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# 1 **A new multifractal-based Grain Size Distribution model**

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3 Filip STANIĆ<sup>1,2</sup>, Ioulia TCHIGUIRINSKAIA<sup>1</sup>, Pierre-Antoine VERSINI<sup>1</sup>, Yu-Jun CUI<sup>2</sup>,  
4 Pierre DELAGE<sup>2</sup>, Patrick AIMEDIEU<sup>3</sup>, Ana Maria TARQUIS<sup>4</sup>, Michel BORNERT<sup>3</sup>, Daniel  
5 SCHERTZER<sup>1</sup>

6

7 <sup>1</sup> Ecole des Ponts ParisTech, HM&Co, Marne la Vallée, France

8 <sup>2</sup> Ecole des Ponts ParisTech, Navier/CERMES, Marne la Vallée, France

9 <sup>3</sup> Ecole des Ponts ParisTech, Navier/Multi-échelle, Marne la Vallée, France

10 <sup>4</sup> Universidad Politécnica de Madrid (UPM), Dept. of Applied Mathematics, Madrid, Spain

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24 Corresponding author:

25 Filip Stanić

26 Ecole des Ponts ParisTech

27 6-8 av. Blaise Pascal, Cité Descartes, Champs-sur-Marne

28 77455 Marne-la-Vallée cedex 2

29 France

30

31 Email: [filip.stanic@enpc.fr](mailto:filip.stanic@enpc.fr)

32 Orcid: 0000-0003-0271-5993

### 33 **Abstract**

34 Previous works related to the application of the multifractal theory for analyzing the grain size  
35 distribution (GSD), showed the potential of this approach to deal with this complex issue.  
36 However, absence of the practical application of this kind of statistical analysis raised some  
37 doubts among the soil scientists. Compared to the experimental dry sieving method, which is  
38 based on mass representations of different grain sizes, the approach presented in this work  
39 relies on the analysis of grain densities (density indicators) scanned by means of X-ray CT  
40 (Computed Tomography). By reducing the resolution of the scanned soil image(s), the  
41 cumulative representation of solid particles equal to or larger than the actual discretization  
42 element can be determined, and described analytically by means of the universal multifractals  
43 (UM).

44 For validation of the new UM approach, the X-ray CT results of three different soils were  
45 used: the volcanic substrate covering Green Wave (a green roof of Champs-sur-Marne in  
46 France), and two horizons of the soil collected from the low land mountain area of Sierra de  
47 Guadarrama in Spain. Comparison between the proposed UM model and the experimental  
48 data of these three materials confirms that the GSD can be reasonably well predicted from the  
49 scanned images of soils covering wide range of grain sizes. The UM model, unlike the fractal-  
50 based models, accounts for fractal dimension that depends on grain size, and hence, based on  
51 the preliminary results presented in this work, it could be rather useful in case of multi-modal  
52 soils whose GSD curves are described with multiple fractal dimensions.

53

54 **Key words:** grain size distribution; multifractals; X-ray computed tomography; granular  
55 media

## 56 **1. Introduction**

57 The grain size distribution (GSD) is one of the fundamental properties of granular soils that,  
58 besides the influence on mechanical characteristics, also affects the packing arrangement of  
59 grains (Nolan & Kavanagh, 1993; He et al., 1999 among the others), and hence the  
60 distribution of pores that further impacts the hydraulic properties of the porous medium (Segal  
61 et al., 2009). Most often the GSD curve is experimentally determined based on the mass  
62 fractions of different grain sizes extracted either by using sieves of different void sizes, for  
63 grains larger than 80 microns (dry sieving method - AFNOR, 1996), or by means of  
64 sedimentation test (AFNOR, 1992; Beuselinck et al., 1998) for finer particles. The alternative  
65 approach proposed to measure GSD is a laser diffraction method (Miller & Schaetzl, 2012).

66 Detailed overview of different approaches used for describing the complexity of GSD curves  
67 can be found in Ghanbarian & Hunt (2017). One of them is the self-similarity principle which  
68 is included in fractal-based models and which assumes occurrence of the same pattern of the  
69 soil structure at all scales. According to Ghanbarian-Alavijeh et al. (2011), the three-phase  
70 PSF (pore-solid-fractal) approach (Perrier et al., 1999; Bird et al., 2000) is the most consistent  
71 and with the strongest physical-basis among the fractal-based approaches. Besides pores and  
72 grains, it assumes one additional “fictive” type of soil elements – fractals - that are  
73 successively broken at smaller scales in a self-similar way, leading finally to the structure  
74 consisting of fractal-distributed pore and grain sizes. Thus, the GSD can be represented by  
75 means of a power (fractal) law, where the fractal coefficient is included in the exponent.  
76 However, unlike assumed in the PSF model, grain densities are non-homogeneous, which also  
77 contributes to the complexity of distribution of different mass fractions that often cannot be  
78 described with a single fractal dimension (Bittelli et al., 1999).

79 Multifractal formalism, that takes into account different fractal coefficients for different  
80 threshold values, was also used for analyzing the complexity of GSD. Grout et al. (1998) and  
81 Posadas et al. (2001) used Renyi dimensions, one of the multifractal parameters, to  
82 characterize the heterogeneous distribution of different mass fractions. Besides this type of  
83 multifractal analysis, the singularity spectra analysis is also applied for analyzing the dry soil  
84 volume-size distribution obtained by using a laser distraction method (Martín & Montero,  
85 2002). Recently, Torre et al. (2016) used a X-ray CT, a non-destructive technique for  
86 obtaining a three-dimensional grey-scale image of a porous material (Hseih, 2003; Banhart,  
87 2008) in order to compare the three-dimensional structural complexity of spatial arrangement  
88 of grains and pores, with that of differently oriented two-dimensional planes. The multifractal  
89 analysis has also proved to be convenient in this case. Even though the multifractal theory  
90 brings great potential for understanding the complexity of GSD (Ghanbarian & Hunt, 2017),  
91 up to date this kind of analysis has not found practical application.

92 This work is focused on development of a new physically-based GSD model founded on the  
93 Universal Multifractal (UM) framework (Schertzer & Lovejoy, 1987; Schertzer & Lovejoy,  
94 1997). Based on a grey-scale soil image scanned by means of X-ray CT, it is possible to  
95 recognize solid particles of different sizes by progressively decreasing the resolution of the  
96 image while keeping the fixed value of the threshold. Change of the representation of solid  
97 particles with the resolution of the image can be directly linked with the grain size  
98 distribution, and described analytically in a mathematically-elegant way by means of the UM  
99 framework. Compared to work of Lai & Chen (2018), where a sophisticated machine learning  
100 tool was used for particle recognition, this approach is much simpler and more convenient for  
101 practical application.

102 The UM framework in combination with X-ray CT imaging was firstly validated for some  
103 artificial volcanic substrate (Stanić et al., 2019; Stanić et al., 2020) used for covering green

104 roof named Green Wave (Versini et al., 2018; Versini et al., 2020). Results of the model,  
105 whose parameters are directly determined from scanned images, were first compared with the  
106 experimental data obtained by means of the standard dry sieving method (AFNOR, 1996) and  
107 sedimentation test (AFNOR, 1992). Furthermore, the UM model was tested on scanned  
108 images of two horizons of an intact soil sample collected from the low land mountain area of  
109 Sierra de Guadarrama (Schmid et al. 2016) called La Herreria. In this case, results of the  
110 model were compared with measured percentages of sand, silt and clay particles, since  
111 detailed GSD curves are lacking. Finally, for published experimental GSD data of the GW  
112 substrate and Walla Walla soil (Bittelli et al., 1999), the UM model was compared with the  
113 fractal-based PSF model.

## 114 **2. Methodology**

115 The GSD model proposed in this work is based on the recognition of solid particles of  
116 different sizes from the scanned soil image, by changing the resolution of the image. This can  
117 be described analytically through the application of the Universal Multifractals (UM)  
118 framework (Schertzer & Lovejoy, 1987; Schertzer & Lovejoy, 1997) which is briefly  
119 described below. Note that two-dimensional scanned soil images (Euclidian dimension  $E = 2$ ),  
120 extracted from the three-dimensional one ( $E = 3$ ), were analyzed in this work in order to  
121 simplify the methodology presented. However, this simplified approach ( $E = 2$ ) is credible  
122 only under certain conditions that are described later in the text, while otherwise the same  
123 methodology should be applied for  $E = 3$ .

### 124 **2.1. Universal Multifractal (UM) theoretical framework**

125 In Figure 1 is presented a renormalized two-dimensional soil density indicator field  $\rho^{ind}(\lambda)$  at  
126 various resolutions  $\lambda$ , which is, due to the better visualization, presented in a three-

127 dimensional form. Here,  $\lambda = \frac{L}{l(\lambda)}$  is equal to the ratio between the size of the image  $L$  [L] and  
 128 the size of a single pixel  $l(\lambda)$  [L], representing the number of pixels along an edge of the  
 129 image. Values of  $\rho^{ind}(\lambda) = \rho(\lambda) / \rho_{bulk}$  are presented as histogram at each  $\lambda$ , where  $\rho_{bulk}$  is the  
 130 constant bulk density of the dry material [M/L<sup>3</sup>] ( $\rho^{ind} = 1$  is a renormalized  $\rho_{bulk}$ ). Clearly,  
 131  $\rho^{ind}(\lambda)$  values mitigate as  $\lambda$  decreases by merging pixels in groups by  $\lambda_l^2$ , where  $\lambda_l$  is an  
 132 integer value equal 2 (check dashed gridlines in Figure 1E). By averaging  $\rho^{ind}(\lambda)$  values of  
 133 each group, attenuated  $\rho^{ind}(\lambda)$  field is obtained with  $\lambda_l$  times smaller  $\lambda$  and  $\lambda_l$  times larger  
 134 pixel size  $l(\lambda)$ .

