_1} \frac{2(\omega_0 u)^2 e^{ik(z_1 - z_0)}}{[(z_1 - z_0)^2 + (\omega_0 u)^2]^2} dz_1 \right) \right] \\
&= K_{\text{lc}}^0 \cdot \frac{1}{2\pi} \int_k \left[ \frac{1}{C_w} \int_u \frac{f_w(u)}{u^{1/2}} \left( \frac{1}{2} - |k|\omega_0 u \right) e^{-|k|\omega_0 u} du \right] \frac{\widehat{\delta\omega}(k)}{\omega_0} e^{ikz_0} dk \\
&\stackrel{\text{I.B.P./}u}{=} K_{\text{lc}}^0 \cdot \frac{1}{2\pi} \int_k \left[ \frac{1}{C_w} \int_u -f_w'(u) u^{1/2} e^{-|k|\omega_0 u} du \right] \frac{\widehat{\delta\omega}(k)}{\omega_0} e^{ikz_0} dk
\end{aligned} \tag{B.10}$$

650 where ‘‘I.B.P/’’ denotes the integration by parts with respect to the variable  $u$ .

651

652 *Terms in  $\delta a$*  – We denote  $\delta K_{\text{czm}}^{a,i}$  the contributions of  $\delta \mathcal{K}_i$  to the variations  $\delta K_{\text{czm}}$  of the cohesive  
653 SIF. From Eqs. (B.3) and (B.7),  $\delta K_{\text{czm}}^{a,1}$  reads:

$$\begin{aligned}
\delta K_{\text{czm}}^{a,1}(z_0) &= \int_u \int_{z_1} \frac{\sqrt{2}}{\pi^{3/2}} \frac{\sigma_c^0 \omega_0^{3/2} u^{1/2}}{(z_0 - z_1)^2 + (\omega_0 u)^2} f_w(u) \left[ \frac{1}{2\pi} \text{PV} \int_z \frac{\delta a(z) - \delta a(z_0)}{(z - z_0)^2} dz \right] dz_1 du \\
&= \int_u \int_{z_1} \frac{\sqrt{2}}{\pi^{3/2}} \frac{\sigma_c^0 \omega_0^{3/2} u^{1/2}}{(z_0 - z_1)^2 + (\omega_0 u)^2} f_w(u) \left[ \frac{1}{2\pi} \text{PV} \int_z \frac{\delta a'(z)}{z - z_0} dz \right] dz_1 du \\
&= \int_u \left[ du \frac{\sqrt{2}}{\pi^{3/2}} \sigma_c^0 \omega_0^{3/2} u^{1/2} f_w(u) \frac{1}{(2\pi)^2} \int_k dk ik \widehat{\delta a}(k) e^{ikz_0} \left( \int_{z_1} \frac{dz_1}{(z_0 - z_1)^2 + (\omega_0 u)^2} \right) \left( \int_z \frac{e^{ik(z - z_0)}}{z - z_0} dz \right) \right] \\
&= K_{\text{lc}}^0 \cdot \frac{1}{2\pi} \int_k -\frac{|k|}{2} \widehat{\delta a}(k) e^{ikz_0} dk
\end{aligned} \tag{B.11}$$

654 One may then show that:

$$\delta K_{\text{czm}}^{a,2}(z) = 0 \tag{B.12}$$

655 because the pre-factor in front of the integral over  $z$  of  $\delta \mathcal{K}_2$  is an even function of  $(z_0 - z_1)$ , so  
656 that the integral over  $z_1$  in  $\delta K_{\text{czm}}^{a,2}$  equates to zero. It is not the case for  $\delta K_{\text{czm}}^{a,3}$ , which reads:

$$\begin{aligned}
\delta K_{\text{czm}}^{a,3}(z_0) &= \int_u \int_{z_1} \frac{\sqrt{2}}{\pi^{3/2}} \sigma_c^0 \omega_0^{3/2} u^{1/2} f_w(u) \frac{2(z_0 - z_1)}{[(z_0 - z_1)^2 + (\omega_0 u)^2]^2} \left[ \frac{1}{2\pi} \text{PV} \int_z \frac{(z - z_1)}{(z - z_1)^2 + (\omega_0 u)^2} \delta a(z) dz \right] dz_1 du \\
&= \int_u \int_{z_1} \frac{\sqrt{2}}{\pi^{3/2}} \sigma_c^0 \omega_0^{3/2} u^{1/2} f_w(u) \frac{2(z_0 - z_1)}{[(z_0 - z_1)^2 + (\omega_0 u)^2]^2} \frac{1}{(2\pi)^2} \int_k dk \widehat{\delta a}(k) e^{ikz_1} \left[ \int_z \frac{(z - z_1)}{(z - z_1)^2 + (\omega_0 u)^2} e^{ik(z - z_1)} dz \right] dz_1 du \\
&= \int_u \frac{\sqrt{2}}{\pi^{3/2}} \sigma_c^0 \omega_0^{3/2} u^{1/2} f_w(u) \frac{1}{2\pi} \int_k dk e^{-|k|\omega_0 u} \widehat{\delta a}(k) e^{ikz_0} i \left[ \int_{z_1} \frac{(z_0 - z_1)}{[(z_0 - z_1)^2 + (\omega_0 u)^2]^2} e^{ik(z_1 - z_0)} dz_1 \right] du \\
&= K_{\text{lc}}^0 \cdot \frac{1}{2\pi} \int_k \left[ \frac{1}{C_w} \int_u \frac{f_w(u)}{u^{1/2}} e^{-2|k|\omega_0 u} du \right] \frac{|k|}{2} \widehat{\delta a}(k) e^{ikz_0} dk
\end{aligned} \tag{B.13}$$

