

Multiscale finite elements for heterogeneous plates

S. Brisard*, V. Ehrlacher^{†,‡}

*Laboratoire Navier (UMR 8205), Université Paris-Est, CNRS, ENPC, IFSTTAR

†CERMICS, École des Ponts ParisTech

‡MATHERIALS, Inria

26 august 2019



Motivation



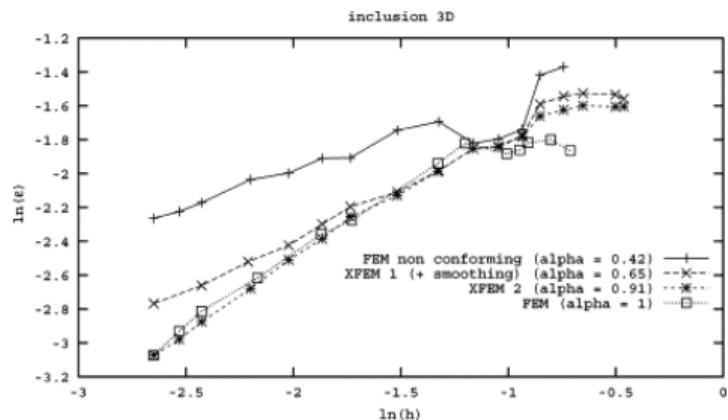
We do not want to use a conforming mesh!

Heterogeneous (Classical) Finite elements

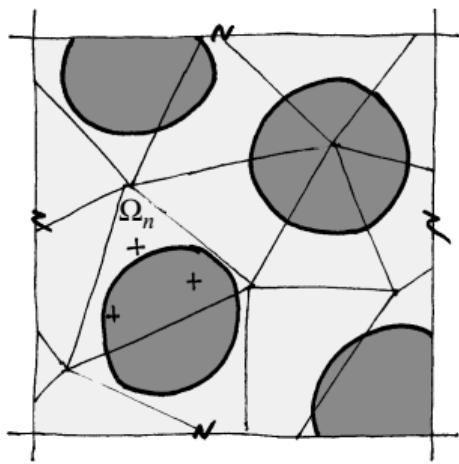
Element stiffness matrix

$$K_{n,ij} = \int_{\Omega_n} \boldsymbol{\epsilon}(\Phi_{n,i})(\mathbf{x}) : \mathbf{C}(\mathbf{x}) : \boldsymbol{\epsilon}(\Phi_{n,j})(\mathbf{x}) \, d\mathbf{x}$$

Φ_i : classical shape functions



Reproduced from Moës et al. [Moë+03]

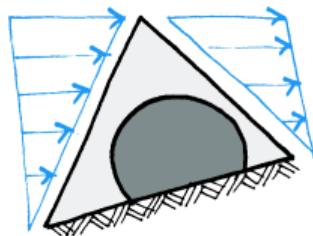


Multiscale Finite Elements in a nutshell

- Classical shape functions → overly stiff heterogeneous elements
- MsFEM uses **enriched** shape functions [EH09]
- MsFEM shape functions should reflect the heterogeneity
- MsFEM shape functions $\Psi_{n,i}$ solve a “similar”, **local** problem

$$\operatorname{div}[\mathbf{C} : \boldsymbol{\epsilon}(\Psi_{n,i})] = \mathbf{0} \text{ in } \Omega_n$$

$$\Psi_{n,i} = \Phi_{n,i} \text{ on } \partial\Omega_n$$



- Reminiscent of the “corrector” problem in homogenization
- Continuity of displacements is ensured

An offline/online strategy

Offline stage

- Solve the local problem for each macro-element and shape function
- Compute element stiffness matrix

$$K_{n,ij} = \int_{\Omega_e} \boldsymbol{\epsilon}(\Psi_{n,i})(\mathbf{x}) : \mathbf{C}(\mathbf{x}) : \boldsymbol{\epsilon}(\Psi_{n,j})(\mathbf{x}) \, d\mathbf{x}$$

- Assemble global stiffness matrix

Online stage

- Assemble right-hand side for each loading
- Solve the macro linear system
- Combine MsFEM shape functions to reconstruct displacement

Comments

- Requires a conforming mesh of each macro-element
- But resulting FE model is small