135 By means of UM (Schertzer & Lovejoy, 1987; Schertzer & Lovejoy, 1997) it is possible to  
 136 compute, at different  $\lambda$ , the probability that  $\rho^{ind}(\lambda)$  exceeds  $\lambda^\gamma$ , a renormalized threshold value  
 137 that changes with  $\lambda$  (a transparent color platform in Figure 1). Threshold value is expressed  
 138 through the fixed dimensionless singularity  $\gamma$  [-], and therefore it decreases together with  $\lambda$   
 139 (see Figure 1a to Figure 1f) until it reaches unit value at  $\lambda = 1$ . For a certain value of  $\gamma$  (=   
 140 0.211 in Figure 1), the previously mentioned probability of exceeding can be computed as:

$$141 \quad P(\rho^{ind}(\lambda) \geq \lambda^\gamma) = \frac{N(\rho^{ind}(\lambda) \geq \lambda^\gamma)}{\lambda^E} \approx \lambda^{-c(\gamma)} \quad (1)$$

$$142 \quad c(\gamma) = C_1 \left( \frac{\gamma}{C_1 \alpha'} + \frac{1}{\alpha} \right)^{\alpha'} ; \quad \alpha' = \left( 1 - \frac{1}{\alpha} \right)^{-1}, \quad \alpha \neq 1 \quad (2)$$

143 where  $N(\rho^{ind}(\lambda) \geq \lambda^\gamma)$  is the number of  $\rho^{ind}(\lambda)$  values that are equal to or higher than  $\lambda^\gamma$ ,  
 144 while  $c(\gamma)$  is the co-dimension function that, besides  $\gamma$ , depends on two parameters,  $C_1$  and  $\alpha$ .

145 Note that in Equation (1) an approximate equal sign is used because term  $\frac{N(\rho^{ind}(\lambda) \geq \lambda^\gamma)}{\lambda^E}$  is  
 146 computed by counting  $N(\rho^{ind}(\lambda) \geq \lambda^\gamma)$  at different  $\lambda$  (discrete form), while  $\lambda^{-c(\gamma)}$  is related  
 147 to the UM analytical form.

148 Parameters  $C_l$  and  $\alpha$  fully characterize  $\rho^{ind}$  field, where  $C_l$  describes the sparseness of the  
149 mean value of the field while  $\alpha$  describes the change of sparseness for values around the  
150 mean. As explained in Schertzer & Lovejoy (1987),  $C_l$  takes values between 0 (mean value is  
151 ubiquitous - homogeneous field) and  $E$  (mean value is too sparse to be observed), while  $\alpha$   
152 takes values between 0 (no occurrence of extremes – fractal field) and 2 (maximal occurrence  
153 of extremes – log-normal field). Equation (1) is presented in Figure 2a for  $C_l = 2.23 \times 10^{-2}$  and  
154  $\alpha = 1.67$ , values that characterize  $\rho^{ind}(\lambda)$  field in Figure 1, and different  $\gamma$  values (including  $\gamma =$   
155 0.211) corresponding to various dashed lines.

## 156 2.2. Adaptation of the UM framework – New GSD model

157 In this work, the presented up-scaling procedure is used for recognizing solid particles of  
158 different sizes from the obtained  $\rho^{ind}(\lambda)$  field. Compared to the previous explanation, where a  
159 resolution dependent threshold  $\lambda^\gamma$  was accounted for, here is used a fixed threshold value  
160  $\rho_{s,min}^{ind}$  (solid platform in Figure 1) related to the renormalized minimal grain density ( $\rho_{s,min}^{ind} >$   
161 1). Therefore,  $\rho^{ind} \geq \rho_{s,min}^{ind}$  values indicate the total area of the image covered with solid  
162 particles (grains). As shown in Figure 1, this area reduces when up-scaling, mostly by getting  
163 rid of isolated  $\rho^{ind}(\lambda) \geq \rho_{s,min}^{ind}$  values that are surrounded by those lower than  $\rho_{s,min}^{ind}$ . On the  
164 contrary, larger continuous zones covered by  $\rho^{ind}(\lambda) \geq \rho_{s,min}^{ind}$  values resist longer to the up-  
165 scaling process, indicating the presence of a large grain on that location (central zone in  
166 Figure 1). Therefore, the total area covered with  $\rho^{ind}(\lambda) \geq \rho_{s,min}^{ind}$  values at certain  $\lambda$  indicates  
167 a cumulative representation of solid particles of diameter equal to or larger than the size of a  
168 single pixel  $l(\lambda) = L / \lambda$ :

$$169 \quad P(\rho^{ind}(\lambda) \geq \rho_{s,min}^{ind}) = \frac{N(\rho^{ind}(\lambda) \geq \rho_{s,min}^{ind})}{\lambda^E} \quad (3)$$

170 In order to transform Equation (3) into the distribution function  $P(d \geq l(\lambda))$ , it is necessary to  
 171 renormalize it with respect to the initial representation of solid particles met at  $\lambda_n \leq \lambda_{up}$ .  
 172 Therefore,  $P(d < l(\lambda)) = 1 - P(d \geq l(\lambda))$  can be expressed as:

$$173 \quad P(d < l(\lambda)) = 1 - \frac{P(\rho^{ind}(\lambda) \geq \rho_{s,min}^{ind})}{P(\rho^{ind}(\lambda_n) \geq \rho_{s,min}^{ind})} = 1 - \frac{N(\rho^{ind}(\lambda) \geq \rho_{s,min}^{ind})}{N(\rho^{ind}(\lambda_n) \geq \rho_{s,min}^{ind})} \left(\frac{\lambda_n}{\lambda}\right)^E \quad (4)$$

174 The analogy between Equation (4) and the dry sieving method is explained in the Appendix.  
 175 Equation (3), and hence Equation (4), can be described analytically through the UM  
 176 framework if expressing  $\rho_{s,min}^{ind}$ , which is independent on  $\lambda$ , through  $\lambda$ :

$$177 \quad \rho_{s,min}^{ind} = \frac{\rho_{s,min}}{\rho_{bulk}} = \lambda^{\gamma(\lambda)} \quad (5)$$

178 where  $\rho_{s,min}$  is the minimal grain density [M/L<sup>3</sup>], and  $\gamma(\lambda)$  differs from a fixed  $\gamma$  used in  
 179 Equations (1) and (2), since it changes with  $\lambda$  in order to maintain fixed value of  $\rho_{s,min}^{ind}$ :

$$180 \quad \gamma(\lambda) = \frac{\ln(\rho_{s,min}^{ind})}{\ln(\lambda)} \quad (6)$$

181 Finally, by introducing Equation (6), instead of  $\gamma$ , into Equation (2), it is possible to express  
 182 Equation (3) analytically:

$$183 \quad P(\rho^{ind}(\lambda) \geq \rho_{s,min}^{ind}) \approx \lambda^{-c(\gamma(\lambda))} = \lambda^{-C_1 \left( \frac{\ln(\rho_{s,min}^{ind})}{\ln(\lambda)} + \frac{1}{\alpha} \right)^{\alpha'}} \quad (7)$$

184 Equation (7) is presented in Figure 2b with solid line which is also fully characterized by  
 185 means of parameters  $C_1$  and  $\alpha$ . Value of  $\gamma(\lambda) = C_1$  corresponds to the upper resolution limit  
 186  $\lambda_{up} = (\rho_{s,min}^{ind})^{1/C_1}$  (see Equation 6) for which, due to the fact that  $c(\gamma(\lambda) = C_1) = C_1$  (see  
 187 Equation 2),  $P(\rho^{ind}(\lambda) \geq \rho_{s,min}^{ind})$  reaches its maximal value equal to  $(\rho_{s,min}^{ind})^{-1}$ . On the

188 contrary, the lower resolution limit  $\lambda_{low} = (\rho_{s,min}^{ind})^{1/\gamma_s}$  can be also computed from Equation  
 189 (6) for  $\gamma(\lambda) = \gamma_s$ , which is known as the most probable singularity:

$$190 \quad \gamma_s = C_1 \alpha' \left( \left( \frac{E}{C_1} \right)^{1/\alpha'} - \frac{1}{\alpha} \right) \quad (8)$$

191 Note that Equation (8) is derived from Equation (2) given that  $c(\gamma_s) = E$ .

192 Finally, having on mind that  $l(\lambda) = d_g$ , and thus  $\lambda = L / d_g$ , the analytical GSD function can be  
 193 derived by introducing Equation (7) into (4):

$$194 \quad P(d < d_g) = 1 - \frac{\left( \frac{L}{d_g} \right)^{-c_1 \left( \frac{\ln(\rho_{s,min}^{ind})}{C_1 \alpha'} + \frac{1}{\alpha} \right) \alpha'}}{\left( \frac{L}{d_{g,min}} \right)^{-c_1 \left( \frac{\ln(\rho_{s,min}^{ind})}{C_1 \alpha'} + \frac{1}{\alpha} \right) \alpha'}} \quad (9)$$

195 where  $d_{g,min} = l(\lambda_n) = L / \lambda_n$  is the minimal grain diameter [L] equal to the size of a pixel at  $\lambda_n$ .

196 From Equation (9) the probability density function can be derived as the first derivative of

197  $P(d \geq d_g) = 1 - P(d < d_g)$  with respect to  $\ln(L/d_g)$ , providing the following expression:

$$198 \quad p(d = d_g) = -c \left( \frac{L}{d_g} \right) \left( \frac{L}{d_{g,min}} \right)^{c \left( \frac{L}{d_{g,min}} \right)} \left[ 1 - \frac{\frac{\ln(\rho_{s,min}^{ind})}{C_1}}{\frac{1}{\alpha} \ln \left( \frac{L}{d_g} \right) + \frac{\ln(\rho_{s,min}^{ind})}{C_1 \alpha'}} \right] e^{-c \left( \frac{L}{d_g} \right) \ln \left( \frac{L}{d_g} \right)} \quad (10)$$

199 where  $c \left( \frac{L}{d_g} \right) = C_1 \left( \frac{\ln(\rho_{s,min}^{ind})}{\ln(L/d_g) C_1 \alpha'} + \frac{1}{\alpha} \right) \alpha'$  is the co-dimension, while  $E - c \left( \frac{L}{d_g} \right)$  describes the

200 change of fractal dimension with  $d_g$ . The approach proposed here would face certain issues

201 mostly related to the way pixels are grouped. Therefore, it is possible to have  $\lambda_l^2$  neighbor

202 pixels that belong to a grain of larger size, but since they are distributed in different groups

203 there is a “good” chance that this larger grain will not be recognized after the aggregation. On

204 the contrary, those pixels can signify separated grains, but if they are aggregated as a part of  
 205 the same group of  $\lambda_l^2$  pixels, they will be recognized as a part of the larger grain.  
 206 Nevertheless, these special cases do not influence the proposed algorithm significantly if  
 207 applied on the sufficiently large  $\lambda$ .