657 The process for  $\delta K_{\text{czm}}^{a,4}$  is similar. One finds:

$$\begin{aligned}
\delta K_{\text{czm}}^{a,4}(z_0) &= \int_u \int_{z_1} \frac{\sqrt{2}}{\pi^{3/2}} \sigma_c^0 \omega_0^{3/2} u^{1/2} f_w(u) \frac{(z_0 - z_1)^2 - (\omega_0 u)^2}{[(z_0 - z_1)^2 + (\omega_0 u)^2]^2} \left[ \frac{1}{2\pi} \text{PV} \int_z \frac{\delta a(z) - \delta a(z_1)}{(z - z_1)^2 + (\omega_0 u)^2} dz \right] dz_1 du \\
&= \int_u \int_{z_1} \frac{\sqrt{2}}{\pi^{3/2}} \sigma_c^0 \omega_0^{3/2} u^{1/2} f_w(u) \frac{(z_0 - z_1)^2 - (\omega_0 u)^2}{[(z_0 - z_1)^2 + (\omega_0 u)^2]^2} \frac{1}{(2\pi)^2} \int_k dk \widehat{\delta a}(k) e^{ikz_1} \left[ \int_z \frac{e^{ik(z-z_1)} - 1}{(z - z_1)^2 + (\omega_0 u)^2} dz \right] dz_1 du \\
&= \int_u \frac{\sqrt{2}}{\pi^{3/2}} \sigma_c^0 \omega_0^{1/2} \frac{f_w(u)}{u^{1/2}} \frac{1}{2\pi} \int_k dk (e^{-|k|\omega_0 u} - 1) \frac{1}{2} \widehat{\delta a}(k) e^{ikz_0} \left[ \int_{z_1} \frac{(z_0 - z_1)^2 - (\omega_0 u)^2}{[(z_0 - z_1)^2 + (\omega_0 u)^2]^2} e^{ik(z_1 - z_0)} dz_1 \right] du \\
&= K_{\text{lc}}^0 \cdot \frac{1}{2\pi} \int_k \left[ \frac{1}{C_w} \int_u \frac{f_w(u)}{u^{1/2}} (1 - e^{-|k|\omega_0 u}) e^{-|k|\omega_0 u} du \right] \frac{|k|}{2} \widehat{\delta a}(k) e^{ikz_0} dk
\end{aligned} \tag{B.14}$$

658 The remaining integral  $\delta K_{\text{czm}}^{a,5}$  yields:

$$\begin{aligned}
\delta K_{\text{czm}}^{a,5}(z_0) &= \int_u \int_{z_1} \frac{\sqrt{2}}{\pi^{3/2}} \sigma_c^0 \omega_0^{3/2} u^{1/2} f_w(u) \frac{\omega_0 u}{[(z_0 - z_1)^2 + (\omega_0 u)^2]^2} [\delta a(z_0) - \delta a(z_1)] dz_1 du \\
&= \int_u \frac{\sqrt{2}}{\pi^{3/2}} \sigma_c^0 \omega_0^{1/2} \frac{f_w(u)}{u^{1/2}} \frac{1}{2\pi} \int_k dk \widehat{\delta a}(k) e^{ikz_0} \left[ \int_{z_1} \frac{\omega_0 u (1 - e^{ik(z_1 - z_0)})}{[(z_0 - z_1)^2 + (\omega_0 u)^2]^2} dz_1 \right] du \\
&= K_{\text{lc}}^0 \cdot \frac{1}{2\pi} \int_k \left[ \frac{1}{C_w} \int_u \frac{f_w(u)}{2u^{3/2}} [(1 + |k|\omega_0 u) e^{-|k|\omega_0 u} - 1] du \right] \frac{\widehat{\delta a}(k)}{\omega_0} e^{ikz_0} dk \\
&\stackrel{\text{I.B.P}/u}{=} -K_{\text{lc}}^0 \cdot \frac{1}{2\pi} \int_k \left[ \frac{1}{C_w} \int_u \frac{f_w(u)}{u^{1/2}} e^{-|k|\omega_0 u} du \right] \frac{|k|}{2} \widehat{\delta a}(k) e^{ikz_0} dk \\
&\quad + K_{\text{lc}}^0 \cdot \frac{1}{2\pi} \int_k \left[ \frac{1}{C_w} \int_u -\frac{f'_w(u)}{u^{1/2}} (1 - e^{-|k|\omega_0 u}) du \right] \frac{\widehat{\delta a}(k)}{\omega_0} e^{ikz_0} dk
\end{aligned} \tag{B.15}$$

659 Regrouping Eqs. (B.11)-(B.15), one finds :

$$\frac{\delta K_{\text{lc}}^a(z_0)}{K_{\text{lc}}^0} = \frac{1}{2\pi} \int_k -\frac{|k|}{2} \widehat{\delta a}(k) e^{ikz_0} dk + \frac{1}{2\pi} \int_k \left[ \frac{1}{C_w} \int_u -\frac{f'_w(u)}{u^{1/2}} (1 - e^{-|k|\omega_0 u}) du \right] \frac{\widehat{\delta a}(k)}{\omega_0} e^{ikz_0} dk \tag{B.16}$$

660 One finally gets Eq. (22) collecting the terms in  $\delta\sigma_c$ ,  $\delta\omega$ , and  $\delta a$  from Eqs.(B.9), (B.10) and  
661 (B.16).

662 Note that Eq. (22) can be expressed in terms of variations of strength  $\delta\sigma_c$  and critical crack  
663 opening  $\delta D_c$  with respect to their average value  $\sigma_c^0$  and  $D_c^0$ . From Eq. (18), one has:

$$\omega_0 = \alpha \frac{\mu}{\sigma_c^0} D_c^0 \text{ and } \frac{\delta\omega}{\omega_0} = \frac{\delta D_c}{D_c^0} - \frac{\delta\sigma_c}{\sigma_c^0} \tag{B.17}$$

664 So that Eq. (22) boils down to:

$$K_{\text{lc}}^0 \left[ 1 + \left( \frac{1}{K_{\text{lc}}^0} \frac{\partial K_{\text{lc}}^0}{\partial a} - \frac{|k|}{2} \right) \widehat{\delta a}(k) \right] = K_{\text{lc}}^0 \left[ 1 + \left( \frac{\widehat{\mathcal{A}}(|k|\omega_0)}{\omega_0} - \frac{|k|}{2} \right) \widehat{\delta a}(k) + \left[ \widehat{\Sigma}(|k|\omega_0) - \frac{\widehat{\Omega}(|k|\omega_0)}{2} \right] \frac{\widehat{\delta\sigma_c}(k)}{\sigma_c^0} + \frac{\widehat{\Omega}(|k|\omega_0)}{2} \frac{\widehat{\delta D_c}(k)}{D_c^0} \right] \tag{B.18}$$

665 Note that, as  $\widehat{\Omega}(|k|\omega_0)$  decreases more strongly than  $\widehat{\Sigma}(|k|\omega_0)$  when  $|k|\omega_0 \rightarrow +\infty$  (see Appendix  
666 C), heterogeneities of strength  $\sigma$  with constant critical opening  $D_c$  have a similar effect to that  
667 with constant process zone size described in Section 4.2. Conversely, heterogeneities of critical  
668 opening  $D_c$  with constant strength  $\sigma$  have a similar effect to that of process zone size described  
669 in Section 4.3.