Comments on convergence

$$\left\{ \int_{\Omega} [\boldsymbol{\epsilon}(\mathbf{u}^H) - \boldsymbol{\epsilon}(\mathbf{u})] : \mathbf{C} : [\boldsymbol{\epsilon}(\mathbf{u}^H) - \boldsymbol{\epsilon}(\mathbf{u})] \right\}^{1/2} \leq C \left(\frac{H}{L} + \frac{a}{L} + \sqrt{\frac{a}{H}} \right)$$

H : size of macro-elements; a : size of heterogeneities

Two point of view on the separation of scales

- The mathematician

$$L_{\text{micro}}/L_{\text{macro}} \rightarrow 0$$

- The mechanician

$$L_{\text{micro}} \ll L_{\text{meso}} \ll L_{\text{macro}}$$

Heterogeneous plates

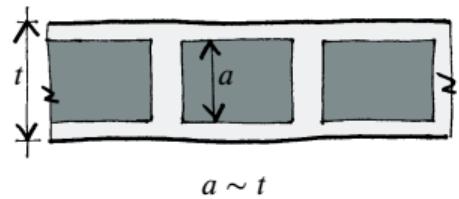
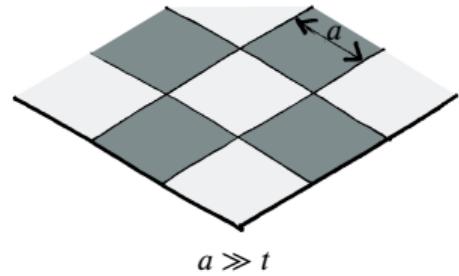
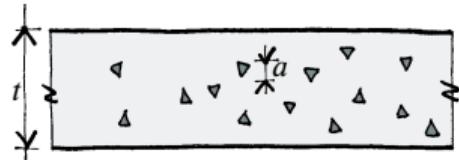
Characteristic length-scales

- L : macroscopic length-scale (span, size of load patch, etc...)
- t : thickness
- a : typical size of heterogeneities

$$t \ll L \quad \text{and} \quad a \ll L$$

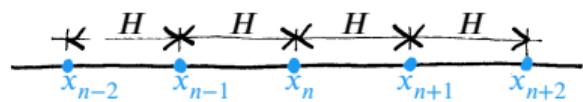
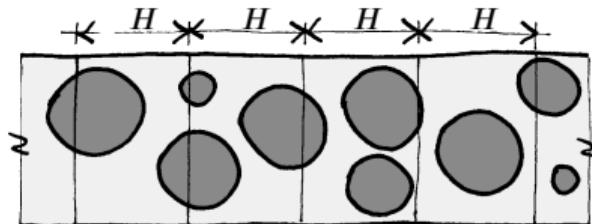
Asymptotic regimes [CN84; KV84]

- $a \ll t$: 3D homogenization
- $a \gg t$: 2D homogenization
- $a \sim t$: the most interesting problem

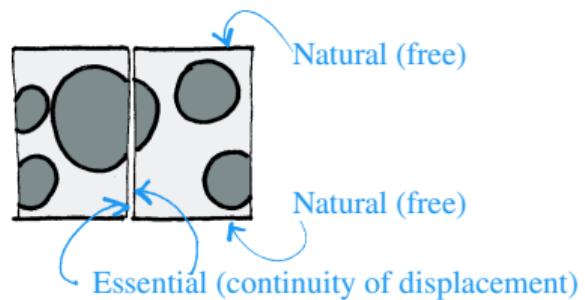


Ms-FEM strategy: guiding principles

One element in the thickness



Boundary conditions

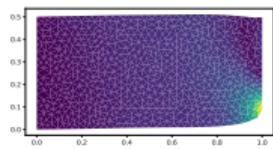
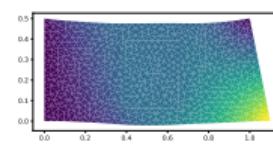
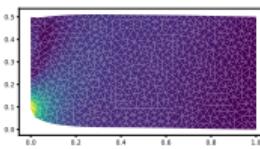
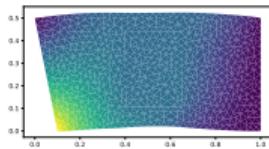
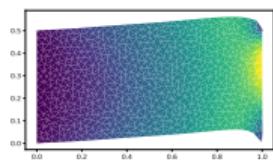
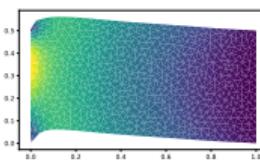
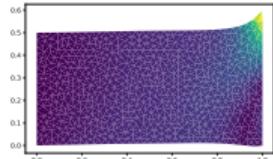
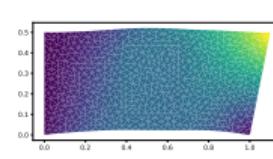
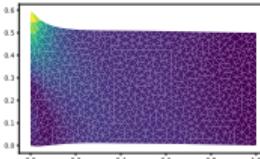
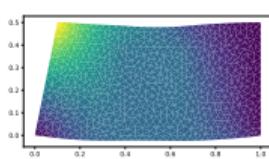