### 208 2.2.1. Determination of the model parameters

209 Parameters of the proposed model (Equation 9) are:  $d_{g,min}$ ,  $\rho_{s,min}^{ind}$ ,  $\alpha$  and  $C_l$ . The first two are  
 210 physical parameters whose values are either estimated based on the type of material ( $d_{g,min}$ ), or  
 211 calculated based on the experimentally determined values of  $\rho_{bulk}$  and  $\rho_{s,min}$  (see Equation  
 212 5), while the last two ( $\alpha$  and  $C_l$ ) are statistical parameters determined by analyzing the scaling  
 213 behavior of  $\rho^{ind}$  field. This can be done by means of Trace Moment (TM) technique (Schertzer  
 214 & Lovejoy, 1987) which assumes that the scaling of the average statistical moments of order  
 215  $p$   $\langle(\rho^{ind}(\lambda))^p\rangle$  can be described through the moment scaling function  $K(p)$ :

$$216 \quad \langle(\rho^{ind}(\lambda))^p\rangle \approx \lambda^{K(p)} \quad (11)$$

$$217 \quad K(p) = \frac{c_1}{\alpha-1}(p^\alpha - p); \quad \alpha \neq 1 \quad (12)$$

218 where  $K(p)$  is described through parameters  $C_l$  and  $\alpha$  (for more details see Schertzer &  
 219 Lovejoy, 1987). Note that  $c(\gamma)$  and  $K(p)$  functions are linked by Legendre transform (Frisch &  
 220 Parisi, 1985), meaning that for each  $\gamma$  there is a corresponding  $p$  (i.e. for  $\gamma = C_l$  and  $\gamma = \gamma_s$  the  
 221 corresponding values are  $p = 1$  and  $p = p_s$ , respectively).

222 To determine values of  $\alpha$  and  $C_l$  for a certain  $\rho^{ind}(\lambda)$  field, the field is firstly up-scaled as  
 223 previously described, and all  $\rho^{ind}(\lambda)$  values are raised on a power  $p$  at each  $\lambda$ . The average  
 224 value of such a modified field  $\langle(\rho^{ind}(\lambda))^p\rangle$  is computed at each  $\lambda$ , and the procedure is  
 225 repeated for variety of  $p \geq 0$  values. After plotting  $\log(\langle(\rho^{ind}(\lambda))^p\rangle)$  against  $\log(\lambda)$ ,

226 different linear regressions depending on  $p$  value are formed. Their slopes are related to  $K(p)$   
227 values that form the moment scaling function. Based on Equation (12), the first derivative of  
228 the obtained  $K(p)$  function at  $p = 1$  is equal to  $C_1 = \frac{dK(p)}{dp} \Big|_{p=1}$  (calculated numerically), while  
229 the ratio between the second and the first derivative at  $p = 1$  is  $\alpha = \frac{1}{C_1} \frac{d^2K(p)}{dp^2} \Big|_{p=1}$ .

### 230 **2.2.2. Influence of the model parameters**

231 To better understand the influence of the four parameters on the model behavior, Equation (9)  
232 has been tested on different values of each parameter, as illustrated in Figure 3. For all cases  
233 presented in Figure 3, value of  $L = 100$  mm is kept constant while changing values of the four  
234 model parameters.

235 The impact of  $C_l$  on the GSD is illustrated in Figure 3a by increasing (dash-dotted line) /  
236 decreasing (dashed line) its initial value (solid line) by 50 % while preserving values of the  
237 three remaining parameters. Similarly, in Figure 3b value of  $\alpha$  is changed by 50 % in both  
238 ways. Figure 3a shows that parameter  $C_l$  mostly affects the break onto the finer particles and  
239 the shape of that part of the curve in a way that smaller  $C_l$  secures higher contribution of fine  
240 grains (dashed line), while the case is opposite for higher  $C_l$  (dash-dotted line). On the  
241 contrary, the change of parameter  $\alpha$  (Figure 3b) is less affecting the representation of small  
242 grains, but it is mainly responsible for the slope of the central part of the GSD curve, where  
243 smaller  $\alpha$  provides steeper curve. Thus, in case of granular soils higher  $\alpha$  and smaller  $C_l$   
244 values describe well-graded, while smaller  $\alpha$  and higher  $C_l$  describe more uniformly graded  
245 materials. Indeed, well-graded materials usually have lower total porosity due to the better  
246 spatial packing of grains, meaning the lower representation of zeros in  $\rho^{ind}$  field that causes  
247 stronger variability of the field (higher  $\alpha$ ) and lower intermittency of its mean value (lower  
248  $C_l$ ).

249 The impacts of  $\rho_{s,min}^{ind}$  and  $d_{g,min}$  on the GSD curve are also tested by varying one of the  
250 parameters while maintaining the rest. As illustrated in Figure 3c, the higher  $\rho_{s,min}^{ind}$  (more  
251 strict threshold value), the higher values of  $P(d < d_g)$  (Equation 9), and vice versa. Unlike the  
252 three other parameters,  $d_{g,min}$  dictates the total range of scales ( $L / d_{g,min}$ ) by affecting mostly  
253 the distribution of small grains (tail of the GSD) - see Figure 3d.

### 254 3. Soil sampling and Image acquisition

255 In this section are given information about soil sampling and image acquisition for three  
256 different materials: Green Wave substrate and Horizons A and A20 of La Herreria soil.

#### 257 3.1. Green Wave substrate

258 Green Wave substrate is an artificial coarse material (VulkaTec Riebensahm GmbH 2016)  
259 with 4 % of organic matter, used for covering green roofs. Due to its volcanic nature (values  
260 of grain and dry bulk densities are 2.35 Mg/m<sup>3</sup> and 1.42 Mg/m<sup>3</sup>, respectively), this material  
261 does not create a significant load on the roof construction which is the reason it has been used  
262 in case of Green Wave (Versini et al., 2018; Versini et al., 2020), a wavy shape green roof  
263 located next to Ecole des Ponts ParisTech in Champs-sur-Marne, France. The substrate  
264 contains 50 % of grains larger than 1.6 mm, with 10 % of particles between 10 and 20 mm in  
265 the coarse range, and 13 % of fine particles smaller than 80  $\mu$ m. Distribution of grains larger  
266 than 80  $\mu$ m was determined by means of the dry sieving method (AFNOR, 1996), while the  
267 sedimentation test (AFNOR, 1992) was used for finer particles. The curvature and uniformity  
268 coefficients are  $C_c = (D_{30})^2 / (D_{60} \times D_{10}) = 1.95$  and  $C_u = D_{60} / D_{10} = 55$ , respectively, and hence  
269 this substrate is regarded as well graded according to the ASTM D2487-06 (2006) standard.  
270 The sample of the GW substrate (10 cm diameter and 15 cm height) was prepared by mixing  
271 and pouring up the material into the plexiglass cylinder (compacted to in situ value of  $\rho_{bulk} =$

272 1.42 Mg/m<sup>3</sup>), simulating the way substrate is placed on the roof to avoid segregation of fine  
273 particles at the bottom. Tomographic scans were conducted with a RX Solutions Ultratom  
274 microtomograph, including a Hamamatsu L10801 X-ray source and a Paxscan Varian 2520V  
275 flat panel detector (1920 x 1560 pix<sup>2</sup>, pixel size 127 µm). X-ray source tension and current  
276 were respectively 200kV and 280µA. The detector was set at 4 fps, each projection resulting  
277 of an average over 25 projections, giving a total number of 4320 averaged projections. The  
278 sample being a long cylinder, stack mode was used and set to three turns. The reconstructed  
279 3D image is finally represented by 1785x1785x3072 voxels with the edge length of 53.7 µm.

### 280 **3.2. La Herreria soil (Horizons A and A20)**

281 Two intact samples (60 mm diameter and 100 mm height) of La Herreria soil were collected  
282 in the low land mountain area of Sierra de Guadarrama in Spain (Schmid et al. 2016)., which  
283 is a highly degraded type of site because of the livestock keeping. One soil sample was  
284 extracted from the top 18 cm layer (Horizon A), being the result of biological alteration with  
285 roots resulting in fertile soil. This layer is moderately acid, with 2.5% of organic matter, 0.8%  
286 of Fe<sub>2</sub>O<sub>3</sub>, sandy texture (65% sand, 25% silt, 10% clay) and bulk density of 1.6 Mg/m<sup>3</sup>. The  
287 second soil sample was extracted from 18-40 cm depth (Horizon A20), also presenting an acid  
288 character (pH = 6) with 0.5% of organic matter, 0.7% of Fe<sub>2</sub>O<sub>3</sub>, 55%, 30% and 15% of sand,  
289 silt and clay particles, respectively, and bulk density of 1.7 Mg/m<sup>3</sup>.

290 X-ray CT scanning was performed using a Phoenix v | tome | x m 240 kV system (GE Sensing  
291 & Inspection Technologies GmbH, Wunstorf, Germany) at the Hounsfield Facility, University  
292 of Nottingham, UK. The scanner consisted of a 240kV microfocus X-ray tube fitted with a  
293 tungsten reflection target and a DXR 250 digital detector array with 200 µm pixel size (GE  
294 Sensing & Inspection Technologies GmbH, Wunstorf, Germany). A maximum X-ray energy  
295 of 140kV and 200 µA was used to scan the soil core. A total of 2400 projection images were

296 acquired over a 360° rotation. Each projection was the average of six images acquired with a  
297 detector exposure time of 200 ms and the resulting isotropic voxel edge length was 32 μm.  
298 The 3D image of the soil samples used in this work is represented by 676×676×300 voxels.

## 299 4. Results and Discussion

300 The approach presented in this work is firstly validated on soil images of the GW substrate,  
301 and the experimental GSD data of the same material. Then, it is applied on two horizons of La  
302 Herreria soil, but in this case only measured percentages of sand, silt and clay particles were  
303 compared with model results because detailed GSD data are lacking. Finally, the comparison  
304 with the fractal-based PSF model (Perrier et al., 1999; Bird et al., 2000) is presented.

### 305 4.1. Obtaining $\rho^{ind}$ field by means of X-ray CT

306 The  $\rho^{ind}$  field is transformed from a grey-scale image carrying the information about different  
307 intensities of grey color - bright shades of grey represent high, and dark shades low density  
308 zones. Since grey-level intensities ( $GL$ ) can be linked with density  $\rho$  by means of the linear  
309 regression (Taina et al., 2008), a linear correlation between  $\rho^{ind}$  [-] and original grey-level  
310 ( $GL$ ) is obtained by:

- 311 - subtracting the  $GL_0$  threshold value from the original  $GL$  values.
- 312 - setting to zero all  $GL$  values lower than  $GL_0$ .
- 313 - renormalizing the modified  $GL$  field.

$$314 \quad \rho^{ind} = \begin{cases} GL > GL_0, & \frac{GL - GL_0}{\langle GL - GL_0 \rangle} \\ GL \leq GL_0, & 0 \end{cases} \quad (13)$$

315 where notation  $\langle \rangle$  indicates the mean value.  $GL_0$  is adjusted based on the locations of pores  
316 that can be reliably identified on the image. By using *Fiji* (<https://fiji.sc/>), an open source

317 Java-based image processing package, it was possible to estimate the value of  $GL_0$  for all three  
318 materials.