670 **Appendix C. Expression and asymptotic behavior of  $\hat{\mathcal{A}}(|k|\omega_0)$ ,  $\hat{\Sigma}(|k|\omega_0)$ ,  $\hat{\Omega}(|k|\omega_0)$  for dif-**  
671 **ferent types of weakening**

672 In this work, we consider four different cohesive laws. The first one is the linear distance-  
673 weakening law of Fig. 4 for which the cohesive stress decays linearly from  $\sigma_c^0$  to 0 along a  
674 distance  $\omega_0$ . The second one is the cohesive law of Dugdale (1960) and Barenblatt (1962) for  
675 which the cohesive stress is constant to  $\sigma_c^0$  along a distance  $\omega_0$ , and 0 elsewhere. The third one  
676 is an exponentially distance-weakening law for which the cohesive stress decays as  $\sigma_c^0 e^{-x/\omega_0}$   
677 behind the crack tip, and never reaches 0. The last one is a linear traction-separation law for  
678 which the stress decay linearly from  $\sigma_c^0$  to 0 with the local opening  $\delta$  up to a critical value  $D_c$ .  
679 In this Appendix, we derive the expressions and the asymptotic behaviors for the three cohesive  
680 pre-factors  $\hat{\mathcal{A}}$ ,  $\hat{\Sigma}$  and  $\hat{\Omega}$  of Eq. (24) that control the front deformations. The first three cohesive  
681 laws yield analytical expressions for  $\hat{\mathcal{A}}$ ,  $\hat{\Sigma}$  and  $\hat{\Omega}$ , while they are computed numerically for the  
682 traction-separation law.

683 *Appendix C.1. Linear-distance weakening*

684 The first one is the linear distance-weakening law of Fig. 4 for which the cohesive stress  
685 decay linearly from  $\sigma_c^0$  to 0 along a distance  $\omega_0$ :

$$f_w(x) = \max(1 - x, 0) \quad (\text{C.1})$$

686 In that case, Eq. (24) yields:

$$\begin{cases} \hat{\mathcal{A}}(|k|\omega_0) = \frac{3}{2} - \frac{3\sqrt{\pi} \operatorname{erf}(\sqrt{|k|\omega_0})}{4\sqrt{|k|\omega_0}} & \hat{\mathcal{A}}(|k|\omega_0) \xrightarrow{|k|\omega_0 \rightarrow +\infty} \frac{3}{2} \\ \hat{\Sigma}(|k|\omega_0) = \frac{3\sqrt{\pi}}{4} \left( |k|\omega_0 - \frac{1}{2} \right) \frac{\operatorname{erf}(\sqrt{|k|\omega_0})}{(|k|\omega_0)^{3/2}} + \frac{3}{4} \frac{e^{-|k|\omega_0}}{|k|\omega_0} & \hat{\Sigma}(|k|\omega_0) \underset{|k|\omega_0 \rightarrow +\infty}{\sim} \frac{3\sqrt{\pi}}{4} (|k|\omega_0)^{-1/2} \\ \hat{\Omega}(|k|\omega_0) = \frac{3\sqrt{\pi} \operatorname{erf}(\sqrt{|k|\omega_0})}{4(|k|\omega_0)^{3/2}} - \frac{3}{2} \frac{e^{-|k|\omega_0}}{|k|\omega_0} & \hat{\Omega}(|k|\omega_0) \underset{|k|\omega_0 \rightarrow +\infty}{\sim} \frac{3\sqrt{\pi}}{4} (|k|\omega_0)^{-3/2} \end{cases} \quad (\text{C.2})$$

687 The evolution of  $\hat{\mathcal{A}}$ ,  $\hat{\Sigma}$  and  $\hat{\Omega}$  with  $|k|\omega_0$  is given in Fig. C.10 for the linear distance-weakening  
688 law.

689 *Appendix C.2. Dugdale-Barenblatt distance weakening*

690 The second one is the cohesive law of Dugdale (1960) and Barenblatt (1962) for which the  
691 cohesive stress is constant to  $\sigma_c^0$  along a distance  $\omega_0$ , and 0 elsewhere:

$$f_w(x) = \chi_{[0,1]}(x) \quad (\text{C.3})$$

692 where  $\chi_{\mathcal{E}}$  is the indicator function of the ensemble  $\mathcal{E}$ . Eq. (24) yields:

$$\begin{cases} \hat{\mathcal{A}}(|k|\omega_0) = \frac{1 - e^{-|k|\omega_0}}{2} & \hat{\mathcal{A}}(|k|\omega_0) \xrightarrow{|k|\omega_0 \rightarrow +\infty} \frac{1}{2} \\ \hat{\Sigma}(|k|\omega_0) = \frac{\sqrt{\pi} \operatorname{erf}(\sqrt{|k|\omega_0})}{2\sqrt{|k|\omega_0}} & \hat{\Sigma}(|k|\omega_0) \underset{|k|\omega_0 \rightarrow +\infty}{\sim} \frac{\sqrt{\pi}}{2} (|k|\omega_0)^{-1/2} \\ \hat{\Omega}(|k|\omega_0) = e^{-|k|\omega_0} & \hat{\Omega}(|k|\omega_0) \underset{|k|\omega_0 \rightarrow +\infty}{\sim} e^{-|k|\omega_0} \end{cases} \quad (\text{C.4})$$

693 The evolution of  $\hat{\mathcal{A}}$ ,  $\hat{\Sigma}$  and  $\hat{\Omega}$  with  $|k|\omega_0$  is given in Fig. C.11 for the Dugdale-Barenblatt weak-  
694 ening law.