Thickness changes

At least quadratic transverse displacement [PR97; Rös+99; BSS11]

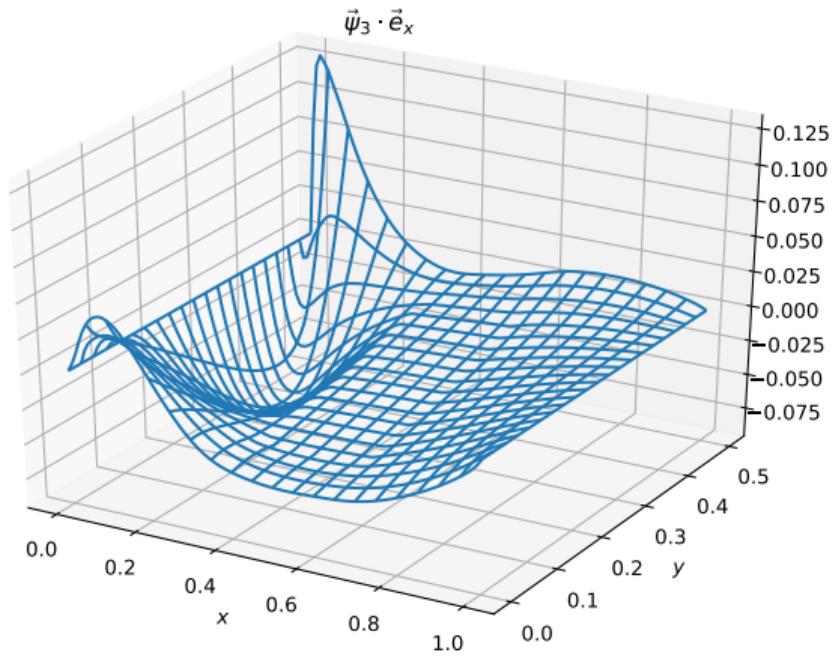
Displacement-based approach (2/3)

Linear-quadratic interpolation of boundary displacements



Displacement-based approach (3/3)

Boundary layers...



Towards mixed MsFEM

A stress-based approach? [Rei47]

- Idea: construct trial stresses from enriched stress shape functions
- Problem: must be statically admissible

A mixed approach?

- See Refs. [CH03] for porous media and XXXXX for conductivity in plates

The Hellinger–Reissner principle

Find \mathbf{u} and $\boldsymbol{\sigma}$ such that, for all $\hat{\mathbf{u}}$ and $\hat{\boldsymbol{\sigma}}$

$$\begin{cases} b(\boldsymbol{\sigma}, \hat{\mathbf{u}}) = \int_{\Omega} \mathbf{b} \cdot \hat{\mathbf{u}} + \int_{\partial\Omega_T} \mathbf{T} \cdot \hat{\mathbf{u}} & a(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}) = \int_{\Omega} \boldsymbol{\sigma} : \mathbf{S} : \hat{\boldsymbol{\sigma}} \\ a(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}) - b(\hat{\boldsymbol{\sigma}}, \mathbf{u}) = \int_{\partial\Omega_u} \mathbf{n} \cdot \hat{\boldsymbol{\sigma}} \cdot \bar{\mathbf{u}} & b(\boldsymbol{\sigma}, \hat{\mathbf{u}}) = \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\hat{\mathbf{u}}) - \int_{\partial\Omega_u} \hat{\mathbf{u}} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \end{cases}$$