319 In *Figure 4* are presented eight horizontal  $\rho^{ind}$  fields of the GW substrate extracted from the  
320 full-three dimensional scanned image. The resolution of the presented planes is 1024x1024  
321 pixels ( $\lambda_n = 2^{10}$ ), and they are equally distant in vertical direction (1.6 cm between two  
322 consecutive images). In *Figure 5A* are presented four horizontal  $\rho^{ind}$  fields of La Herreria soil  
323 – Horizon A, while in *Figure 5B* are presented horizontal fields of Horizon A20 (all images  
324 are 512x512 pixels -  $\lambda_n = 2^9$ ).

## 325 **4.2. UM model vs. Experimental data**

326 The two-dimensional fields can be analyzed instead of the full-three dimensional one only if  
327 the statistical isotropy within the soil specimen is secured, which is the case here (explained  
328 further in the text). Note that  $\rho_{s,min}^{ind}$  and  $d_{g,min}$  are physical properties of the particular material  
329 and they are considered as unique for the whole sample (and every two-dimensional slice  
330 within it), while  $\alpha$  and  $C_l$  are statistical parameters determined for every horizontal plane  
331 individually.

### 332 **4.2.1. UM model parameters**

333 In case of the GW substrate, value of  $\rho_{s,min}^{ind}$  is computed as  $\rho_{s,min}^{ind} = \frac{\rho_{s,min}}{\rho_{bulk}} = \frac{2.2}{1.42} = 1.55$ ,  
334 where  $\rho_{s,min}$  and  $\rho_{bulk}$  are experimentally determined, while in case of La Herreria soils  $\rho_{s,min}^{ind}$   
335 values are adjusted to fit measured percentages of sand, silt and clay particles (explained later  
336 in the text) because the corresponding  $\rho_{s,min}$  values are missing (only  $\rho_{bulk}$  values are given).  
337 Clearly, due to its physical basis, value of  $\rho_{s,min}^{ind}$  should not change significantly regardless of

338 the soil type (adjusted values are  $\rho_{s,min}^{ind} = 1.73$  and  $1.54$  for Horizon A and A20,  
339 respectively).

340 Value of  $d_{g,min}$  depends on the range of scales that is analyzed. If focusing on the range of  
341 scales covered by the scanned image,  $d_{g,min}$  corresponds to the pixel size at  $\lambda_n$  ( $= 1024$  and  $512$   
342 for GW substrate and La Herreria soils, respectively), and thus it is equal to  $d_{g,min} = L / \lambda_n \approx 60$   
343 /  $1024 \approx 50 \mu\text{m}$  in case of the GW substrate, and  $d_{g,min} = 16 / 512 \approx 32 \mu\text{m}$  for La Herreria  
344 soils. For the full-range of scales,  $d_{g,min}$  represents the minimal grain size that needs to be  
345 approximately estimated if not measured. For the GW substrate,  $d_{g,min} = 1 \mu\text{m}$  is adopted  
346 based on the GW experimental data, while lower values of  $d_{g,min} = 0.1 \mu\text{m}$  and  $0.05 \mu\text{m}$  are  
347 adopted for Horizons A and A20 of La Herreria soil, respectively, having on mind significant  
348 percentages of clay particles in both cases (10% and 15%, respectively).

349 Finally, the UM parameters are determined by performing TM analysis on every  $\rho^{ind}(\lambda)$  field  
350 presented in Figure 4 and Figure 5. In Figure 6 is presented  $\log(\langle(\rho^{ind}(\lambda))^p\rangle)$  versus  $\log(\lambda)$   
351 for four different  $p$  values (0.2, 1.5, 2 and 3) and eight horizontal  $\rho^{ind}(\lambda)$  fields of the GW  
352 substrate presented in Figure 4. In all cases, scaling of  $\log(\langle(\rho^{ind}(\lambda))^p\rangle)$  (different symbols)  
353 can be reasonably well interpreted with linear regressions. This is also the case with La  
354 Herreria soils whose moment scaling behavior is illustrated in Figure 7A for Horizon A, and  
355 in Figure 7B for Horizon A20. The same kind of analysis is applied on the eight vertically  
356 oriented fields ( $\lambda_n = 2^{10}$ ) of the GW substrate (equal horizontal distance between the two  
357 consecutive images). The quality of scaling is slightly better for vertical planes, but they are  
358 not presented in the Figure. Vertical planes for La Herreria soils were not analyzed since the  
359 maximal resolution of the vertically oriented image is  $256 \times 256$  pixels ( $\lambda_n = 2^8$ ), which is  
360 regarded as insufficient.

361 Based on the slopes of the obtained linear regressions, in Figure 8 are presented  $K(p)$   
362 functions for all analyzed horizontal planes (different solid lines) of three different soils,  
363 together with  $K(p)$  functions related to the vertical planes of the GW substrate (dashed lines in  
364 Figure 8A). Since vertical and horizontal  $K(p)$  functions are overlapping in case of the GW  
365 substrate, indicating similar values of  $\alpha$  and  $C_I$  (see **Table 1**), it is reasonable to assume the  
366 statistical isotropy within the GW specimen. Even though vertical images are not analyzed,  
367 the same assumption is adopted for two remaining soils, having on mind the obtained  
368 horizontal  $K(p)$  functions are rather similar (values of  $\alpha$  and  $C_I$  are presented in **Table 2**).

369 Low values of  $C_I$  (order of magnitude  $10^{-2}$ ) obtained for all three soils indicate the narrow  
370 range of  $\rho^{ind}$  values, meaning that  $\rho$  cannot be significantly larger than  $\rho_{bulk}$  (should be the case  
371 regardless of the soil type). Also, higher  $\alpha$  values (closer to 2) point out more significant  
372 fluctuations of  $\rho$  around  $\rho_{bulk}$ , indicating the presence of different grain sizes with different  
373 densities.

#### 374 4.2.2. Comparison with Experimental data

375 After determining values of the four model parameters, Equation (9) is firstly tested on the  
376  $\rho^{ind}$  field of the Hor. plane 4 of the GW substrate (see Figure 4). Since focusing on the range  
377 of scales covered by the image,  $d_{g,min} = 50 \mu\text{m}$  is used. Figure 9a-top illustrates comparison  
378 between Equation (9) (UM model - solid line), Equation (4) (connected dots) that uses the  
379 counted number of  $\rho^{ind}(\lambda) \geq \rho_{s,min}^{ind}$  values at different  $\lambda$ , and the truncated experimental  
380 GSD data of the GW substrate (triangles). In Figure 9a-bottom are compared probability  
381 density functions coming from the UM model (Equation 10 – solid lines) and measurements  
382 (triangles). Good agreement between different analytical curves and truncated experimental  
383 data that consider only  $d_g \geq 50 \mu\text{m}$  is obtained.

384 In Figure 9b-top is presented comparison between the same truncated experimental data  
385 (triangles) and Equation (9) applied on every  $\rho^{ind}$  field of Figure 4 (different solid lines), while  
386 in Figure 9b-bottom are illustrated the corresponding probability density functions (Equation  
387 10).

388 Finally, in Figure 9c-top and Figure 9c-bottom the full range experimental data (squares) are  
389 compared with Equations (9) and (10), respectively, by using the same parameter values as in  
390 Figure 9b, with only difference that  $d_{g,min} = 1 \mu\text{m}$  is adopted. The agreement between the  
391 Equation (9) / Equation (10) and both truncated and full-range experimental data is considered  
392 as satisfactory. The obtained family of curves creates reasonably narrow confidential zone  
393 around experimental points, verifying that way the analytical model proposed.

394 In Figure 10 is presented the same kind of analysis as in Figure 9, but for Horizons A and A20  
395 of La Herreria soil. In Figure 10a and b are illustrated results obtained from  $\rho^{ind}$  fields of  
396 Figure 5a (Horizon A) and Figure 5b (Horizon A20), respectively, where  $d_{g,min} = 32 \mu\text{m}$  is  
397 adopted, while in Figure 10c are compared full-range results for Horizon A (solid lines) and  
398 Horizon A20 (dashed lines) using  $d_{g,min} = 0.1 \mu\text{m}$  and  $d_{g,min} = 0.05 \mu\text{m}$ , respectively. As  
399 mentioned earlier in the text, values of  $\rho_{s,min}^{ind}$  (1.73 and 1.54 for Horizons A and A20,  
400 respectively) are adjusted so that the average percentages of sand, silt and clay particles,  
401 computed based on the model results from Figure 10c, fit well with measured values. For  
402 Horizon A, the average computed values of sand, silt and clay particles are 64%, 27% and 8%  
403 (about 1% of particles  $> 2 \text{ mm}$ ), respectively, while for Horizon A20 those values are 54%,  
404 32% and 14%, respectively. The results obtained with a model are rather close to  
405 measurements, confirming the UM model is valid.

406 The proposed UM approach was successfully evaluated on soils that cover wide range of  
407 grain sizes with significant percentages of both coarse and fine particles. To see limitations of

408 the model, it was additionally tested on a material with a rather uniform GSD curve, the  
409 Hostun sand HN31 extracted in the Drôme region in France (Bruchone et al, 2013). This sand  
410 is made of about 98% of quartz (grain densities are uniform), and it covers rather narrow  
411 spectrum of grain sizes (0.2 ÷ 0.8 mm). Preliminary results showed the proposed approach is  
412 not applicable on such material, having on mind the scaling of statistical moments in log-log  
413 scale significantly deviates from linear regression, and hence parameters  $\alpha$  and  $C_I$  cannot be  
414 determined.

### 415 **4.3. UM model vs. PSF model**

416 The PSF approach, a three-phase fractal-based GSD model firstly introduced by Perrier et al.  
417 (1999), is also used for interpreting the experimental GSD curve of the GW substrate.  
418 According to this model, the GSD can be described using the following expression (Bird et al.  
419 2000):

$$420 \quad P(d < d_g) = \left( \frac{d_g}{d_{g,max}} \right)^{3-D_{f,PSF}} \quad (14)$$

421 where  $D_{f,PSF}$  is the fractal coefficient [-] whose optimal value can be determined from the  
422 slope of the best fitting linear regression in logarithmic scale that goes through the  
423 experimental GSD data and reaches 100 % at  $d_{g,max}$ . By adopting  $d_{g,max} = 18$  mm in case of the  
424 GW substrate, the optimal value of  $D_{f,PSF} = 2.57$  is obtained (dashed line in Figure 8a).

