

Cracks at the interface of circular inclusions: propagation and homogenization

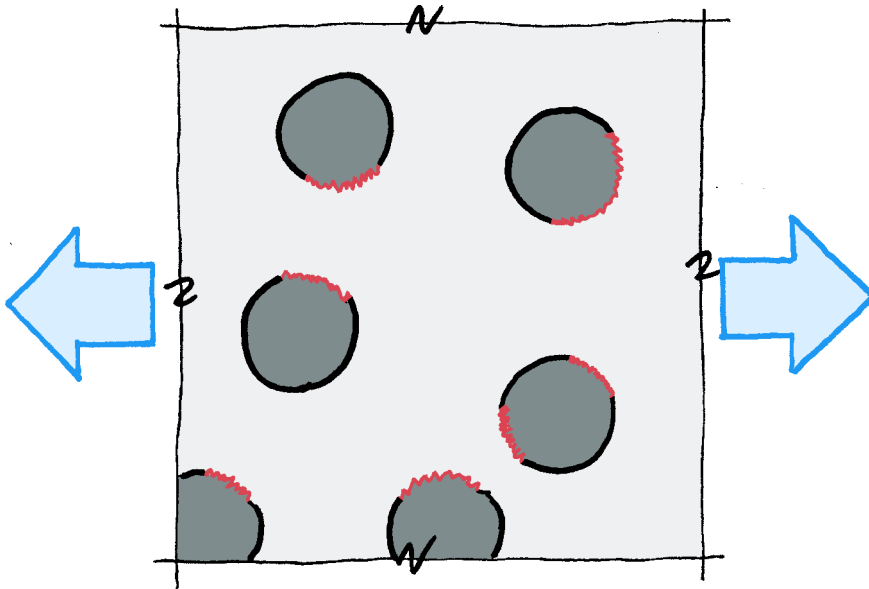
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Motivation



References: Le, Brisard, and Pouya [1] and Le [2]

- [1] V. H. T. Le, S. Brisard, A. Pouya, *International Journal of Solids and Structures* **2019**, 167, 71–78.
- [2] V. H. T. Le, PhD thesis, Université Paris-Est, **2019**.

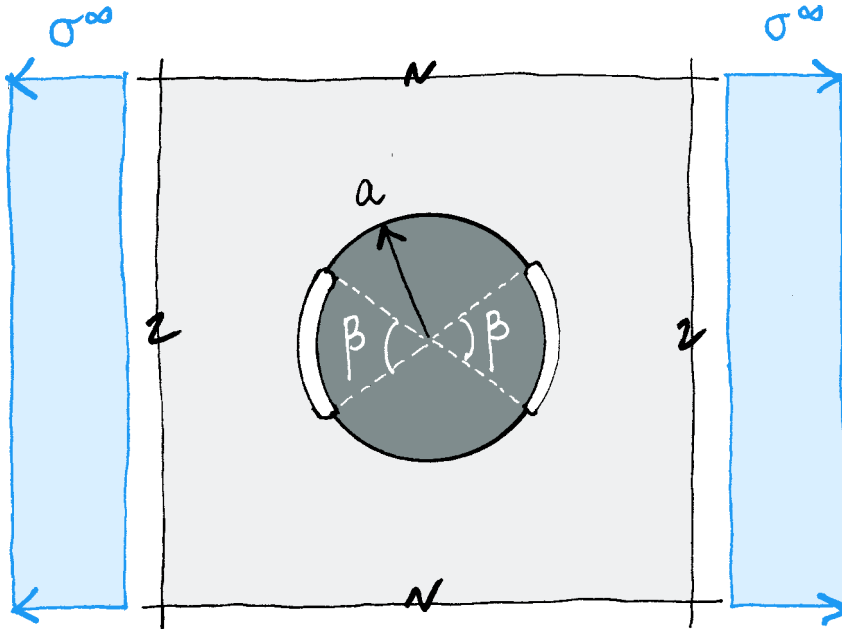
Outline of this talk

- Solution to the auxiliary problem: overview of the derivation
- Propagation of cracks around single inclusion
- Homogenization of elastic moduli
- Damage propagation

