SDCA-Powered Inexact Dual Augmented Lagrangian Method for Fast CRF Learning

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Abstract

We propose an efficient *dual* augmented Lagrangian formulation to learn conditional random fields (CRF). Our algorithm, which can be interpreted as an inexact gradient descent algorithm on the multiplier, does not require to perform global inference iteratively, and requires only a fixed number of stochastic clique-wise updates at each epoch to obtain a sufficiently good estimate of the gradient w.r.t. the Lagrange multipliers. We prove that the proposed algorithm enjoys global linear convergence for both the primal and the dual objectives. Our experiments show that the proposed algorithm outperforms state-of-the-art baselines in terms of speed of convergence.

1 Introduction

Learning in graphical models has historically relied on the computation of the (sub)gradient of the log-likelihood w.r.t. to the canonical parameters, which requires to solve a MAP or probabilistic inference problem at each iteration. This approach is slow given that the inference problem is itself computationally expensive. The difficulty of inference and learning in graphical models is related to the fact that the log-partition function is in general intractable.

Recent progress on the optimization problems whose objective is a large finite sum of convex terms has shown that they could be optimized very efficiently by stochastic algorithms that sample one term at a time (Defazio et al., 2014; Roux et al., 2012; Shalev-Shwartz and Zhang, 2016). It turns out that the dual objective of the maximum likelihood estimation of CRF (a.k.a. the maximum entropy principle) decomposes additively over all cliques if a decomposable entropy surrogate is used. Even though this dual formulation has a potential to take advantage of stochastic algorithms, and can be optimized without resorting to solve a global inference on the entire graph per iteration, all dual parameters (i.e. *mean parameters*) are coupled by the *marginal polytope* constraints, which are in general intractable. Even its most commonly used relaxation, namely the *local consistency polytope*, is itself in practice difficult to optimize over. Recently, Meshi et al. (2015a,b) proposed to replace the marginalization constraints, which are part of the local consistency polytope, by quadratic penalty terms. The relaxed problem has then only separable constraints over the cliques that makes it possible to use efficient block coordinate optimization schemes.

Following these ideas, we consider a dual formulation for CRF learning in which the marginalization constraints are replaced by an augmented Lagrangian term, and the intractable Shannon entropy is replaced by a quadratic surrogate so that stochastic dual coordinate ascent (SDCA) can be used to optimize over the mean parameters, with similar guarantees as in Shalev-Shwartz and Zhang (2016). We finally show that by periodically updating the Lagrangian multipliers as we are optimizing the relaxed dual, we can gradually enforce the marginalization constraints, while retaining global linear convergence. In terms of the primal problem associated with the Lagrange multipliers, our algorithm is an inexact gradient descent algorithm using stochastic approximation of the multiplier gradients.

Our paper is organized as follows. We review CRF learning in Section 3. A dual augmented Lagrangian formulation is presented in Section 4. The proposed algorithm is presented in Section 5, followed by its convergence analysis in Section 6. Finally, we present experiments on three applications in Section 7 (Most notations used in the paper can be found in Appendix F).

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2 Related Work

Due to the independent interest of inference problem in discrete graphical models, in particular in computer vision, a significant amount of work has been devoted to develop efficient approximate inference algorithms (Komodakis et al., 2007; Martins et al., 2015; Savchynskyy et al., 2011; Sontag et al., 2008). However, the learning problem is not necessarily easier (can even fail to converge) with an approximate inference approach as the subroutine (Kulesza and Pereira, 2007).

There is a large body of research on efficient algorithms for structured learning. For the max-margin formulation, the fastest algorithms to date rely on block coordinate Frank-Wolfe updates (Lacoste-Julien et al., 2013; Meshi et al., 2015b; Tang et al., 2016). Using dual decomposition in the inner inference problem, Hazan and Urtasun (2010); Komodakis (2011); Meshi et al. (2010) proposed to solve the classical saddle-point formulation for structured learning problem with algorithms that alternate between message passing and model parameter updates. Going further Meshi et al. (2015b); Yen et al. (2016) work on a purely dual formulation to enable clique-wise updates. For maximum likelihood learning, exponentiated gradient and its block variants can be applied (Collins et al., 2008). Other recent work have relied on incremental algorithms (Schmidt et al., 2015) and the fact that the Gauss-Southwell rule can be applied efficiently for coordinate descent in some forms of graphical models (Nutini et al., 2015).