425 To better illustrate the difference between the PSF (Equation 14) and the UM model  
426 (Equation 9), probability functions are presented in a log-log scale (Figure 11a). For the same  
427 values of  $L$ ,  $d_{g,min}$  and  $\rho_{s,min}^{ind}$  as in Figure 9c, the best agreement between Equation (9) and  
428 measurements is obtained for  $\alpha = 1.60$  and  $C_I = 2.25 \times 10^{-2}$ . These values are rather close to  
429 those describing the horizontal plane 4 of the GW substrate (see Table 1), confirming the  
430 physical basis of the proposed approach.

431 To test whether the UM model can be used for soils with multi-modal GSD curves that are  
432 described with multiple fractal dimensions, Equation (9) is compared with experimental data  
433 of Walla Walla (WW) soil, described in Bittelli et al. (1999) by means of three different  
434 fractal dimensions related to distributions of sandy, silty and clayey particles. According to  
435 the U.S. soil taxonomy, this soil is Typic Haploxeroll with 8.3% of sand, 78.4% of silt and  
436 13.3% of clay. Due to the lack of scanned soil images and other relevant information, the UM  
437 model parameters could not be determined as explained earlier in the text, but only roughly  
438 estimated / adjusted. Figure 11b shows that for  $\rho_{s,min}^{ind} = 1.72$ ,  $\alpha = 1.05$ ,  $C_l = 4.5 \times 10^{-2}$ , and  $L /$   
439  $d_{g,min} = 1 \text{ mm} / 1 \times 10^{-4} \text{ mm}$  (from the graph), Equation (9) provides rather good agreement  
440 with WW experimental points. Since the relevant information are missing, these results  
441 should be taken with a grain of salt and understood only as a test if the model is capable of  
442 interpreting multi-modal GSD curves.

443

444 Compared to Equation (14) which considers the fixed fractal dimension  $D_{f,PSF}$ , Equation (9)  
445 takes into account the fractal dimension that changes with  $d_g$  (the co-dimension function).  
446 Therefore, for the UM approach it is quite important to know the total range of scales  
447 investigated ( $L / d_{g,min}$ ). Even though the PSF model (Equation 14) is more convenient for  
448 practical application, since it uses only two parameters that can be determined quite easily, the  
449 UM model proposed in this work (Equations 9 and 10) shows better agreement with  
450 experimental GSDs for soils with wide spectrum of grain sizes whose distribution cannot be  
451 accurately described with a simple fractal (power) law.

## 452 **5. Conclusion**

453 This work shows that the up-scaling approach presented can be used for predicting the GSD  
454 of a certain material based on its scanned micro-structure which represents a density indicator

455 field. By reducing the resolution of the scanned image, density indicator values above the  
456 fixed threshold are treated at each resolution as a cumulative representation of solid particles  
457 of diameter equal to or larger than the corresponding pixel size. The quantity of values above  
458 the threshold at different resolutions can be analytically described by means of the Universal  
459 Multifractals (UM), leading to the new multifractal-based GSD model. The model uses four  
460 parameters, where two of them are physical (the minimal grain diameter and the ratio between  
461 the minimal grain density and dry bulk density), while the other two are statistical UM  
462 parameters that characterize the spatial heterogeneity of the soil density field.

463 An innovative approach proposed in this work was tested on three different materials  
464 containing significant percentages of both coarse and fine particles: an unconventional  
465 volcanic granular material used for covering green roofs, and two horizons of La Herreria soil  
466 collected from the low land mountain area of Sierra de Guadarrama in Spain. By performing  
467 the Trace Moment analysis on two-dimensional scanned soil images (density indicator field)  
468 of different materials, the statistical parameters ( $C_I$  and  $\alpha$ ) were determined and families of  
469 analytical GSD curves were obtained for each material, showing a good agreement with  
470 experimental data. Additionally, the UM approach was tested on Houston quartz sand and  
471 results showed that such a uniform GSD cannot be reliably estimated by means of the  
472 presented methodology. In all cases, the analysis was applied on two-dimensional images to  
473 save computational time and memory, which is legit if the statistical isotropy within the  
474 specimen is secured.

475 Results showed the values of  $C_I$  are the same order of magnitude for different soils ( $10^{-2}$ )  
476 because density of an individual grain cannot be significantly larger than the dry bulk density,  
477 regardless of the soil type. Also, it was proved that lower  $C_I$  and higher  $\alpha$  are related to GSD  
478 curves that change gradually with respect to grain sizes, while lower  $\alpha$  and higher  $C_I$  are

479 related to the steeper GSD curve in its central part and stronger curvature close to the break  
480 onto the finer particles.

481 Finally, the proposed four parameters UM model was compared with the fractal-based two  
482 parameters PSF model, showing better agreement with the Green Wave experimental data. Its  
483 advantage was additionally emphasized by showing excellent agreement with multi-modal  
484 GSD curve of Walla Walla soil that is described in the literature by means of three different  
485 fractal dimensions related to sandy, silty and clayey particles.

486

## 487 **Appendix: Analogy between the up-scaling approach and the dry sieving** 488 **method**

489 The analogy between Equation (4) (the up-scaling approach) and the dry sieving method can  
490 be derived under the following assumptions:

- 491 1. Three-dimensional space is considered ( $E = 3$ ) since the dry sieving method is based on  
492 the grain masses.
- 493 2. All grains have the same shape and density  $\rho_s$ .
- 494 3. A discrete number of sieves is used, and hence the size of voids on each sieve follows the  
495 size of voxels  $l(\lambda_{n-i})$  at different  $\lambda_{n-i}$ , where  $i = [0 \div n]$ .

496 The total mass of the specimen ( $M_{total}$ ) is placed on the cascade of sieves arranged in a  
497 descending order (the largest void size is on the top, while the smallest one is on the bottom).

498 If all grains are distributed on the corresponding sieves  $i = [0 \div n]$ , it can be assumed that  
499 those staying on a certain sieve have diameter equal to the size of sieve voids. Therefore, the  
500 following can be written for  $i = 0$ :

$$501 \quad M(d \geq l(\lambda_n)) = M_{total} = C_V \rho_s \sum_{j=0}^n N_g(\lambda_{n-j}) l(\lambda_{n-j})^3 \quad (A1)$$

502 where  $C_V$  is the volume shape coefficient [-] (for cube  $C_V = 1$ , for sphere  $C_V = \pi / 6$ ),  
 503  $N_g(\lambda_{n-j})$  is the number of grains [-] that stay on the sieve of void size  $l(\lambda_{n-j})$  [L] ( $j = 0 \div n$ ).  
 504 Since grains that stay on the sieve  $l(\lambda_n)$  are assumed to have identical diameters,  $l(\lambda_n)^3$  can  
 505 be pulled outside the sum, and the following is obtained:

$$506 \quad M(d \geq l(\lambda_n)) = C_V \rho_s l(\lambda_n)^3 \sum_{j=0}^n N_g(\lambda_{n-j}) \left( \frac{l(\lambda_{n-j})}{l(\lambda_n)} \right)^3 \quad (\text{A2})$$

$$507 \quad N_g^{cum}(\lambda_n) \approx C_V \sum_{j=0}^n N_g(\lambda_{n-j}) \left( \frac{l(\lambda_{n-j})}{l(\lambda_n)} \right)^3 \quad (\text{A3})$$

508 where  $N_g^{cum}(\lambda_n)$  is the cumulative number of grains equal to or larger than  $d_{g,min} = l(\lambda_n)$ .  
 509 Following Equation (A2), the cumulative mass of all grains equal to or larger than  $l(\lambda_{n-i})$  can  
 510 be expressed as the following:

$$511 \quad M(d \geq l(\lambda_{n-i})) = C_V \rho_s l(\lambda_{n-i})^3 \sum_{j=i}^n N_g(\lambda_{n-j}) \left( \frac{l(\lambda_{n-j})}{l(\lambda_{n-i})} \right)^3 \quad (\text{A4})$$

$$512 \quad N_g^{cum}(\lambda_{n-i}) \approx C_V \sum_{j=i}^n N_g(\lambda_{n-j}) \left( \frac{l(\lambda_{n-j})}{l(\lambda_{n-i})} \right)^3 \quad (\text{A5})$$

513 By introducing Equation (A3) into (A2) and Equation (A5) into (A4), and having on mind  
 514 that  $d_g = l(\lambda_{n-i})$ , the GSD can be expressed as:

$$515 \quad P(d < d_g) = 1 - \frac{M(d \geq l(\lambda_{n-i}))}{M(d \geq l(\lambda_n))} \quad (\text{A6})$$

$$516 \quad P(d < d_g) = 1 - \frac{N_g^{cum}(\lambda_{n-i})}{N_g^{cum}(\lambda_n)} \left( \frac{l(\lambda_{n-i})}{l(\lambda_n)} \right)^3 \quad (\text{A7})$$

517 Note that  $N_g^{cum}(\lambda_{n-i})$  and  $N_g^{cum}(\lambda_n)$  correspond to  $N(\rho^{ind}(\lambda) \geq \rho_{s,min}^{ind})$  and

518  $N(\rho^{ind}(\lambda_n) \geq \rho_{s,min}^{ind})$  in Equation (4), respectively, while  $\frac{l(\lambda_{n-i})}{l(\lambda_n)} = \frac{\lambda_n}{\lambda_{n-i}}$ . Therefore,