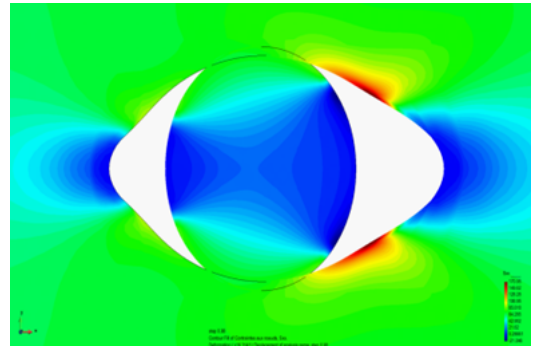
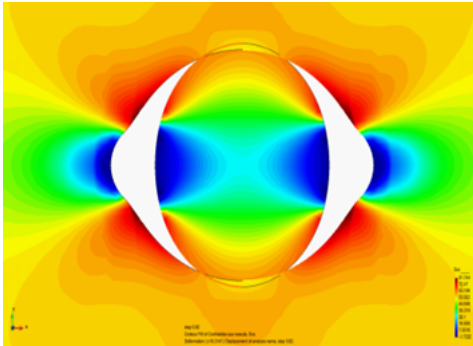
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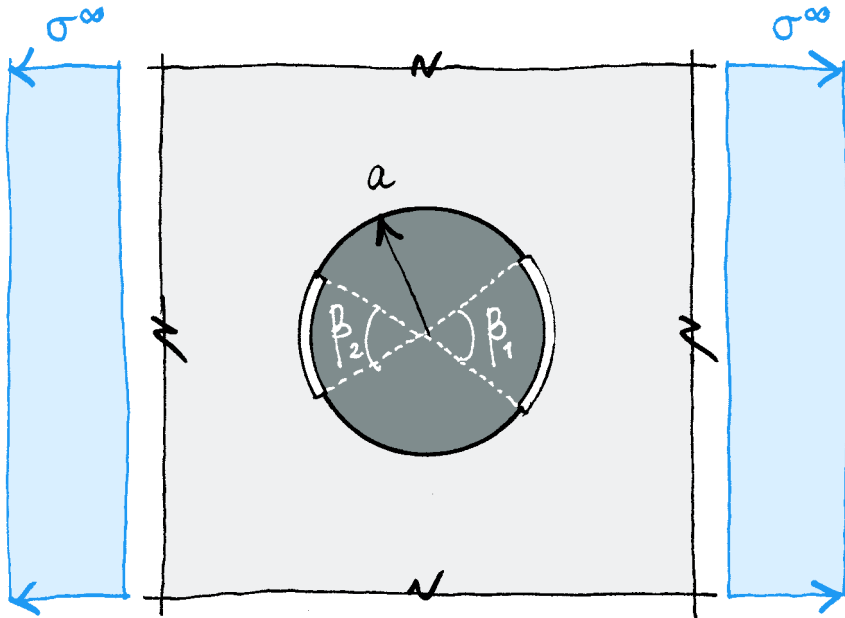
Auxiliary problem (1/3)



Auxiliary problem (2/3)



Auxiliary problem (3/3)



Solution to the auxiliary problem (1/4)

Complex potentials of Kolosov and Muskhelishvili [1]

Solution (stresses and displacements) fully defined by two complex potentials, Φ and Ψ (holomorphic functions).

$$\begin{aligned}\sigma_{rr} + \sigma_{\theta\theta} &= 2(\Phi + \bar{\Phi}) \\ \sigma_{rr} + i\sigma_{r\theta} &= \Phi + \bar{\Phi} - \bar{z}\Phi' - \frac{\bar{z}}{z}\bar{\Psi} \\ \partial_{\theta}(u_x + iu_y) &= \frac{iz}{2\mu}(\kappa\Phi - \bar{\Phi} + \bar{z}\Phi' + \frac{\bar{z}}{z}\bar{\Psi})\end{aligned}$$