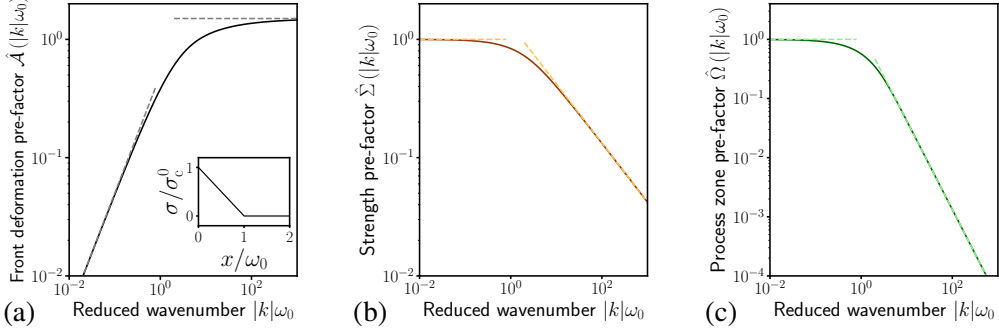


Figure C.10: Evolution of the cohesive pre-factors for a linear distance-weakening cohesive law: (a) the front deformations pre-factor  $\hat{\mathcal{A}}(|k|\omega_0)$  scales as  $\propto |k|\omega_0/2$  when  $|k|\omega_0 \rightarrow 0$  (dashed gray line on the left) and saturates to  $\mathcal{A}_w^\infty$  when  $|k|\omega_0 \rightarrow +\infty$  (horizontal dashed gray line on the right). (b) The strength variation pre-factor  $\hat{\Sigma}(|k|\omega_0)$  goes to 1 in the brittle limit  $|k|\omega_0 \rightarrow 0$  and decays as  $(|k|\omega_0)^{-1/2}$  when  $|k|\omega_0 \rightarrow +\infty$ . (c) The process zone size variations pre-factor  $\hat{\Omega}(|k|\omega_0)$  goes to 1 when  $|k|\omega_0 \rightarrow 0$  and decays as  $(|k|\omega_0)^{-3/2}$  when  $|k|\omega_0 \rightarrow +\infty$ .

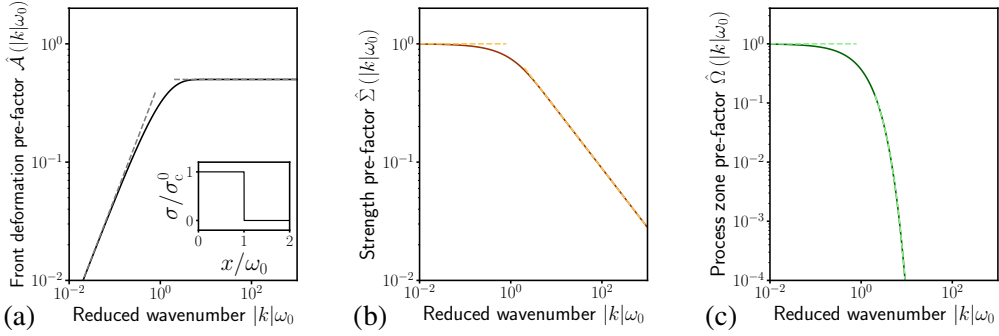


Figure C.11: Evolution of the cohesive pre-factors for a Dugdale-Barenblatt cohesive law: (a) the front deformations pre-factor  $\hat{\mathcal{A}}(|k|\omega_0)$  scales as  $\propto |k|\omega_0/2$  when  $|k|\omega_0 \rightarrow 0$  (dashed gray line on the left) and saturates to  $\mathcal{A}_w^\infty$  when  $|k|\omega_0 \rightarrow +\infty$  (horizontal dashed gray line on the right). (b) The strength variation pre-factor  $\hat{\Sigma}(|k|\omega_0)$  goes to 1 in the brittle limit  $|k|\omega_0 \rightarrow 0$  and decays as  $(|k|\omega_0)^{-1/2}$  when  $|k|\omega_0 \rightarrow +\infty$ . (c) The process zone size variations pre-factor  $\hat{\Omega}(|k|\omega_0)$  goes to 1 when  $|k|\omega_0 \rightarrow 0$  and decays as  $e^{-|k|\omega_0}$  when  $|k|\omega_0 \rightarrow +\infty$ .

### 695 Appendix C.3. Exponential-distance weakening

696 The third one is an exponentially distance-weakening law for which the cohesive stress de-  
 697 cays as  $\sigma_c^0 e^{-x/\omega_0}$  behind the crack tip, and never reaches 0:

$$f_w(x) = e^{-x} \quad (\text{C.5})$$

698 In that case, Eq. (24) yields:

$$\begin{cases} \hat{\mathcal{A}}(|k|\omega_0) = 1 - \frac{1}{(1 + |k|\omega_0)^{1/2}} & \hat{\mathcal{A}}(|k|\omega_0) \xrightarrow{|k|\omega_0 \rightarrow +\infty} 1 \\ \hat{\Sigma}(|k|\omega_0) = \frac{1}{(1 + |k|\omega_0)^{1/2}} & \Rightarrow \hat{\Sigma}(|k|\omega_0) \underset{|k|\omega_0 \rightarrow +\infty}{\sim} (|k|\omega_0)^{-1/2} \\ \hat{\Omega}(|k|\omega_0) = \frac{1}{(1 + |k|\omega_0)^{3/2}} & \hat{\Omega}(|k|\omega_0) \underset{|k|\omega_0 \rightarrow +\infty}{\sim} (|k|\omega_0)^{-3/2} \end{cases} \quad (\text{C.6})$$

699 The evolution of  $\hat{\mathcal{A}}$ ,  $\hat{\Sigma}$  and  $\hat{\Omega}$  with  $|k|\omega_0$  is given in Fig. C.12 for the exponential distance-  
700 weakening law.

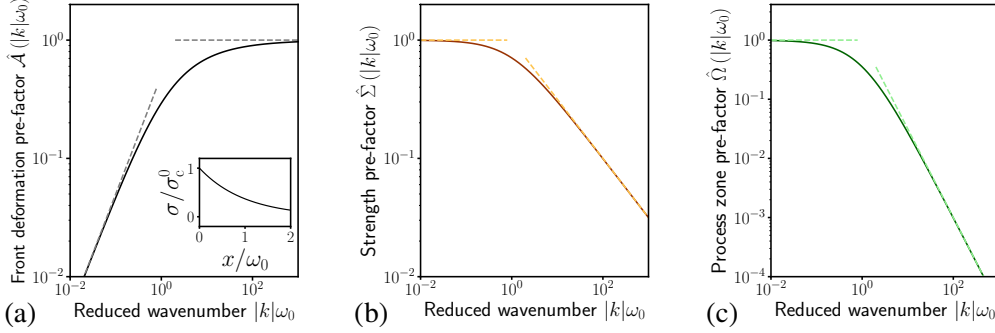


Figure C.12: Evolution of the cohesive pre-factors for an exponential distance-weakening cohesive law: (a) the front deformations pre-factor  $\hat{\mathcal{A}}(|k|\omega_0)$  scales as  $\propto |k|\omega_0/2$  when  $|k|\omega_0 \rightarrow 0$  (dashed gray line on the left) and saturates to  $\mathcal{A}_w^\infty$  when  $|k|\omega_0 \rightarrow +\infty$  (horizontal dashed gray line on the right). (b) The strength variation pre-factor  $\hat{\Sigma}(|k|\omega_0)$  goes to 1 in the brittle limit  $|k|\omega_0 \rightarrow 0$  and decays as  $(|k|\omega_0)^{-1/2}$  when  $|k|\omega_0 \rightarrow +\infty$ . (c) The process zone size variations pre-factor  $\hat{\Omega}(|k|\omega_0)$  goes to 1 when  $|k|\omega_0 \rightarrow 0$  and decays as  $(|k|\omega_0)^{-3/2}$  when  $|k|\omega_0 \rightarrow +\infty$ .