The BCMM algorithm of Hong et al. (2014) which uses stochastic block coordinate updates inside ADMM inspired our approach. But our algorithm performs multiple passes over all blocks before updating the multiplier; and we prove stronger convergence rates.

We list related structured learning methods with their main characteristics in Table 1 in Appendix B.5.

Yen et al. (2016) is the most similar work to ours: the proposed algorithm constructs greedily an (initially sparse) working set of cliques, which is incremented at each epoch, while we perform stochastic updates on all cliques and possibly several passes over the data between each update of all Lagrange multipliers. Also, our work is leveraging the connection with SDCA, and we prove both linear convergence in the primal and the dual whereas Yen et al. (2016) prove only linear convergence in the dual. Finally, our algorithm is outperforming other methods in experiments.

3 CRF Learning

A discrete *conditional random field* (CRF) is a family of conditional distributions over a vector of discrete random variables $Y := (Y_1, \dots, Y_m)$ given the observation X. The form of the CRF is assumed to be a product of *local functions* (a.k.a. *factors* or *clique functions*) that each depends on only a small number of random variables (i.e. a *clique*). If there exists multiple cliques that share the same local function, then we group cliques by *clique types*. Specifically, let $w_\tau \in \mathbb{R}^{d_\tau}$ be the parameter vector associated with the clique type $\tau \in \mathcal{T}$, where $\mathcal{T}$ is the set of clique types. Let $\mathcal{C}$ denote the set of all cliques, and $\mathcal{C}_\tau$ the set of cliques of type $\tau \in \mathcal{T}$. Note that each clique c has a unique clique type, which we denote by τ_c. With these notations the density function of the CRF can be written as

$$p(y|x; w) := \frac{1}{Z(x, w)} \prod_{\tau \in \mathcal{T}} \prod_{c \in \mathcal{C}_\tau} \exp\Big(\langle w_\tau, \phi_c(x, y_c)\rangle\Big),$$

where $w = (w_\tau)_{\tau \in \mathcal{T}}$; we denoted $Z(x, w)$ the *partition function* and $\phi_c(x, y_c) \in \mathbb{R}^{d_{\tau_c}}$ the *feature map* for clique c. Since all random variables are discrete, we use a one-hot vector $y_i \in \mathcal{Y}_i := \{u \in \{0,1\}^{k_i} \colon \|u\|_1 = 1\}$ to represent the value of Y_s. Here k_i is the cardinality of $\mathcal{Y}_i$. For a clique c, the value for the corresponding random variables is $y_c = \otimes_{i \in c}\, y_i \in \mathcal{Y}_c := \bigotimes_{i \in c} \mathcal{Y}_i$, where $\otimes$ (resp. $\bigotimes$) denotes the tensor product of vectors (resp. of spaces). Similarly, $y \in \mathcal{Y}$ is of the form $y = \otimes_{i \in \mathcal{V}}\, y_i$. W.l.o.g., we consider in the paper only cliques of size at most 2, that is $\mathcal{C} = \mathcal{V} \cup \mathcal{E}$, with $\mathcal{V}$ and $\mathcal{E}$ respectively the set of nodes and of edges of the graph; the framework generalizes easily to higher-order cliques. Notations used in the paper are listed in Appendix F.