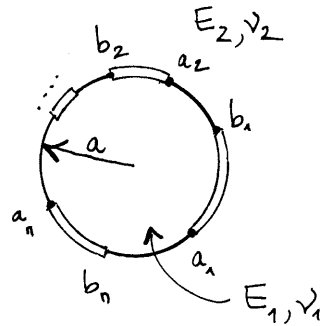
$$\kappa = \begin{cases} 3 - 4\nu & \text{plane strain} \\ \frac{3 - \nu}{1 + \nu} & \text{plane stress} \end{cases}$$

Two-phase materials

$$\alpha = \frac{\mu_1 + \kappa_1\mu_2}{\mu_2 + \kappa_2\mu_1} \quad m = \frac{1 + \kappa_2\mu_1}{1 + \kappa_1\mu_2} \quad \epsilon = \frac{\ln \alpha}{2\pi}$$

The Ω potential [2]

$$\Omega_j(z) = \overline{\Phi_j\left(\frac{a^2}{z}\right)} - \frac{a^2}{z}\overline{\Phi_j'\left(\frac{a^2}{z}\right)} - \frac{a^2}{z^2}\overline{\Psi_j\left(\frac{a^2}{z}\right)}$$



[1] N. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, 1st, Springer Netherlands, 1977, 732 pp.

[2] A. B. Perlman, G. C. Sih, *International Journal of Engineering Science* 1967, 5, 845–867.

Solution to the auxiliary problem (2/4)

Tractions and displacements at the interface

$$\begin{aligned}
 T &= \sigma_{rr} + i \sigma_{r\theta} & T_1^- &= \Phi_1^- + \Omega_1^+ & T_2^+ &= \Phi_2^+ + \Omega_2^- \\
 u &= u_x + i u_y & \partial_\theta u_1^- &= \frac{iz}{2\mu_1} (\kappa_1 \Phi_1^- - \Omega_1^+) & \partial_\theta u_2^+ &= \frac{iz}{2\mu_2} (\kappa_2 \Phi_2^+ - \Omega_2^-)
 \end{aligned}$$

$$F^-(ae^{i\theta}) = \lim_{r \rightarrow a^-} F(re^{i\theta}) \quad F^+(ae^{i\theta}) = \lim_{r \rightarrow a^+} F(re^{i\theta})$$

Boundary conditions

$$\text{At infinity: } \begin{cases} \Phi_2(z) = \frac{1}{2}(\sigma_{xx}^\infty + \sigma_{yy}^\infty) + \mathcal{O}(z^{-2}) \\ \Psi_2(z) = \frac{1}{2}(\sigma_{yy}^\infty - \sigma_{xx}^\infty) + i \sigma_{xy}^\infty + \mathcal{O}(z^{-2}) \end{cases}$$

$$\text{Bonded interfaces: } \begin{cases} \Phi_1^- + \Omega_1^+ = \Phi_2^+ + \Omega_2^- \\ \mu_2(\kappa_1 \Phi_1^- - \Omega_1^+) = \mu_1(\kappa_2 \Phi_2^+ - \Omega_2^-) \end{cases}$$

$$\text{Cracks: } \Phi_1^- + \Omega_1^+ = \Phi_2^+ + \Omega_2^- = 0$$

Solution to the auxiliary problem (3/4)