#### 701 Appendix C.4. Linear-slip weakening

702 The fourth and last one is a linear traction-separation law for which the stress decays linearly  
703 from  $\sigma_c^0$  to 0 with the local opening  $\delta$  up to a critical value  $D_c$ :

$$f_w(x) = \max(1 - \delta(x)/D_c, 0) \quad (\text{C.7})$$

704 This type of weakening is widely used in numerical simulations of cohesive fracture. One cannot  
705 find an analytical formula in that case, but  $f_w$  can be computed numerically using the method of  
706 Viesca and Garagash (2018) that builds on a Gauss–Chebyshev quadrature. The resulting  $f_w$  is  
707 plotted in Fig. E.17a.

708  
709 The finiteness of the stress at the crack tip implies that  $f_w(x) \underset{x \rightarrow 0}{\sim} 1 - a_w x^{3/2}$  where  $a_w$  is a  
710 cohesive constant related to the weakening shape (Rice, 1980). This very peculiar behavior leads  
711 a change in the asymptotic behavior of  $\hat{\Omega}$  when  $|k|\omega_0 \rightarrow +\infty$ :

$$\hat{\Omega}(|k|\omega_0) \underset{|k|\omega_0 \rightarrow +\infty}{\sim} \frac{\Omega_w^\infty}{(|k|\omega_0)^2} \text{ if } f_w(x) \underset{x \rightarrow 0}{\sim} 1 - a_w x^{3/2}, \text{ with } \Omega_w^\infty = \frac{2a_w}{C_w} \quad (\text{C.8})$$

712 The behavior of  $\hat{\mathcal{A}}$  and  $\hat{\Sigma}$  is left unchanged. For the linear traction-separation law, one finds:

$$\begin{cases} \hat{\mathcal{A}}(|k|\omega_0) & \underset{|k|\omega_0 \rightarrow +\infty}{\simeq} 1.045 \\ \hat{\Sigma}(|k|\omega_0) & \underset{|k|\omega_0 \rightarrow +\infty}{\simeq} 1.209 (|k|\omega_0)^{-1/2} \\ \hat{\Omega}(|k|\omega_0) & \underset{|k|\omega_0 \rightarrow +\infty}{\simeq} 1.946 (|k|\omega_0)^{-2} \end{cases} \quad (\text{C.9})$$

713 The evolution of  $\hat{\mathcal{A}}$ ,  $\hat{\Sigma}$  and  $\hat{\Omega}$  with  $|k|\omega_0$  is given for the linear traction-separation cohesive law in  
714 Fig. C.13.

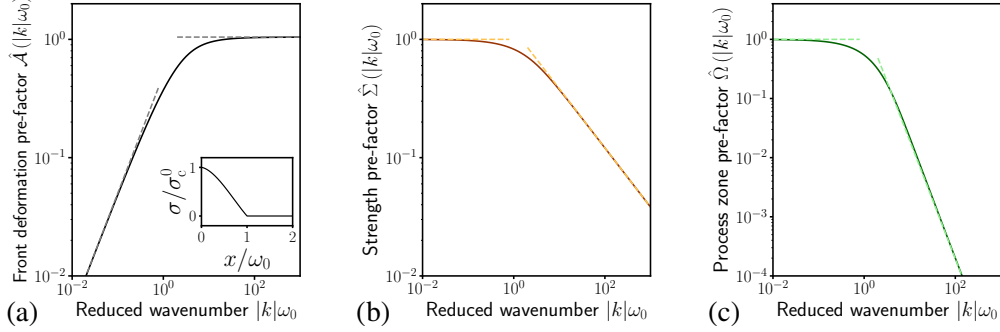


Figure C.13: Evolution of the cohesive pre-factors for a linear traction-separation law: (a) the front deformations pre-factor  $\hat{\mathcal{A}}(|k|\omega_0)$  scales as  $\propto |k|\omega_0/2$  when  $|k|\omega_0 \rightarrow 0$  (dashed gray line on the left) and saturates to  $\mathcal{A}_w^\infty$  when  $|k|\omega_0 \rightarrow +\infty$  (horizontal dashed gray line on the right). (b) The strength variation pre-factor  $\hat{\Sigma}(|k|\omega_0)$  goes to 1 in the brittle limit  $|k|\omega_0 \rightarrow 0$  and decays as  $(|k|\omega_0)^{-1/2}$  when  $|k|\omega_0 \rightarrow +\infty$ . (c) The process zone size variations pre-factor  $\hat{\Omega}(|k|\omega_0)$  goes to 1 when  $|k|\omega_0 \rightarrow 0$  and decays as  $(|k|\omega_0)^{-2}$  when  $|k|\omega_0 \rightarrow +\infty$ .

## 715 Appendix D. Influence of the inclusion spacing

716 In this Appendix, we explore the influence of the inclusion spacing on the deformations of a  
717 crack front, interacting with heterogeneities of strength and process zone size.

### 718 Appendix D.1. Heterogeneities of strength and process zone size

719 We consider the situations of Sections 4.2 and 4.3. The crack front interacts with hetero-  
720 geneities of (i) varying strength but uniform process zone size, or (ii) uniform strength but vary-  
721 ing process zone size. In the former case, an increase of the average process zone size  $\omega_0$  was  
722 associated with an increase in the front amplitude. In the latter, it was linked to a decrease in  
723 amplitude. We observe in Figs. D.14 and D.15 that the two different behaviors are left unchanged  
724 by the inclusion spacing.