519 Equation (A7) is identical to Equation (4) for  $E = 3$ , just in a discrete form.

520

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524

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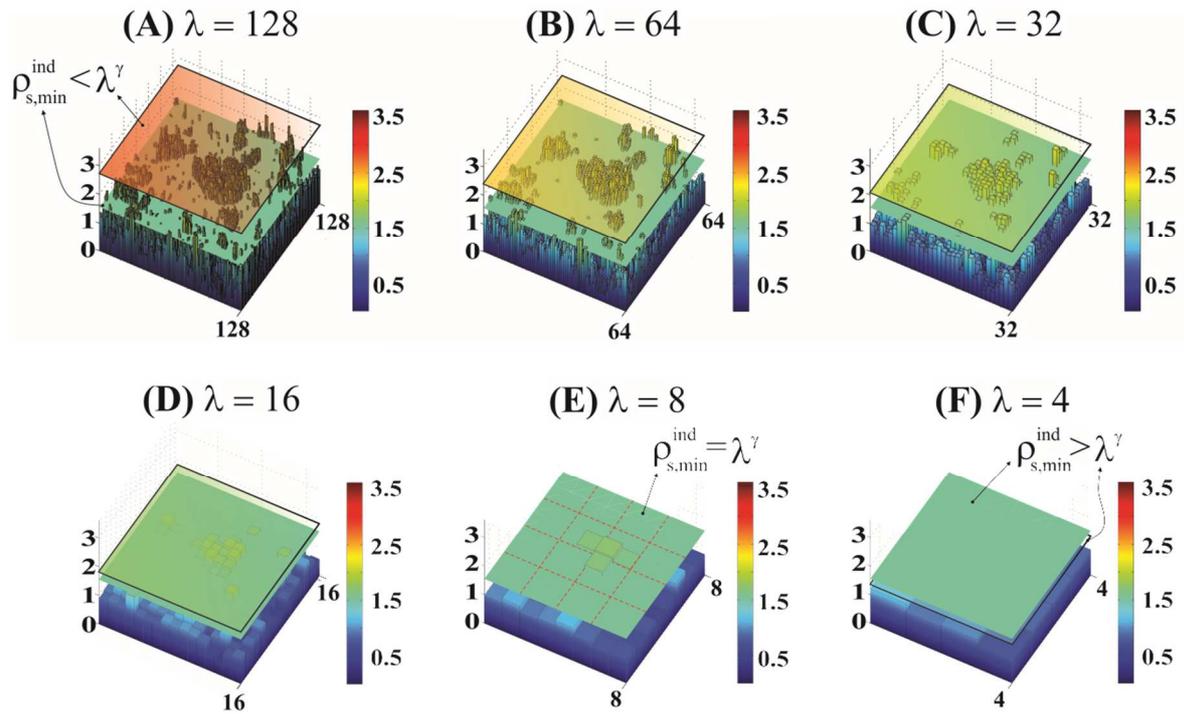
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644 Figure 1. Change of the two-dimensional  $\rho^{ind}(\lambda)$  field with  $\lambda$  at: (A)  $\lambda = 128$ ; (B)  $\lambda = 64$ ; (C)  $\lambda$

645  $= 32$ ; (D)  $\lambda = 16$ ; (E)  $\lambda = 8$ ; (F)  $\lambda = 4$ . A turquoise color platform signifies a fixed threshold

646  $\rho_{s,min}^{ind} = 1.55$ , while the one with transparent color signifies a resolution dependent threshold

647  $\lambda^\gamma$  ( $\gamma = 0.211$ )

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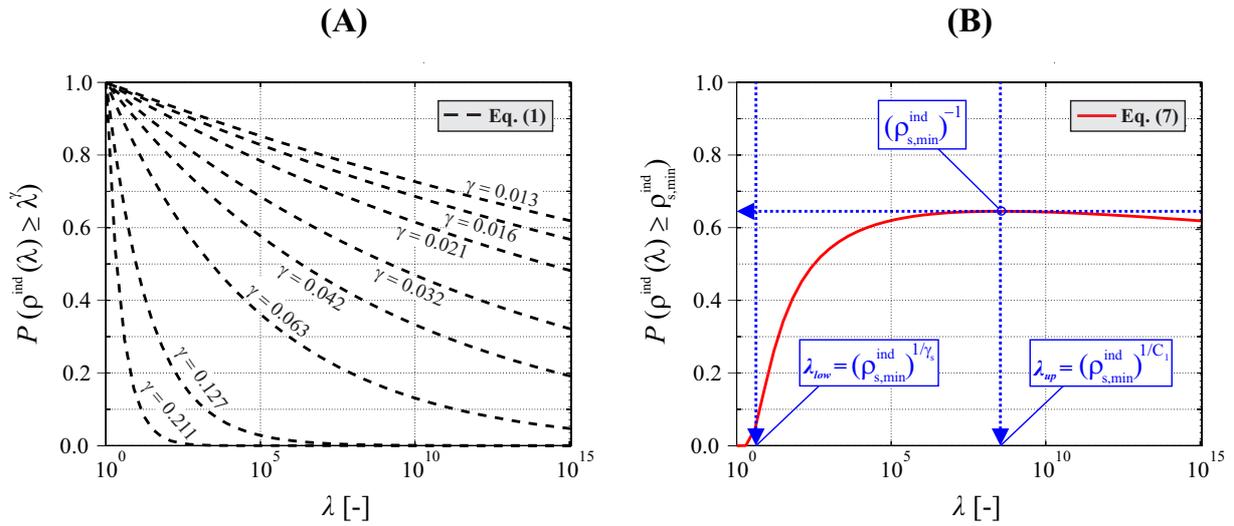
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656 Figure 2. A) Equation (1) (dashed lines) calculated for:  $C_I = 2.23 \times 10^{-2}$ ,  $\alpha = 1.67$  (values that

657 characterize  $\rho^{\text{ind}}(\lambda)$  field in Figure 1) and different values of  $\gamma$  (including  $\gamma = 0.211$ ); B)

658 Equation (5) (solid line) calculated for the same values of  $C_I$  and  $\alpha$  and the fixed value of

659  $\rho_{s,\text{min}}^{\text{ind}} = 1.55$ .

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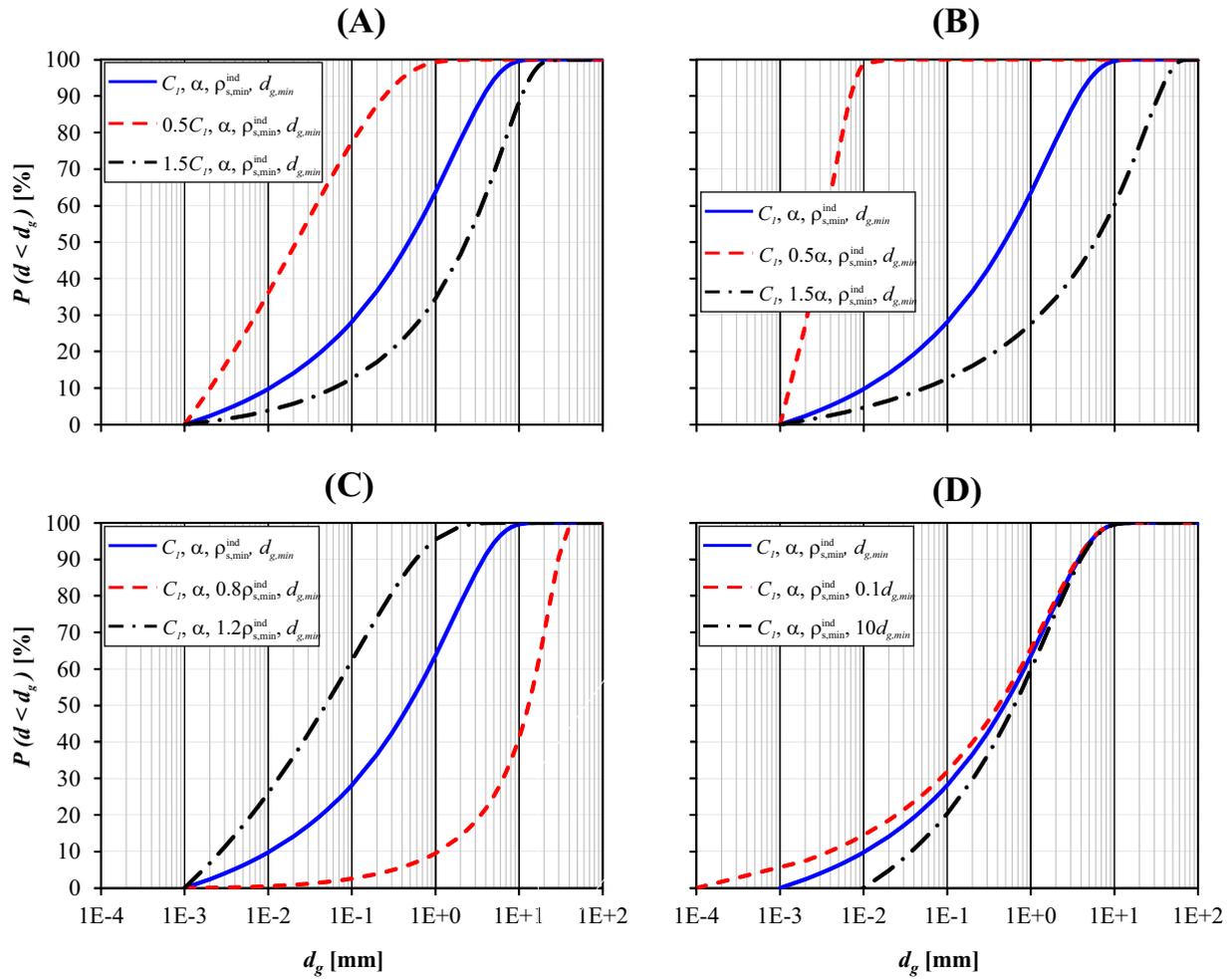
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672 Figure 3. Behavior of the proposed GSD model when changing values of: a)  $C_I$ ; b)  $\alpha$ ; c)

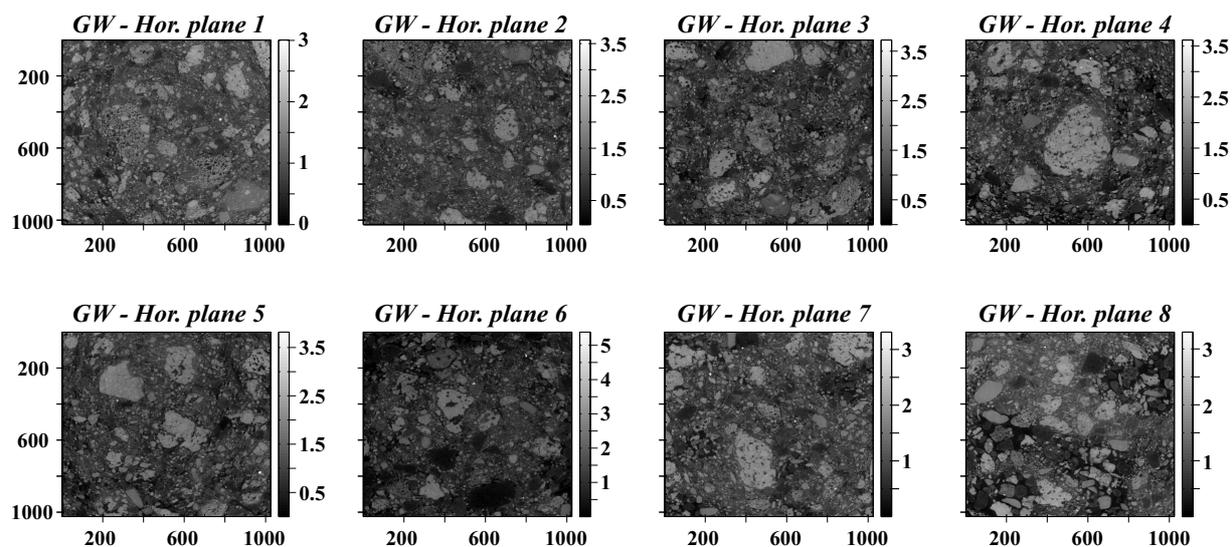
673  $\rho_{s,min}^{ind}$ ; d)  $d_{g,min}$ . Initial parameter values (solid line in each graph) are  $C_I = 1.85 \times 10^{-2}$ ,  $\alpha = 1.3$ ,