The formal solution of Perlman and Sih [1]

$$(1 + \alpha)\Phi_1 = \chi P + mQ$$

$$(1 + \alpha)\Omega_1 = \alpha\chi P - mQ$$

$$(1 + \alpha)\Phi_2 = \alpha\chi P + Q$$

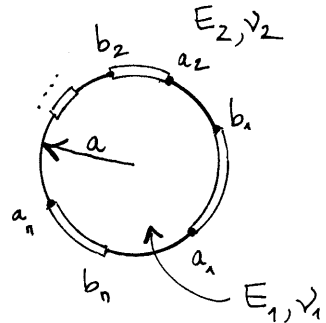
$$(1 + \alpha)\Omega_2 = \chi P - Q$$

$$\chi(z) = \prod_{k=1}^n (z - a_k)^{-\frac{1}{2} + i\epsilon} (z - b_k)^{-\frac{1}{2} - i\epsilon}$$

$$P(z) = P_{-2}z^{-2} + P_{-1}z^{-1} + P_0 + P_1z + \dots + P_nz^n$$

$$Q(z) = Q_{-2}z^{-2} + Q_{-1}z^{-1} + Q_0$$

(n + 6) constants to be determined!



[1] A. B. Perlman, G. C. Sih, *International Journal of Engineering Science* 1967, 5, 845–867.

Solution to the auxiliary problem (4/4)

Determination of the integration constants

- Boundary conditions at infinity (4 equations)
- Analyticity of Φ_1 and Ψ_1 at $z = 0$ (3 equations)
- Single-valuedness of displacements ($n - 1$ equations)

($n + 6$) equations for ($n + 6$) unknowns!

Implementation of the solution

- Perlman and Sih [1] and Toya [2]: one single crack
- Prasad and Simha [3, 4]: two symmetric cracks
- Le, Brisard, and Pouya [5]: two coaxial, asymmetric cracks

[1] A. B. Perlman, G. C. Sih, *International Journal of Engineering Science* **1967**, 5, 845–867.

[2] M. Toya, *Journal of the Mechanics and Physics of Solids* **1974**, 22, 325–348.

[3] P. B. N. Prasad, K. R. Y. Simha, *International Journal of Fracture* **2002**, 117, 39–62.

[4] P. B. N. Prasad, K. R. Y. Simha, *Engineering Fracture Mechanics* **2003**, 70, 285–307.

[5] V. H. T. Le, S. Brisard, A. Pouya, *International Journal of Solids and Structures* **2019**, 167, 71–78.

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- **Propagation of cracks around single inclusion**
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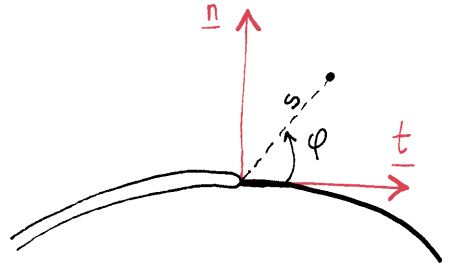
Complex stress-intensity factor

Stresses and displacements near crack-tip [1]

$$\sigma_{nn} + i\sigma_{nt} \sim \frac{K}{\sqrt{2\pi s}} s^{i\epsilon} \quad (\varphi = 0, s \rightarrow 0)$$

$$[u_n + i u_t] \sim \frac{8K}{E^*} \sqrt{\frac{s}{2\pi}} \frac{s^{i\epsilon}}{1 + 2i\epsilon \cosh(\pi\epsilon)}$$

$$E_j^\sigma = E_j \quad E_j^\epsilon = E_j(1 - \nu_j^2)^{-1} \quad \frac{2}{E^*} = \frac{1}{E_1^*} + \frac{1}{E_2^*}$$



Oscillations of displacements [2]

$s^{i\epsilon} = \cos(\epsilon \ln s) + i \sin(\epsilon \ln s) \Rightarrow$ Interpenetration of crack walls!

SIF has “awkwardly complex” physical dimensions [2]

$$\hat{K} = \hat{L}^{i\epsilon} \mathcal{K} \quad \hat{L} : \text{arbitrary reference length [3]}$$

- [1] M. L. Williams, *Bulletin of the Seismological Society of America* **1959**, 49, 199–204.
- [2] J. R. Rice, *Journal of Applied Mechanics* **1988**, 55, 98–103.
- [3] L. Banks-Sills, *International Journal of Fracture* **2015**, 191, 131–146.