### 725 Appendix D.2. Limit case of the single defect in an infinite matrix

726 A general expression of the front deformations  $\delta a$  with the position  $z$  cannot be found ex-  
727 plicitly. Yet, it is possible to derive it in some specific cases. In the limit of a single obstacle  
728 ( $\Delta L \gg d$ ) embedded in a rather brittle medium ( $\omega_0 \ll d$ ), one retrieves the solution of (Chopin  
729 et al., 2011):

$$(\delta a(z) - \delta a(0))/d = \frac{1}{2\pi} \frac{\Delta G_c}{G_c^0} \left[ \left(1 + \frac{2z}{d}\right) \ln \left|1 + \frac{2z}{d}\right| + \left(1 - \frac{2z}{d}\right) \ln \left|1 - \frac{2z}{d}\right| \right] \quad (\text{D.1})$$

730 where  $d$  is the obstacle width,  $\Delta G_c$  is the contrast in fracture energy and  $G_c^0$  its average value.  
731 The evolution of the front deformations  $\delta a$  with the position is plotted in Fig. D.16a.

732  
733 On the contrary, it is also possible to derive an analytical expression of the front deformation  
734 when the same obstacle lies in a very ductile medium ( $\omega_0 \gg d$ ). For heterogeneities of strength,  
735 one finds:

$$(\delta a(z) - \delta a(0))/d = \frac{2}{\sqrt{\pi}} \frac{\Sigma_w^\infty}{\mathcal{A}_w^\infty} \frac{\Delta \sigma_c}{\sigma_c^0} \left(\frac{\omega_0}{d}\right)^{1/2} \left[ 2 - \sqrt{1 + \frac{2|z|}{d}} + \text{sgn}\left(|z| - \frac{d}{2}\right) \sqrt{\left|1 - \frac{2|z|}{d}\right|} \right] \quad (\text{D.2})$$



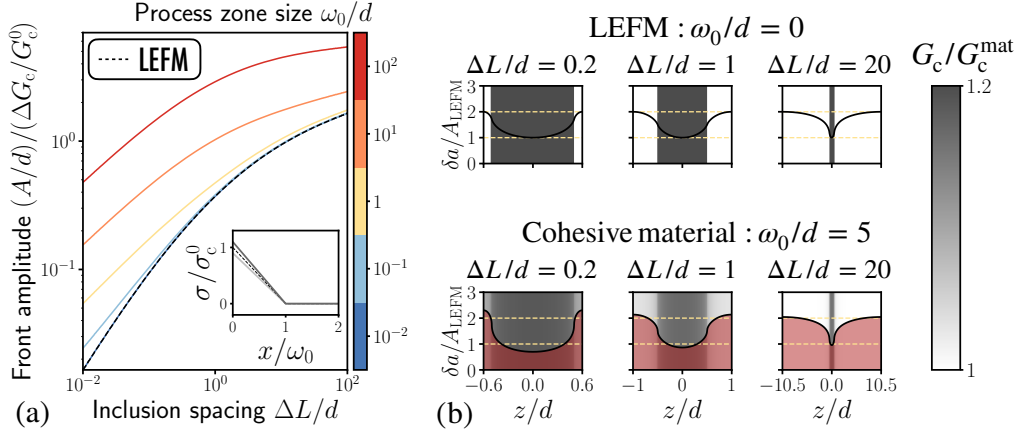


Figure D.14: (a) Influence of the inclusion spacing  $\Delta L$  and the process zone size  $\omega_0$  on the amplitude  $A$  of the front deformations for a periodic array of obstacles of width  $d$ ; Inset: the obstacles (in solid dark gray line) are stronger than the matrix  $\sigma_c^{obs} = 1.1\sigma_c^{mat}$  (in solid light gray line), but they are equally brittle  $\omega_0^{obs} = \omega_0^{mat}$ , so that  $G_c^{obs} \simeq 1.2G_c^{mat}$ . (b) The deformation amplitude is always larger than in the LEFM case (in yellow dashed line) and increases with the average process zone size  $\omega_0$  (in red).

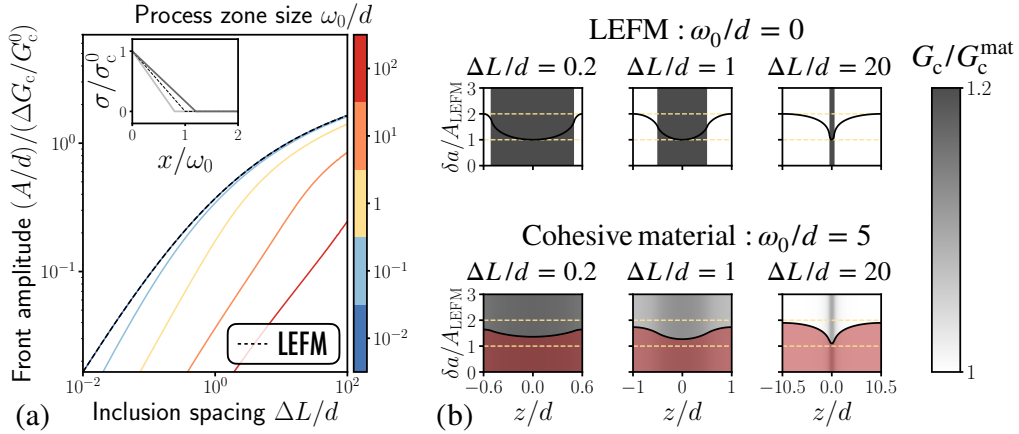


Figure D.15: (a) Influence of the inclusion spacing  $\Delta L$  and the process zone size  $\omega_0$  on the amplitude  $A$  of the front deformations for a periodic array of obstacles of width  $d$ ; Inset: the obstacles (in solid dark gray line) are more ductile than the matrix  $\omega_0^{obs} = 1.2\omega_0^{mat}$  (in solid light gray line), but they are equally strong  $\sigma_c^{obs} = \sigma_c^{mat}$ , so that  $G_c^{obs} \simeq 1.2G_c^{mat}$ . (b) The deformation amplitude is always smaller than in the LEFM case (in yellow dashed line) and decreases with the average process zone size  $\omega_0$  (in red).