674  $\rho_{s,min}^{ind} = 1.55$  and  $d_{g,min} = 1 \times 10^{-3}$  mm

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679 Figure 4. Eight two-dimensional horizontal  $\rho^{nd}$  fields (1024 x 1024 pixels), extracted from  
 680 the original three-dimensional grey scale image, that are equally distant along the specimen  
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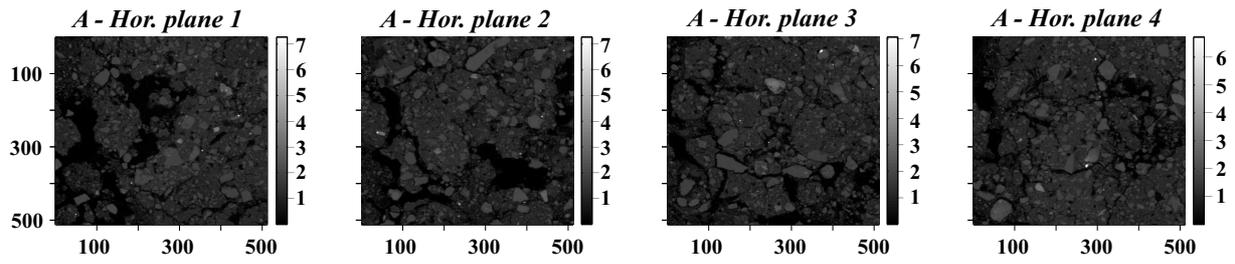
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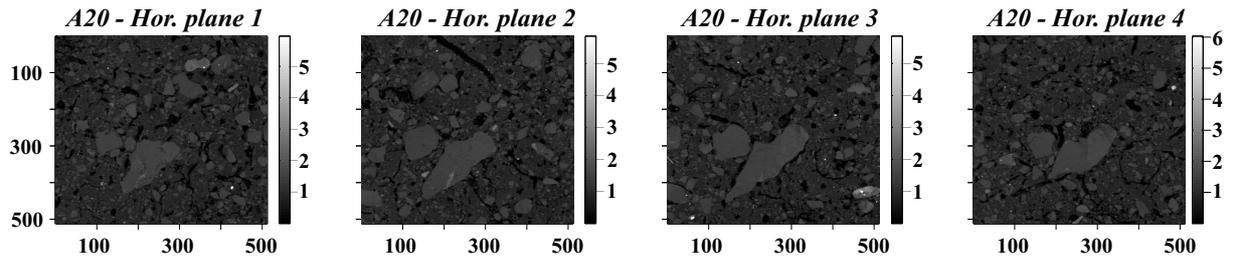
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(A) *La Herreria - Horizon A*



(B) *La Herreria - Horizon A20*



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699 Figure 5. (A) Four equally distant horizontal  $\rho^{\text{ind}}$  fields (512 x 512 pixels) of Horizon A of La

700 Herreria soil, extracted from the original three-dimensional grey scale image; (B) same as in

701 (A) just for Horizon A20

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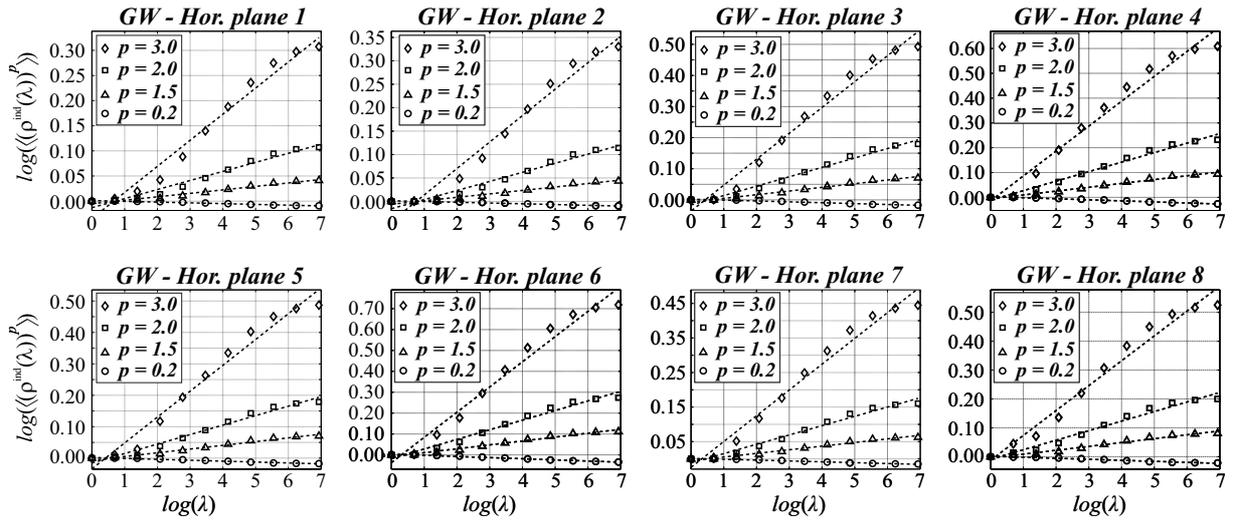
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719 Figure 6. Scaling of statistical moments of eight horizontal  $p^{\text{ind}}(\lambda)$  fields of the GW substrate  
 720 presented in Figure 4

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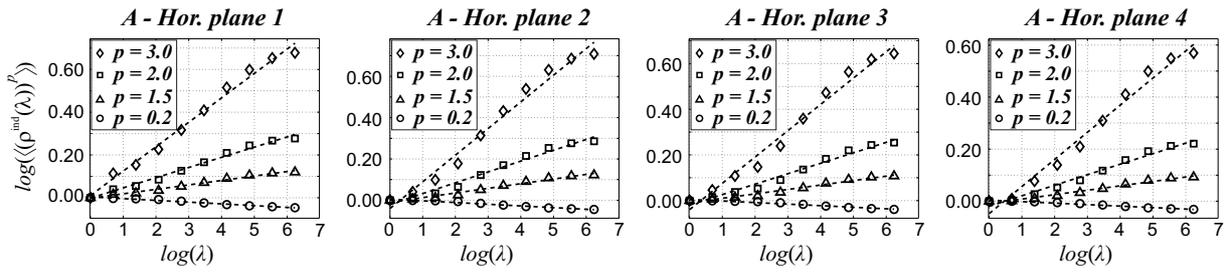
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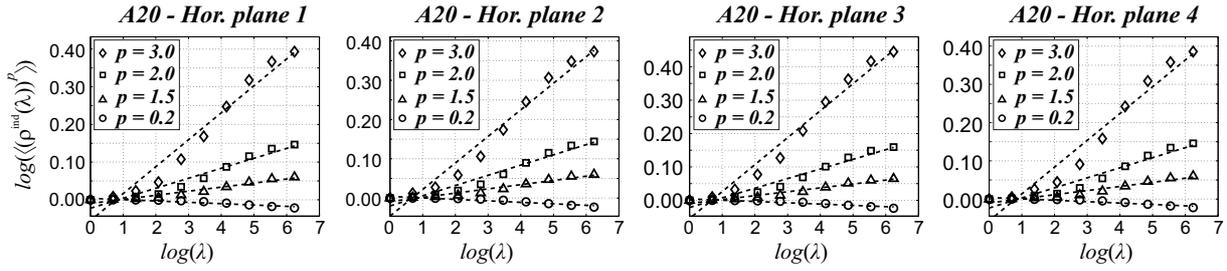
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(A) La Herreria - Horizon A



(B) La Herreria - Horizon A20



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736 Figure 7. (A) Scaling of statistical moments of four horizontal  $\rho^{\text{ind}}(\lambda)$  fields of La Herreria soil

737 (Horizon A) presented in Figure 5a; (B) same as in (A) just for Horizon A20 (fields presented

738 in Figure 5b)

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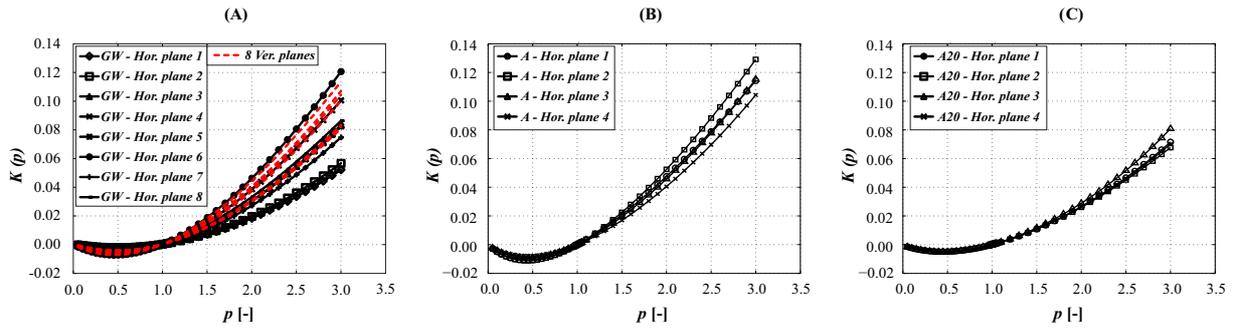
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759 Figure 8. Moment scaling functions  $K(p)$  obtained by applying TM technique on: (A) eight

760 horizontal fields from Figure 4 (different solid lines), and eight vertically oriented fields

761 (dashed lines); (B) four horizontal fields from Figure 5a; (C) four horizontal fields from