Fracture criterion

Energy release rate: extension of Irwin's formula [1]

$$G = \frac{|K|^2}{E^* \cosh^2(\pi\epsilon)}$$

Simple fracture criterion

$$G = G_c = \text{const.}$$

More elaborate fracture criterion [2]

$$G = G_{c,\min} (1 + \tan^2 \psi) \quad \psi = \arg \hat{K}$$

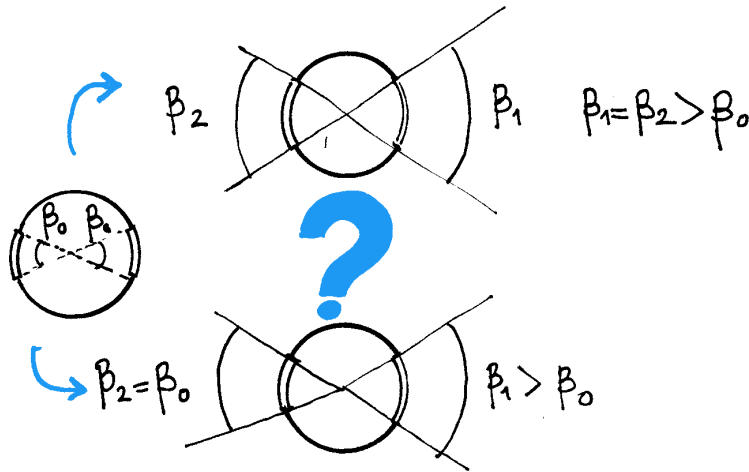
This criterion depends on \hat{L} , which becomes a material parameter.

[1] B. M. Malyshev, R. L. Salganik, *International Journal of Fracture Mechanics* **1965**, *1*, 114–128.

[2] L. Banks-Sills, *International Journal of Fracture* **2015**, *191*, 131–146.

Symmetric vs. asymmetric propagation

- System initially fully symmetric: is symmetry preserved?
- First analysis concluded in ref. [1] to **conditional asymmetry**
- New analysis based on the framework of Fedelich and Ehrlicher [2].



[1] V. H. T. Le, S. Brisard, A. Pouya, *International Journal of Solids and Structures* **2019**, 167, 71–78.

[2] B. Fedelich, A. Ehrlicher, *Comptes Rendus de l'Académie des Sciences Série II* **1989**, 308, 1391–1394.

A tribute to A. Ehlacher...

Minimum principle of Fedelich and Ehlacher [1], see also [2–4]

At each time t , for all $\delta\ell_\alpha \geq 0$ ($\alpha = 1, 2$)

$$\Pi(\ell_1, \ell_2, t) + G_{c1}\ell_1 + G_{c2}\ell_2 \leq \Pi(\ell_1 + \delta\ell_1, \ell_2 + \delta\ell_2, t) + G_{c1}(\ell_1 + \delta\ell_1) + G_{c2}(\ell_2 + \delta\ell_2)$$

First-order conditions

$$G_\alpha(\ell_1, \ell_2, t) \leq G_c \quad G_\alpha = -\frac{\partial \Pi}{\partial \ell_\alpha}$$
$$\dot{\ell}_\alpha(t) [G_c - G_\alpha(\ell_1, \ell_2, t)] = 0$$

(no summation)

Second-order condition

$$(\hat{\ell}_1, \hat{\ell}_2) \in \arg \min_{\hat{\ell}_1, \hat{\ell}_2 \geq 0} \left(\frac{1}{2} H_{\alpha\beta} \hat{\ell}_\alpha \hat{\ell}_\beta - K_\alpha \hat{\ell}_\alpha \right) \quad K_\alpha = \partial_t G_\alpha$$

- [1] B. Fedelich, A. Ehlacher, *Comptes Rendus de l'Académie des Sciences Série II* **1989**, 308, 1391–1394.
- [2] G. Francfort, J.-J. Marigo, *Journal of the Mechanics and Physics of Solids* **1998**, 46, 1319–1342.
- [3] A. Chambolle, G. Francfort, J.-J. Marigo, *Journal of the Mechanics and Physics of Solids* **2009**, 57, 1614–1622.
- [4] B. Bourdin, G. A. Francfort in *Variational Models and Methods in Solid and Fluid Mechanics*, (Eds.: F. dell'Isola, S. Gavrilyuk), CISM Courses and Lectures, Springer Vienna, Vienna, **2011**, pp. 107–161.