3.1 CRF as exponential family

Given a sample $(x^{(n)}, y^{(n)})$, for each clique c, let $\eta_c^{(n)}(w) := [\langle w_{\tau_c}, \phi_c(x^{(n)}, y_c)\rangle \colon y_c \in \mathcal{Y}_c]$; then a *natural parameter* for the exponential family form of the conditional distribution $p(y \mid x^{(n)})$ is $\eta^{(n)}(w) := [\eta_c^{(n)}(w) \colon c \in \mathcal{C}]$. The associated *sufficient statistics* is $T(y) := [y_c \colon c \in \mathcal{C}]$, and $\langle \eta^{(n)}(w), T(y)\rangle = \sum_c \langle \eta_c^{(n)}(w), y_c\rangle$. With these notations, $p(y \mid x^{(n)})$ has the exponential family form:

$$p\big(y \mid \eta^{(n)}(w)\big) = \exp\Big[\langle \eta^{(n)}(w), T(y)\rangle - F\big(\eta^{(n)}(w)\big)\Big],$$

where $F(\eta) := \log \sum_y \exp\langle \eta, T(y)\rangle = \log Z(x^{(n)}, w)$ is the log-partition function.

Given i.i.d. samples $\{(x^{(n)}, y^{(n)})\}_{1 \le n \le N}$, the maximum likelihood estimator for w is computed by the maximizing $\sum_n \log p(y^{(n)} \mid x^{(n)}; w)$. Using the exponential family representation, we can rewrite this problem in two equivalent forms:

$$\max_w \sum_{n=1}^N \Big[\langle \eta^{(n)}(w), T(y^{(n)})\rangle - F\big(\eta^{(n)}(w)\big)\Big],$$

and $\min_w \sum_{n=1}^N F\big(\theta^{(n)}(w)\big)$ with $\theta^{(n)}(w)$ another natural parameter obtained via the affine transformation $\theta^{(n)}(w) = \eta^{(n)}(w) - \langle \eta^{(n)}(w), T(y^{(n)})\rangle \mathbf{1}$. Alternatively, by defining $\Psi^{(n)}$ as a sparse block matrix with $|\mathcal{T}| \times |\mathcal{C}|$ blocks, whose (τ_c, c)-th block is the matrix $\Psi_c^{(n)} \in \mathbb{R}^{d_{\tau_c} \times k_c}$ with

$$\Psi_c^{(n)} = [\phi_c(x^{(n)}, y_c) - \phi_c(x^{(n)}, y_c^{(n)}) \colon y_c \in \mathcal{Y}_c],$$

we have $\theta_c^{(n)}(w) = \Psi_c^{(n)\intercal} w_{\tau_c}$ and $\theta^{(n)}(w) = \Psi^{(n)\intercal} w$.

W.l.o.g., we assume $N = 1$ and drop the superscript (n) from now on, since one may view N graphs as a single large graph with several connected components.

Regularized maximum likelihood estimation with a regularization constant $\lambda > 0$ is thus formulated as

$$\min_w F\big(\theta(w)\big) + \frac{\lambda}{2}\|w\|_2^2. \tag{1}$$

In order to extend this formulation to cover as well max-margin learning (i.e., structured SVMs), we consider the loss-augmented CRF learning introduced by Pletscher et al. (2010) and Hazan and Urtasun (2010), which leads to a slightly generalized formulation:

$$\min_w \gamma F\big(\tfrac{1}{\gamma}\theta_\ell(w)\big) + \frac{\lambda}{2}\|w\|_2^2, \tag{2}$$

where $\theta_\ell(w) := \theta(w) + \ell$ is then the natural parameter, with $\ell = \big[[\ell_c(y_c^\star, y_c) \colon y_c \in \mathcal{Y}_c] \colon c \in \mathcal{C}\big]$ the user-defined loss and $\gamma \in (0, +\infty)$ the temperature hyperparameter. For a derivation for the loss-augmented CRF see Appendix A.