736 where  $d$  is the obstacle width,  $\omega_0$  is the average process zone size,  $\Delta\sigma_c$  is the contrast in strength  
737 and  $\sigma_c^0$  its average value.  $\mathcal{A}_w^\infty$  and  $\Sigma_w^\infty$  relate to the cohesive law and are given in Eqs. (33) and  
738 (47). The evolution of the front deformations  $\delta a$  with the position is plotted in Fig. D.16b, and  
739 successfully compared to the numerical solution of Eq. (46) for  $L_z/d = 10^5$  and  $\omega_0/d = 10^5$ .  
740 Eq. (D.2) validates the linear dependence in  $(\Delta\sigma_c/\sigma_c^0)$  and that in  $(\omega_0/d)^{1/2}$  that was mentioned  
741 in Section 4.2. Moreover, we observe that the deformation localizes at the edges of the obsta-

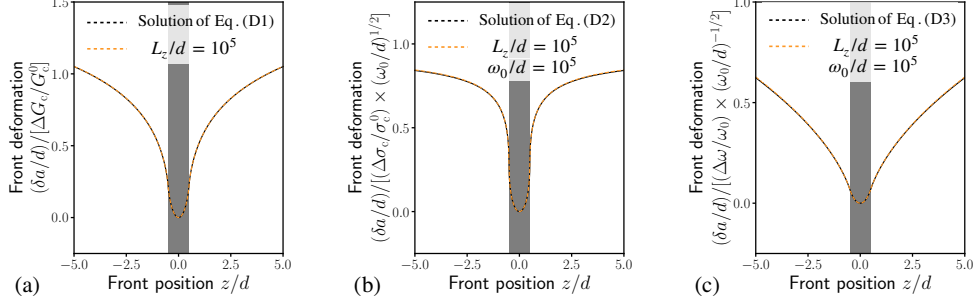


Figure D.16: Deformation profiles of a crack front pinned by a single tough heterogeneity ( $\Delta L/d \rightarrow +\infty$ ): (a) In the perfectly brittle limit  $\omega_0/d \rightarrow +\infty$ , the deformations of the front follow Eq. (D.1) of Chopin et al. (2011). In the fully cohesive limit  $\omega_0/d \rightarrow +\infty$ , (b) the deformations of the front by heterogeneities of strength follow Eq. (D.2) and increase as  $\propto (\omega_0/d)^{1/2}$ , while (c) that induced by heterogeneities of process zone size follow Eq. (D.3) and vanish as  $\propto (\omega_0/d)^{-1/2}$ . The analytical solution (in solid black line) are compared to numerical solutions (in dashed orange line) computed from Eq. (25).

742 cle, where  $\delta a$  is non-differentiable. It may challenge the assumptions lying under our first-order  
 743 theory as we assumed  $\partial \delta a / \partial z \ll 1$ , justifying future derivation of a second-order theory for the  
 744 quasi-static cohesive front deformations to rationalize the experimental results of Chopin et al.  
 745 (2011).

746  
 747 For heterogeneities of process zone size, one finds:

$$(\delta a(z) - \delta a(0)) / d = \frac{1}{6\sqrt{\pi}} \frac{\Omega_w^\infty}{\mathcal{A}_w^\infty} \frac{\Delta\omega}{\omega_0} \left(\frac{\omega_0}{d}\right)^{-1/2} \left[ -2 + \left(1 + \frac{2|z|}{d}\right)^{3/2} - \operatorname{sgn}\left(|z| - \frac{d}{2}\right) \left|1 - \frac{2|z|}{d}\right|^{3/2} \right] \quad (\text{D.3})$$

748 where  $d$  is the obstacle width,  $\Delta\omega$  is the contrast in process zone size and  $\omega_0$  its average value.  
 749  $\mathcal{A}_w^\infty$  and  $\Omega_w^\infty$  relate to the cohesive law and are given in Eqs. (33) and (51). The evolution of the  
 750 front deformations  $\delta a$  with the position is plotted in Fig. D.16c. Eq. (D.3) confirms the linear  
 751 dependence in both  $(\Delta\omega/\omega_0)$  and  $(\omega_0/d)^{-1/2}$ .

752 Note that Eq. (D.3) strongly depends on the weakening shape  $f_w$ , as it controls the asymptotic  
 753 behavior of  $\hat{\Omega}$ . For example, one finds for the linear traction-separation law:

$$(\delta a(z) - \delta a(0)) / d = \begin{cases} \frac{\Omega_w^\infty}{4\mathcal{A}_w^\infty} \frac{\Delta\omega}{\omega_0} \left(\frac{\omega_0}{d}\right)^{-1} \left(\frac{z}{d}\right)^2 & \text{if } |z| \leq d/2 \\ \frac{\Omega_w^\infty}{4\mathcal{A}_w^\infty} \frac{\Delta\omega}{\omega_0} \left(\frac{\omega_0}{d}\right)^{-1} \left(\frac{|z|}{d} - \frac{1}{4}\right) & \text{if } |z| > d/2 \end{cases} \quad (\text{D.4})$$

## 754 Appendix E. Influence of the nature of weakening

755 In this Appendix, we explore the influence of the shape  $f_w$  of the cohesive law on the sta-  
 756 bility of crack fronts and their interaction with periodic arrays of tough obstacles. We consider  
 757 the four weakening shapes studied in Appendix C: the reference linear distance-weakening,  
 758 a Dugdale-Barenblatt weakening, an exponential distance-weakening, and a more conventional  
 759 linear traction-separation cohesive law.

760 *Appendix E.1. Influence of the nature of weakening on the crack front stability*

761 We saw in Eqs. (35) and (36) that the stability of a perturbed crack strongly depends on the  
 762 evolution of  $\hat{\mathcal{A}}$  and its asymptotic behavior as  $|k|\omega_0 \rightarrow +\infty$ . We thus expect that the stability of  
 763 crack fronts somehow depends on the spatial distribution  $f_w$  of cohesive stress in the crack wake.

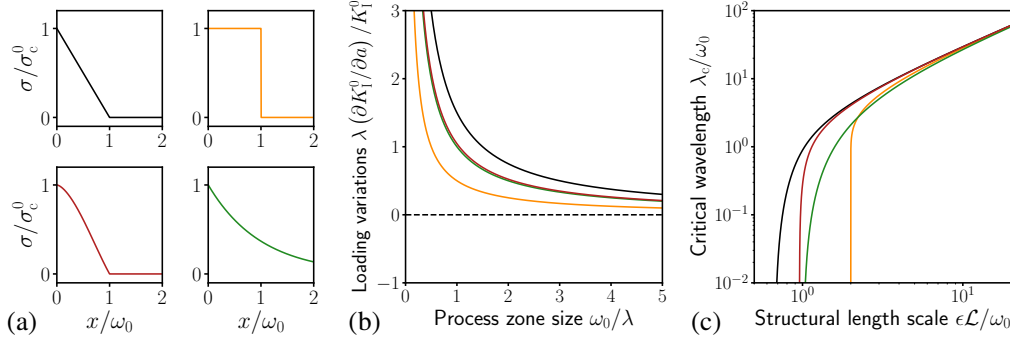


Figure E.17: Influence of the nature of weakening  $f_w$  on the stability of a sinusoidal crack front: (a) linear distance-weakening (in black line), Dugdale-Barenblatt (in orange line), linear traction-separation (in red line), and exponential distance-weakening (in green line) cohesive laws are considered. (b) The frontier delimiting the region of conditional (Regime III) to unconditional instability (Regime II) is controlled by the evolution of  $\hat{\mathcal{A}}$  which is set by  $f_w$ . (c) It also controls the value of the critical wavelength  $\lambda_c$  above which modal perturbations are unstable, and the value of the critical structural length scale  $\mathcal{L}_c$  below which perturbations of any wavelengths are unstable.