The case of a symmetric system

$$\ell_1 = \ell_2 = \ell_0 \quad G_1 = G_2 = G_c \quad H_{11} = H_{22} = H \quad K_1 = K_2 = K$$

Second-order condition

$$(\hat{\ell}_1, \hat{\ell}_2) \in \arg \min_{\hat{\ell}_1, \hat{\ell}_2 \geq 0} \left[\frac{1}{2} H (\hat{\ell}_1^2 + \hat{\ell}_2^2) + H_{12} \hat{\ell}_1 \hat{\ell}_2 - K (\hat{\ell}_1 + \hat{\ell}_2) \right]$$

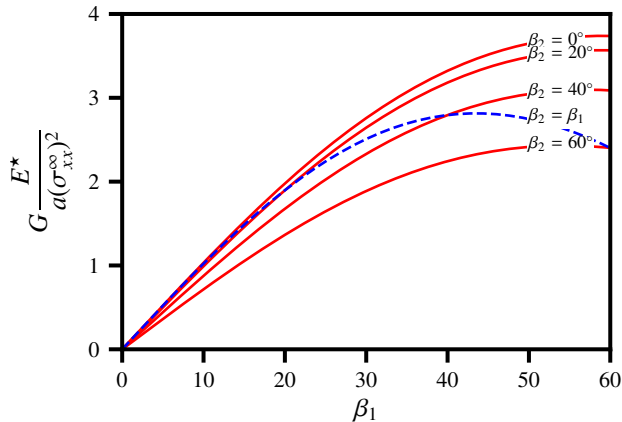
1. If $|H_{12}| < H$

- Quadratic form is positive definite
- **Symmetric** propagation always favorable!

2. If $|H_{12}| > H$

- Quadratic form has a strictly negative eigenvalue
- Propagation is **unstable**!

Application



$$E_1 = 10E_2, \nu_1 = 0.2, \nu_2 = 0.3 \quad [1]$$

$$H = \frac{\partial G_1}{\partial \ell_1} \quad H_{12} = \frac{\partial G_1}{\partial \ell_2} \quad |H_{12}| > H \quad \Rightarrow \quad \text{unstable!}$$

[1] V. H. T. Le, S. Brisard, A. Pouya, *International Journal of Solids and Structures* **2019**, 167, 71–78.

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Homogenization of partially debonded inclusions

Use morphologically representative patterns [1]

MRP = inclusion + surrounding crack(s)

Solution to Eshelby's inhomogeneity problem

Use previously derived solution

Average strain in MRP

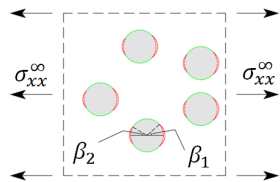
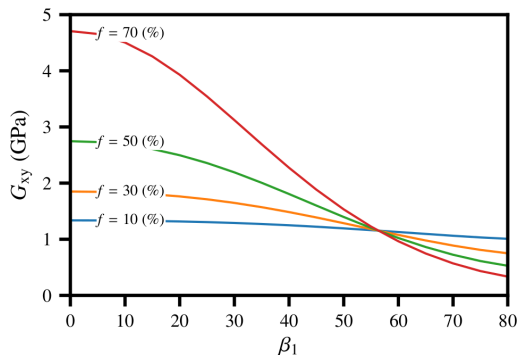
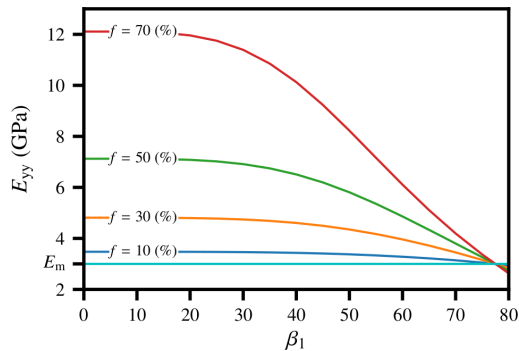
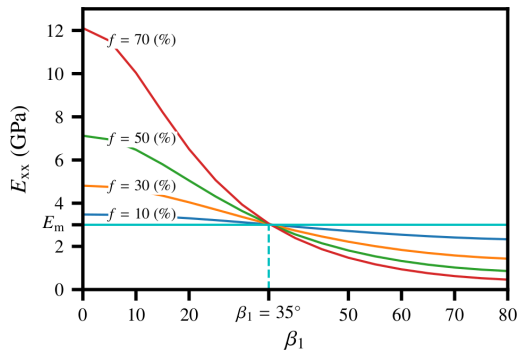
Contribution of strain in inclusion & displacement discontinuities [2]