It is well known that the cost of gradient descent to optimize either (1) or (2) (for $\gamma > 0$) is prohibitive since $\nabla_{w_\tau} F(\theta(w)) = \sum_{c \in \mathcal{C}_\tau} \Psi_c \mathbb{E}_\theta[Y_c]$ involves an expectation over the exponentially large space $\mathcal{Y}$. To exploit the underlying structure of the function F it is useful to work on the dual problem. Indeed, since F is convex, it has a variational representation based on conjugate duality:

$$F(\theta) = \max_\mu \langle \mu, \theta\rangle - F^*(\mu),$$

where F^* is the Fenchel conjugate of F, and the dual variable μ called the *mean parameter* is defined by $\mu = (\mu_c)_{c \in \mathcal{C}}$ with $\mu_c = \mathbb{E}_\theta[Y_c]$. The set of valid mean parameters form the so called *marginal polytope* $\mathcal{M}$, which is defined as the convex hull of $\{T(y) : y \in \mathcal{Y}\}$. Moreover, if let $H_{\text{Shannon}}(\mu)$ denote the Shannon entropy of a CRF with mean parameter μ, it is a classical result (Wainwright, 2008, Thm 3.4) that

$$F^*(\mu) = -H_{\text{Shannon}}(\mu) + \iota_{\mathcal{M}}(\mu),$$

where $\iota_{\mathcal{M}}(\mu)$ equal to 0 if $\mu \in \mathcal{M}$ and $+\infty$ otherwise.

4 Relaxed Formulations

In this section, we derive general relaxed dual, primal and corresponding saddle-point formulations for the CRF learning problem: first, we use the classical local polytope relaxation (Sec. 4.1). Second, we further relax the marginalization constraints via an augmented Lagrangian (Sec. 4.2). Third, we propose a surrogate for the entropy, which is decomposable, and retains good properties even when the aforementioned constraints are relaxed (Sec. 4.3). The resulting formulation is convex and is amenable to fast optimization algorithm that are presented in Section 5.

4.1 Classical local polytope relaxation

Both $\mathcal{M}$ and $H_{\text{Shannon}}(\mu)$ are in general intractable due to the exponentially large structured-output space $\mathcal{Y}$ and they are typically replaced by decomposable surrogates.

It is common to relax $\mathcal{M}$ to the *local consistency polytope* (Wainwright, 2008)

$$\mathcal{L} := \Big\{\mu \in \mathcal{I} \colon \sum_{y_j \in \mathcal{Y}_j} \mu_{ij}(y_i, y_j) = \mu_i(y_i), \forall \{i,j\} \in \mathcal{E}, \forall y_i\Big\},$$

where $\mathcal{I}$ denotes the Cartesian product of *simplex constraints* on each clique. Note that $\mathcal{L} \supseteq \mathcal{M}$, since any set of true marginals must satisfy the simplex constraints and the *marginalization constraints*, but not vice versa. Equivalently, if we define $A_i = I_{k_i} \otimes \mathbf{1}_{k_i}^\intercal$, the equality constraints can be written in a matrix form as $\mu_i - A_i \mu_{ij} = 0$ for all $\{i,j\} \in \mathcal{E}$. Combining all equations, we have $A\mu = 0$, where A is a $|\mathcal{E}| \times |\mathcal{C}|$ block matrix (see Appendix F). So, we have equivalently $\mathcal{L} = \mathcal{I} \cap \{\mu \colon A\mu = 0\}$.

Since H_{Shannon} is also intractable for graphs with large tree-width, we will use an approximation H_{Approx} which will be constructed so as to be defined and concave on the whole set $\mathcal{I}$. We propose several entropy approximations suited to our needs in Section 4.3.