The trial and test stresses

The general form over Ω_n

$$\boldsymbol{\sigma} = N_n \boldsymbol{\sigma}_{n,1} + V_n \boldsymbol{\sigma}_{n,2} + M_n \boldsymbol{\sigma}_{n,3} + N_{n+1} \boldsymbol{\sigma}_{n,4} + V_{n+1} \boldsymbol{\sigma}_{n,5} + M_{n+1} \boldsymbol{\sigma}_{n,6}$$

The local problem

$\boldsymbol{\sigma}_{n,i}$ is the solution to

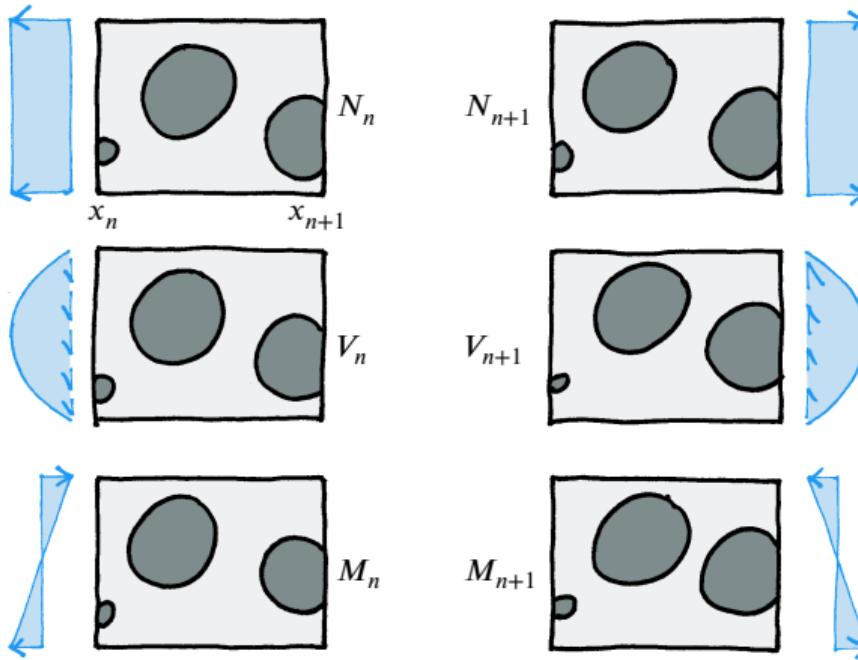
$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \mathbf{b}_{n,i} = \mathbf{0} \\ \boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\epsilon}(\mathbf{u}) \end{cases} \quad \text{and} \quad \begin{cases} \boldsymbol{\sigma}(x = x_n) \cdot \mathbf{n} = \mathbf{T}_{n,i}^{\text{left}} \\ \boldsymbol{\sigma}(x = x_{n+1}) \cdot \mathbf{n} = \mathbf{T}_{n,i}^{\text{right}} \\ \boldsymbol{\sigma}(y = \pm t/2) \cdot \mathbf{n} = \mathbf{0} \end{cases}$$

Continuity of traction

$$\mathbf{T}_{n,3+i}^{\text{right}} + \mathbf{T}_{n+1,i}^{\text{left}} = \mathbf{0} \quad (i = 1, 2, 3)$$

The trial and test stresses

The six elementary load cases



+ body forces!

The trial and test displacements

General form

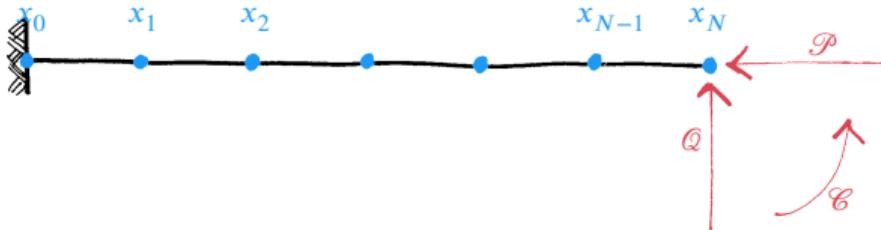
$$\mathbf{u} = [u(x) - y \theta(x)]\mathbf{e}_x + v(x)\mathbf{e}_y$$

How to discretize u , v and θ ?