Mori–Tanaka homogenization [3]

Watch-out for loss of symmetry! [4–6]

- [1] M. Bornert, C. Stolz, A. Zaoui, *Journal of the Mechanics and Physics of Solids* **1996**, 44, 307–331.
- [2] Y. Benveniste, *Mechanics of Materials* **1985**, 4, 197–208.
- [3] Y. Benveniste, *Mechanics of Materials* **1987**, 6, 147–157.
- [4] Y. Benveniste, G. J. Dvorak, T. Chen, *Journal of the Mechanics and Physics of Solids* **1991**, 39, 927–946.
- [5] M. Ferrari, *Mechanics of Materials* **1991**, 11, 251–256.
- [6] J. Schjødt-Thomsen, R. Pyrz, *Mechanics of Materials* **2001**, 33, 531–544.

Effective moduli vs. crack length



$E_m = 3\text{GPa}$ $\nu_m = 0.3$ $\beta_1 = \beta_2$
 $E_i = 95\text{GPa}$ $\nu_i = 0.27$

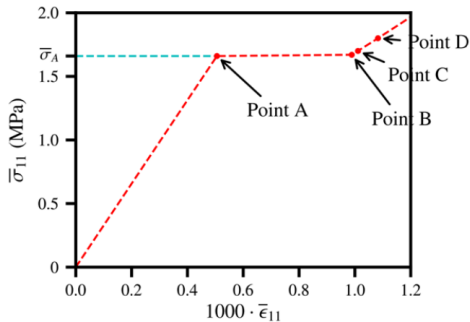
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Damage propagation

Forward Euler approach

1. Propose macroscopic stress increment $\Delta\bar{\sigma}$
2. Localization
 - Compute stress increment in matrix, $\Delta\sigma_m$
 - Analysis of propagation: compute $\Delta\beta_1, \Delta\beta_2$
3. Homogenization: update effective moduli



Point	β_1^i	β_2^i
A	30°	30°
B	113.69°	30°
C	114.85°	30°
D	118.36°	30°

$$\begin{aligned}E_m &= 3\text{GPa} \\v_m &= 0.3 \\E_i &= 95\text{GPa} \\v_i &= 0.27 \\f &= 24\% \\G_c &= 0.05\text{J m}^{-2}\end{aligned}$$

To conclude

Homogenization of partially debonded, circular inclusions

Perspectives – local problem

- Nucleation?
- Allow for arbitrary number of cracks?
- More realistic propagation criterion?
- Extension to 3D?

Perspectives – macroscopic model

- Better account for interactions
- Introduce some diversity!
- Allow for symmetry breaking!
- Relation to prior works on bulk damage [1]

[1] L. Dormieux, D. Kondo, F.-J. Ulm, *Comptes Rendus Mécanique* **2006**, 334, 440–446.

Thank you for your attention

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<https://navier-lab.fr/en/equipe/brisard-sebastien>

<https://cv.archives-ouvertes.fr/sbrisard>

<https://sbrisard.github.io>



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