Definition 1. *Let $F_{\mathcal{I}}$ and $F_{\mathcal{L}}$ be the counterparts of F obtained by relaxing $\mathcal{M}$ to $\mathcal{I}$ and $\mathcal{L}$ respectively, which, in other words, are the Fenchel conjugates of $F_{\mathcal{I}}^*$ and $F_{\mathcal{L}}^*$ when these are defined by H_{Approx}:*

$$F_{\mathcal{I}}(\theta_\ell) := \max_\mu \langle \mu, \theta_\ell\rangle - F_{\mathcal{I}}^*(\mu),$$
$$F_{\mathcal{L}}(\theta_\ell) := \max_\mu \langle \mu, \theta_\ell\rangle - F_{\mathcal{L}}^*(\mu),$$

where $F_{\mathcal{I}}^*(\mu) := -H_{\text{Approx}}(\mu) + \iota_{\mathcal{I}}(\mu)$ and $F_{\mathcal{L}}^*(\mu) := F_{\mathcal{I}}^*(\mu) + \iota_{\{A\mu=0\}}$.

Replacing F with $F_{\mathcal{L}}$ in (2) yields the relaxed primal

$$P(w) := \gamma F_{\mathcal{L}}\big(\tfrac{1}{\gamma}\theta_\ell(w)\big) + \frac{\lambda}{2}\|w\|_2^2. \tag{3}$$

The corresponding dual objective function is given by

$$D(\mu) := \langle \mu, \ell \rangle - \gamma F_{\mathcal{L}}^*(\mu) - \frac{1}{2\lambda}\|\Psi\mu\|_2^2. \qquad (4)$$

See Appendix B.1 for a derivation.

4.2 A dual augmented Lagrangian

It is difficult to optimize $D(\mu)$, since the optimization requires some form of projection onto $\mathcal{L}$, which can be shown to be equivalent to perform graph-wise marginal inference (Collins et al., 2008). The difficulty is due to the coupling equality constraint $A\mu = 0$. Meshi et al. (2015b) proposed to relax $\iota_{\{A\mu=0\}}$ by a quadratic term $\frac{1}{2\rho}\|A\mu\|_2^2$, which corresponds to employ the *penalty method* (Bertsekas, 1982). They argue that it is not crucial to enforce exact $A\mu = 0$ in learning, since the relaxed problem works well in practice and enables an efficient optimization with only clique-wise updates. However, the penalty method is known to have issues associated with the choice of ρ: unless we use a carefully designed scheduling to update ρ, for a reasonably small ρ, the algorithm will be slow; on the other hand, using a large fixed value of ρ degrades the problem to independent logistic regression problems, and, thereby, leads to suboptimal solutions.

Instead, we propose to solve problem (4) as a saddle problem of the form $\max_\mu \min_\xi D_\rho(\mu, \xi)$ where D_ρ is the augmented Lagrangian

$$D_\rho(\mu,\xi) := \Big[\langle \ell, \mu\rangle - \gamma F_{\mathcal{I}}^*(\mu) + \langle \xi, A\mu\rangle\Big] - \Big[\frac{1}{2\rho}\|A\mu\|_2^2 + \frac{1}{2\lambda}\|\Psi\mu\|_2^2\Big], \qquad (5)$$

with ξ is the Lagrangian multiplier and $\rho > 0$.

Using duality again, we can derive an associated relaxed primal objective

$$\tilde{P}_\rho(w,\delta,\xi) := \gamma F_{\mathcal{I}}\Big(\frac{\theta_\ell(w) + A^\intercal\delta}{\gamma}\Big) + \frac{\lambda}{2}\|w\|_2^2 + \frac{\rho}{2}\|\delta - \xi\|_2^2,$$

so that $\min_{(w,\delta)} \tilde{P}_\rho(w,\delta,\xi)$ is a primal problem associated with the dual problem $\max_\mu D_\rho(\mu,\xi)$.