762 Figure 5b

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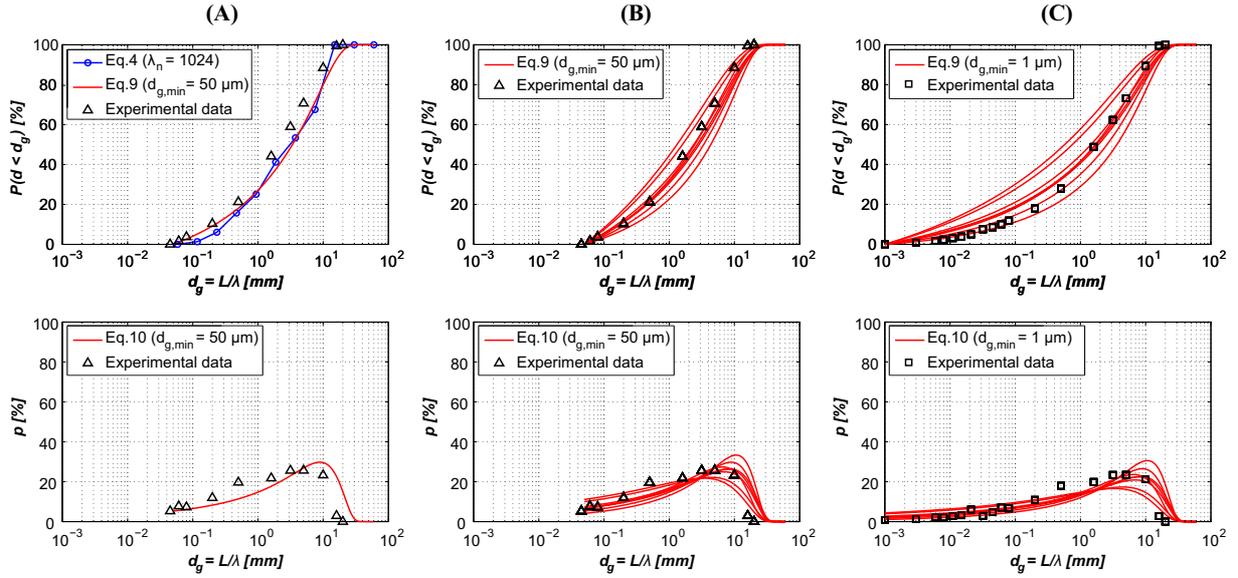
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779 Figure 9. Comparison between: A-top) Equation (9) applied on the GW Hor. plane 4 – solid

780 line ( $L = 60$  mm,  $\rho_{s,min}^{ind} = 1.55$ ,  $d_{g,min} = 50$   $\mu\text{m}$ ,  $C_l = 2.23 \times 10^{-2}$ ,  $\alpha = 1.67$ ), Equation (4)

781 applied on the same field – connected dots ( $\rho_{s,min}^{ind} = 1.55$ ,  $\lambda_n = 1024$ ), and truncated

782 experimental GSD data ( $d_g \geq 50$   $\mu\text{m}$ ) - triangles; A-bottom) Equation (10) computed with the

783 same parameter values as in A-top, and truncated experimental GSD data (triangles); B-top)

784 Equation (9) applied on eight  $\rho^{ind}(\lambda)$  fields from Figure 4 – solid lines ( $L$ ,  $\rho_{s,min}^{ind}$  and  $d_{g,min}$

785 identical as in A,  $\alpha$  and  $C_l$  presented in **Table 1**), and truncated experimental GSD data

786 (triangles); B-bottom) Same as in A-bottom just for all fields from Figure 4; C-top) Same as

787 in B-top just for  $d_{g,min} = 1$   $\mu\text{m}$  (solid lines), and full-range experimental GSD data ( $d_g \geq 1$   $\mu\text{m}$ )

788 – squares; C-bottom) Same as in B-bottom just for  $d_{g,min} = 1$   $\mu\text{m}$  (solid lines), and full-range

789 experimental GSD data ( $d_g \geq 1$   $\mu\text{m}$ ) – squares

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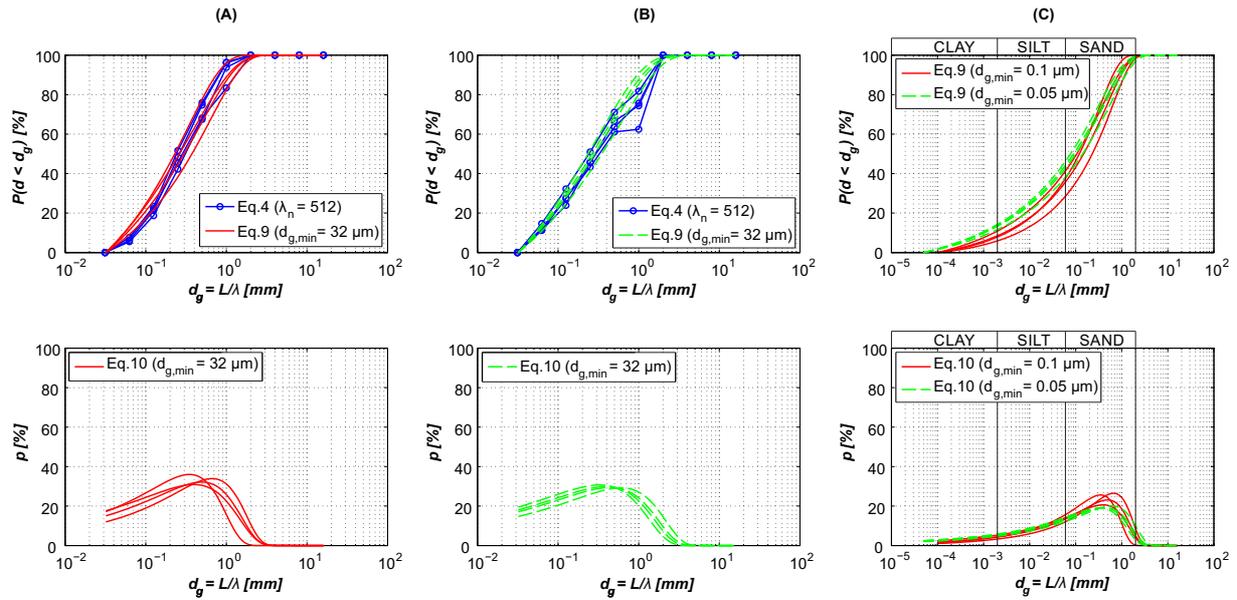
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797 Figure 10. Comparison between: (A) Equation (9) applied on four Horizon A fields from

798 Figure 5A ( $L = 16 \text{ mm}$ ,  $\rho_{s,min}^{ind} = 1.73$ ,  $d_{g,min} = 32 \text{ }\mu\text{m}$ ,  $\alpha$  and  $C_I$  from Table 2), and Equation

799 (4) (box-counting method) applied on same fields ( $\lambda_n = 512 \approx L/d_{g,min}$ ); (B) Same as in (A)

800 just for Horizon A20 ( $\rho_{s,min}^{ind} = 1.54$ ); (C) Equation (9) applied on four fields from Figure 5A

801 (parameters same as in (A), only  $d_{g,min} = 0.1 \text{ }\mu\text{m}$  – solid lines) and on four fields from Figure

802 5B (parameters same as in (A), only  $d_{g,min} = 0.05 \text{ }\mu\text{m}$  – dashed lines)

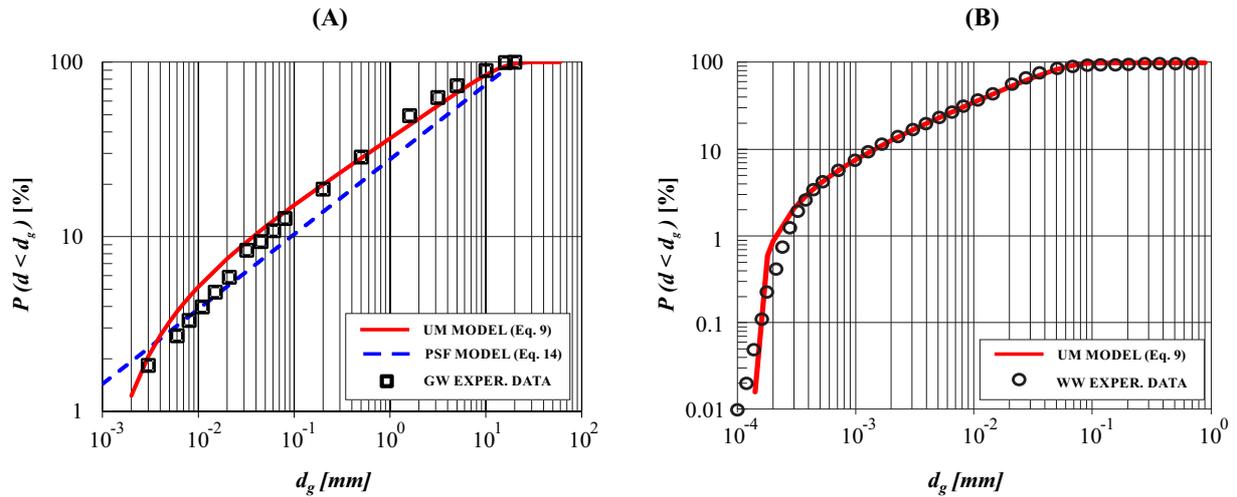
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 809 Figure 11. (A) Comparison between the experimental GSD data of the GW substrate (squares)  
 810 and Equations (9) (solid lines) and (14) (dashed lines) computed with the adjusted parameter  
 811 values that fit the measurements ( $C_l = 2.25 \times 10^{-2}$  and  $\alpha = 1.60$  for UM model, and  $D_{f,PSF} =$   
 812  $2.57$  for PSF model); (B) same as in (A) just for Walla Walla soil taken from Bittelli et al.  
 813 (1999) –  $L = 1$  mm,  $d_{g,min} = 1 \times 10^{-4}$  mm,  $\rho_{s,min}^{ind} = 1.72$ ,  $\alpha = 1.05$ ,  $C_l = 4.5 \times 10^{-2}$  (PSF model is  
 814 not illustrated since it clearly deviates from the measurements)

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831 **Table 1.** Determined UM parameters for eight  $\rho^{ind}(\lambda)$  fields of the GW substrate presented in

832 Figure 4

Horizontal plane								
	1	2	3	4	5	6	7	8
$C_I$	9.34E-03	9.93E-03	1.64E-02	2.23E-02	1.66E-02	2.72E-02	1.45E-02	1.93E-02
$\alpha$	1.93	1.96	1.83	1.67	1.80	1.61	1.85	1.66
Vertical plane								
	1	2	3	4	5	6	7	8
$C_I$	2.66E-02	2.21E-02	1.74E-02	1.72E-02	1.67E-02	1.63E-02	2.06E-02	2.37E-02
$\alpha$	1.56	1.72	1.75	1.75	1.76	1.82	1.78	1.60

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858 **Table 2.** Determined UM parameters for eight  $\rho^{ind}(\lambda)$  fields of La Herreria soil (Horizons A  
 859 and A20) presented in Figure 5

<b>Horizon A</b>				
	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
<b><math>C_I</math></b>	3.36E-02	3.55E-02	3.03E-02	2.60E-02
<b><math>\alpha</math></b>	1.15	1.25	1.30	1.37
<b>Horizon A20</b>				
	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>
<b><math>C_I</math></b>	1.62E-02	1.63E-02	1.73E-02	1.62E-02
<b><math>\alpha</math></b>	1.46	1.40	1.48	1.44

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