The most simple discretization strategy

- Piecewise linear interpolation
- Consistent with discretization of stresses (same number of unknowns)
- **BUT** major risk of shear locking: $\theta = v'$ for thin plates
- Reduced integration is not an option

The problem with shear



Generalized stresses

$$N_n = \mathcal{P} \quad V_n = \mathcal{Q} \quad M_n = \mathcal{Q}(L - x_n) + \mathcal{C}$$

Generalized displacements (homogeneous plate)

$$u_n = \frac{\mathcal{P}(1 - \nu^2)}{Et} x_n \quad \theta_n = \frac{12\mathcal{C}(1 - \nu^2)}{Et^3} x_n$$

But oscillatory solution for v_n !

Can we get rid of shear?

Attempt #1: Kirchhoff–Love approach

$$\boldsymbol{\sigma} = N_n \boldsymbol{\sigma}_{n,1} + M_n \boldsymbol{\sigma}_{n,3} + N_{n+1} \boldsymbol{\sigma}_{n,4} + M_{n+1} \boldsymbol{\sigma}_{n,6}$$

$$\mathbf{u} = [u(x) - y v'(x)] \mathbf{e}_x + v(x) \mathbf{e}_y$$

Cannot apply a vertical force at the end of a cantilever!

Attempt #2: introduce shear forces that do not produce work

$$\boldsymbol{\sigma} = N_n \boldsymbol{\sigma}_{n,1} + V_n \boldsymbol{\sigma}_{n,2} + M_n \boldsymbol{\sigma}_{n,3} + N_{n+1} \boldsymbol{\sigma}_{n,4} + V_{n+1} \boldsymbol{\sigma}_{n,5} + M_{n+1} \boldsymbol{\sigma}_{n,6}$$

$$\mathbf{u} = [u(x) - y v'(x)] \mathbf{e}_x + v(x) \mathbf{e}_y$$

Inconsistent number of unknowns!

3 generalized stresses and 2 generalized displacements per node

Summary and outlook

Summary

- Application of MsFEM strategies to plates is a natural extension
- The specific features of plates (essential vs. natural BCs, thickness changes) must be accounted for
- Improper discretization may lead to problems that are reminiscent of shear-locking issues for thin plates

Outlook

- Quadratic variation of v and M through bubble nodes
- Quadratic variation of v (and M ?) through end slopes and continuity

References

- [BSS11] D. Braess et al. “On the Justification of Plate Models”. In: *Journal of Elasticity* 103.1 (2011), pp. 53–71.
- [CH03] Z. Chen and T. Hou. “A Mixed Multiscale Finite Element Method for Elliptic Problems with Oscillating Coefficients”. In: *Mathematics of Computation* 72.242 (2003), pp. 541–576.
- [CN84] D. Caillerie and J. C. Nedelec. “Thin Elastic and Periodic Plates”. In: *Mathematical Methods in the Applied Sciences* 6.1 (1984), pp. 159–191.
- [EH09] Y. Efendiev and T. Y. Hou. *Multiscale Finite Element Methods – Theory and Applications*. Vol. 4. Surveys and Tutorials in the Applied Mathematical Sciences. New York: Springer-Verlag, 2009.
- [KV84] R. V. Kohn and M. Vogelius. “A New Model for Thin Plates with Rapidly Varying Thickness”. In: *International Journal of Solids and Structures* 20.4 (1984), pp. 333–350.
- [Moë+03] N. Moës et al. “A Computational Approach to Handle Complex Microstructure Geometries”. In: *Computer Methods in Applied Mechanics and Engineering* 192.28–30 (2003), pp. 3163–3177.
- [PR97] J.-C. Paumier and A. Raoult. “Asymptotic Consistency of the Polynomial Approximation in the Linearized Plate Theory”. In: *ESAIM: Proceedings* 2 (1997), pp. 203–213.

- [Rei47] E. Reissner. “On Bending of Elastic Plates”. In: *Quarterly of Applied Mathematics* 5.1 (1947), pp. 55–68.
- [Rös+99] A. Rössle et al. “On the Mathematical Foundation of the (1,1,2)-Plate Model”. In: *International Journal of Solids and Structures* 36.14 (1999), pp. 2143–2168.

Thank you for your attention

sebastien.brisard@ifsttar.fr

<http://navier.enpc.fr/BRISARD-Sebastien>

<http://sbrisard.github.io>



This work is licensed under the Creative Commons Attribution 4.0 International License. To view a copy of this license, visit <http://creativecommons.org/licenses/by/4.0/> or send a letter to Creative Commons, PO Box 1866, Mountain View, CA 94042, USA.