Strong duality between these two problems yields a representer theorem

$$w^\star = -\frac{1}{\lambda}\Psi\mu^\star, \quad \delta^\star = \xi^\star - \frac{1}{\rho}A\mu^\star \qquad (6)$$

which provides a duality gap

$$\mathrm{gap}(w,\delta,\mu,\xi) := \tilde{P}_\rho(w,\delta,\xi) - D_\rho(\mu,\xi)$$

for the convergence of the maximization of $D_\rho(\mu,\xi)$ with respect to μ. Moreover, it is easy to check that $\min_{\xi,\delta} \tilde{P}_\rho(w,\delta,\xi) = P(w)$ because $\min_\delta F_{\mathcal{I}}(\theta(w) + A^\intercal\delta) = F_{\mathcal{L}}(\theta(w))$ for any w (see Appendix B.2). This shows that $w^\star$ defined in (6) is also an optimum of the original primal problem $\min_w P(w)$. As a consequence, if a sequence μ^t converges to $\mu^\star$ then the corresponding $w^t = -\frac{1}{\lambda}\Psi\mu^t$ converges to a solution of (2). For more details, see Appendix B.

4.3 Gini entropy surrogate

We seek a concave entropy surrogate H_{Approx} that decomposes additively on the cliques. Since the constraint $A\mu = 0$ is relaxed, we need a surrogate well defined on the whole set $\mathcal{I}$. The Bethe entropy (Yedidia et al., 2005) is generally non-concave. Its concave counterparts, such as the tree-reweighted entropy (Wainwright et al., 2005) or the region-based entropy (London et al., 2015; Yedidia et al., 2005), are only concave on the local consistency polytope, but non-concave on $\mathcal{I}$.

Moreover, a generic difficulty with these entropies is that they do not have Lipschitz gradients, which prevents the direct application of proximal methods with usual quadratic proximity terms. We thus propose a coarse but convenient entropy surrogate of the form:

$$H_{\text{Approx}}(\mu) = \sum_{c\in\mathcal{C}} h_c(\mu_c) \quad \text{with} \quad h_c(\mu_c) := (1 - \|\mu_c\|_2^2).$$

Another surrogate with the same separable form is the second-order Taylor expansion of the *oriented tree-reweighted entropy* (OTRW, Globerson and Jaakkola, 2007) around the uniform distribution. This surrogate is also concave on $\mathcal{I}$ (although not strongly concave) and smooth. Preliminary experiments however did not show that using this more sophisticated entropy improved the results. See Appendix C for more details.

5 Algorithm

Given the form of the entropy surrogate proposed, D_ρ decomposes as a sum of convex separable terms over the block associated to cliques plus a smooth term:

$$\begin{aligned} D_\rho(\mu,\xi) &= -\sum_{c\in C} f_c^*(\mu_c) - r(\mu) \qquad \text{with} \qquad (7)\\ f_c^*(\mu_c) &:= -\gamma h_c(\mu_c) + \iota_{\triangle_c}(\mu_c)\\ r(\mu) &:= -\langle A^\intercal\xi + \ell, \mu\rangle + \frac{1}{2\lambda}\|\Psi\mu\|^2 + \frac{1}{2\rho}\|A\mu\|^2, \end{aligned}$$

where $\triangle_c := \{\mu_c \in \mathbb{R}_+^{d_c} \mid \mu_c^\intercal\mathbf{1} = 1\}$ is the canonical simplex. It can thus be maximized efficiently by a block-coordinate proximal scheme, such as the proximal stochastic dual coordinate descent (SDCA, Shalev-Shwartz and Zhang, 2016), which has linear convergence guarantees both in the primal and the dual.

Algorithm 1 IDAL scheme

1: **Input**: $T_{\text{in}}, T_{\text{ex}}, \epsilon$
2: **Initialize**: $\hat{\mu}_c^0 = \frac{1}{k_c}\mathbf{1}$ for all $c \in \mathcal{C}$ and $\xi^1 = 0$
3: **for** $t = 1, \dots, T_{\text{ex}}$ **do**
4: $\quad\hat{\mu}^t = \mathcal{A}(\hat{\mu}^{t-1}, T_{\text{in}}, t)$
5: $\quad$Stop if $G_t \leq \epsilon$ and $\|A\hat{\mu}^t\|^2 \leq \epsilon$
6: $\quad\xi^{t+1} = \xi^t - \frac{1}{L_d}A\hat{\mu}^t$
7: **end for**
8: **Output**: $\hat{\mu}^{T_{\text{ex}}}, \xi^{T_{\text{ex}}}$