764 We observe in Fig. E.17 that all four cohesive laws lead to a similar stability behavior. The  
 765 main difference is a shift of the frontier separating Regimes II and III, as it is controlled by  
 766 the asymptotic value  $\mathcal{A}_w^\infty$  of the operator  $\hat{\mathcal{A}}$ . Namely, the shift from the two regimes occurs for  
 767 smaller process zone size  $\omega_0$  for the Dugdale-Barenblatt cohesive law than for the linear distance-  
 768 weakening law. The cases of exponentially distance-weakening and linear traction-separation  
 769 laws are found in between.

770 *Appendix E.2. Influence of the nature of weakening on the crack front deformations*

771 We address next the question of the influence of the nature of weakening, characterized by  
 772 the weakening shape  $f_w$ , on the front deformations.

773 We first observe in Fig. E.18 that the influence of the weakening shape on the front deforma-  
 774 tion is rather weak for heterogeneities of strength. The deformations are overall larger for a  
 775 Dugdale-Barenblatt law, and lower for a linear distance-weakening law. The other two laws are  
 776 found in-between. They all display a similar scaling  $A/d \propto (\omega_0/d)^{1/2}$  for large process zone sizes  
 777  $\omega_0/d \rightarrow +\infty$ , as the asymptotic behavior of  $\hat{\Sigma}$  of Eq. (47) depends very weakly on  $f_w$ .

778 The case of heterogeneities of process zone size is more interesting, as the asymptotic behav-  
 779 ior of  $\hat{\Omega}$  changes significantly with the definition of  $f_w$ . Indeed, the decrease in magnitude of the  
 780 effective fluctuations of fracture energy is sharper for traction-separation laws  $A/d \propto (\omega_0/d)^{-1}$   
 781 than for distance-weakening ones  $A/d \propto (\omega_0/d)^{-1/2}$  (as long as  $f_w'(0) \neq 0$ ). Consequently, the  
 782 amplitude of front deformations vanishes at a much higher rate as the average process zone size  
 783  $\omega_0$  gets larger than the obstacle width  $d$  (see Fig. E.19). This effect is even stronger for the  
 784 Dugdale-Barenblatt law, for which  $A/d \propto (\omega_0/d) \times e^{-\omega_0/d}$ .

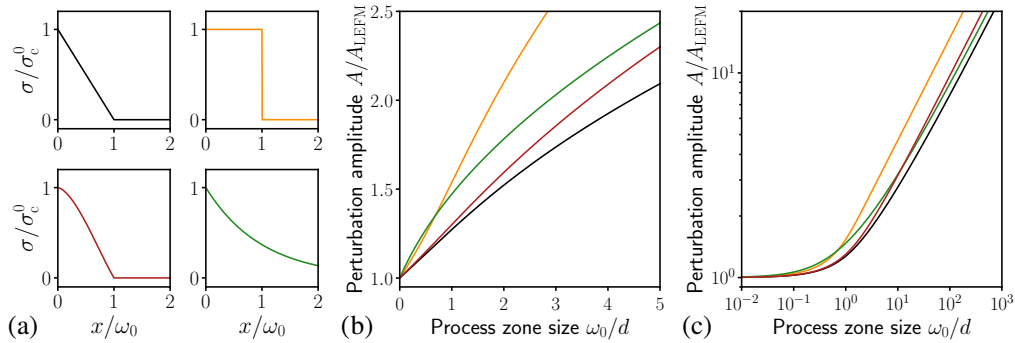


Figure E.18: Influence of the nature of weakening  $f_w$  on the front deformations associated with spatial variations of strength: (a) linear distance-weakening (in black line), Dugdale-Barenblatt (in orange line), linear traction-separation (in red line), and exponential distance-weakening (in green line) cohesive laws are considered. (b) The perturbation amplitude always increases with the average process zone size, but at a rate controlled by the weakening  $f_w$ . (c) It scales as  $\propto (\omega_0/d)^{1/2}$  when  $\omega_0/d \rightarrow +\infty$  no matter how the material weakens.

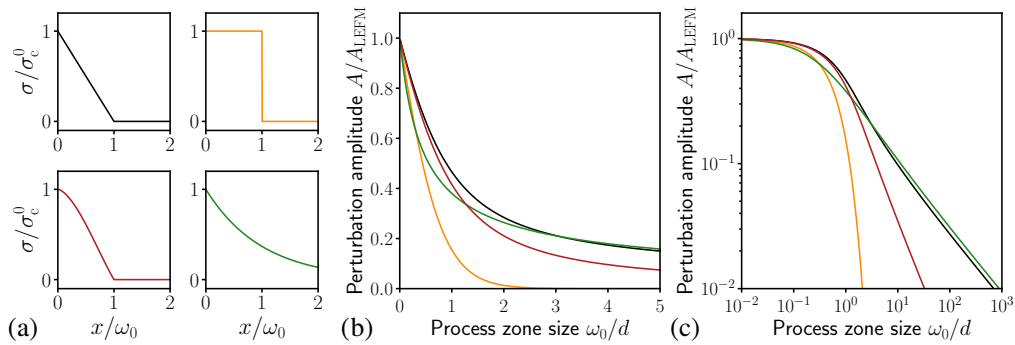


Figure E.19: Influence of the nature of weakening  $f_w$  on the front deformation emerging from spatial variations of process zone size: (a) linear distance-weakening (in black line), Dugdale-Barenblatt (in orange line), linear traction-separation (in red line), and exponential distance-weakening (in green line) cohesive laws are considered. (b) The perturbation amplitude always decreases with the average process zone size  $\omega_0$  but at a rate controlled by the weakening  $f_w$ . (c) Its scaling as when  $\omega_0/d \rightarrow +\infty$  depends on how the material weakens. For distance-weakening laws, it scales as  $\propto (\omega_0/d)^{-1/2}$ , while it decreases as  $\propto (\omega_0/d)^{-1}$  for traction-separation laws. The decrease is even stronger for the Dugdale-Barenblatt cohesive law  $\propto (\omega_0/d)e^{-\omega_0/d}$ .

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