Algorithm 2 SDCA version of $\mathcal{A}(\mu, T_{\text{in}}, t)$

1: $\mu^{t,0} = \mu$
2: **for** $s = 1, \dots, T_{\text{in}}$ **do**
3: $\quad$Draw a clique c uniformly at random
4: $\quad\mu_c^{t,s} = \text{Prox}_{\frac{1}{L_c}f_c^*}\big(\mu_c^{t,s-1} - \frac{1}{L_c}\nabla_{\mu_c} r(\mu^{t,s-1})\big)$
5: $\quad\mu_{-c}^{t,s} = \mu_{-c}^{t,s-1}$
6: **end for**
7: **Output**: $\mu^{t,T_{\text{in}}}$

To solve $\min_\xi \max_\mu D_\rho(\mu, \xi)$ we thus propose an algorithm similar to the block coordinate method of multipliers (BCMM) of Hong et al. (2014): perform dual stochastic block coordinate ascent (SDCA) on the variables μ_c to partially maximize $D_\rho(\mu, \xi)$ in μ and regularly take a gradient descent step in ξ. Our algorithm, is an inexact dual augmented Lagrangian (IDAL) method, in the sense that it is an inexact gradient descent algorithm on the function $\xi \mapsto d(\xi) := \max_\mu D_\rho(\mu, \xi)$. To be precise, if at epoch t, ξ takes the value ξ^t and $\hat{\mu}^{t-1}$ is the value of μ from the previous epoch, Algorithm 2 takes T_{in} stochastic block-coordinate proximal gradient steps on μ to obtain $\hat{\mu}^t$. Denoting L_c the Lispchitz constant of r w.r.t. μ_c, μ_c is then updated by a partial gradient step, and an application of the proximal operator of $\frac{1}{L_c}f_c^*$. Then, by Danskin's theorem[1], applied to equation (5), we have that $A\hat{\mu}^t$ is an approximate gradient of $d(\xi^t)$, and so, Algorithm 1 updates ξ with $\xi^{t+1} = \xi^t - \frac{1}{L_d}A\hat{\mu}^t$, where L_d is the Lispchitz constant of $d(\xi)$. As for the stopping criteria, we use $G_t := \text{gap}(w(\hat{\mu}^t), \delta(\hat{\mu}^t, \xi^t), \hat{\mu}^t, \xi^t) \leq \epsilon$ and $\|A\hat{\mu}^t\|^2 \leq \epsilon$, where $w(\hat{\mu}^t), \delta(\hat{\mu}^t, \xi^t)$ are defined via the representer theorem (6) (see Appendix B.4).

6 Convergence Analysis

In this section, we study the convergence rate of our algorithm. First, we show that if we use an iterative and linearly convergent algorithm $\mathcal{A}$ to approximately solve $\min_\mu D_\rho(\mu, \xi)$, and if we use warm starts, that is, following the notations of the previous section, we use $\hat{\mu}^{t-1}$ as the initial value to solve $\min_\mu D_\rho(\mu, \xi^t)$, then running $\mathcal{A}$ for a fixed number of iterations is sufficient to guarantee global linear convergence in the primal and in the dual. We show that SDCA or simple block-coordinate proximal gradient descent are applicable as the algorithm $\mathcal{A}$.

6.1 Conditions for global linear convergence

To study the convergence, we consider:

- $\bar{\mu}^t := \mu^\star(\xi^t) = \text{argmax}_\mu D_\rho(\mu, \xi^t)$.
- $\mu^{t,s}$, the value of μ after s inner steps at epoch t.
- $\hat{\mu}^t := \mu^{t,T_{\text{in}}}$ the value of μ at the end of epoch t.
- D_ρ-suboptimality: $\Delta_t^s := D_\rho(\bar{\mu}^t, \xi^t) - D_\rho(\mu^{t,s}, \xi^t)$, with at the end of each epoch $\hat{\Delta}_t := \Delta_t^{T_{\text{in}}} = \Delta_{t+1}^0$.
- d-suboptimality: $\Gamma_t := d(\xi^t) - d(\xi^\star)$.

Lemma 1 (Linear convergence of the outer iteration)**.** *Let $\mathcal{A}$ be an algorithm that approximately solves $\max_\mu D_\rho(\mu, \xi^t)$ in the sense that*

$$\exists \beta \in (0, 1), \qquad \mathbb{E}[\hat{\Delta}_t] \leq \beta\, \mathbb{E}[\Delta_t^0].$$

Then, $\exists \kappa \in (0, 1)$ characterizing $d(\xi)$ and $C > 0$, such that, if $\lambda_{\max}(\beta)$ is the largest eigenvalue of the matrix

$$M(\beta) = \begin{bmatrix} 6\beta & 3\beta \\ 1 & 1-\kappa \end{bmatrix},$$

then after T_{ex} iterations of Algorithm 1 we have

$$\left\| \begin{matrix} \mathbb{E}[\hat{\Delta}_{T_{\text{ex}}}] \\ \mathbb{E}[\Gamma_{T_{\text{ex}}}] \end{matrix} \right\| \leq C\, \lambda_{\max}(\beta)^{T_{\text{ex}}} \left\| \begin{matrix} \mathbb{E}[\hat{\Delta}_0] \\ \mathbb{E}[\Gamma_0] \end{matrix} \right\|.$$

The constant κ in the theorem is of the form $\kappa = \frac{\tau}{L_d}$ with L_d the Lipschitz constant of $d(\xi)$ and τ a restricted strong convexity constant for $d(\xi)$ obtained by Hong and Luo (2017) (see Lemma D.3 in Appendix D.2).

Corollary 1. *If $\mathcal{A}$ is a linearly convergent algorithm with rate π and if it is run for T_{in} iterations, such that, for some β: $\lambda_{\max}(\beta) < 1$, we have $(1-\pi)^{T_{\text{in}}} \leq \beta$, then $\mathbb{E}[\hat{\Delta}_t]$ and $\mathbb{E}[\Gamma_t]$ converge linearly to 0.*

Note that linear convergence of the expectations implies that Δ_t and Γ_t converge linearly to 0 almost surely, as a classical consequence of Markov's inequality and the Borel-Cantelli lemma. We will show in the next section that when $\mathcal{A}$ is SDCA it is linearly convergent.

Note that the convergence of the gaps Δ_t and Γ_t imply the linear convergence for the augmented Lagrangian formulation, in the following sense:

Corollary 2. *Let $D_\infty(\mu) := \langle \ell, \mu \rangle - \gamma F_{\mathcal{I}}^*(\mu) - \frac{1}{2\lambda}\|\Psi\mu\|_2^2$, so that we have $D(\mu) = D_\infty(\mu) - \iota_{\{A\mu=0\}}$. If Δ_t and Γ_t converge linearly to 0, then $|D_\infty(\hat{\mu}^t) - D_\infty(\mu^\star)|$ and $\|A\hat{\mu}^t\|_2^2$ both converge to 0 linearly.*

[1](see e.g. Bertsekas, 1999, Prop. B.25)

Furthermore, if $\mathcal{A}$ is linearly convergent as in Corollary 2, the algorithm is linearly convergent in terms of the total number of inner steps (for SDCA this is the total number of clique updates) performed by algorithms $\mathcal{A}$ throughout:

Corollary 3. *With the notations of the previous corollary, for any $\beta \in (0,1)$ such that $\lambda_{\max}(\beta) < 1$, it is possible to obtain $\mathbb{E}[\hat{\Delta}_t] \leq \epsilon$ and $\mathbb{E}[\Gamma_t] \leq \epsilon$ with a total number of inner iterations $T_{\text{tot}} := T_{\text{in}} T_{\text{ex}}$ such that*

$$T_{\text{tot}} \geq \frac{\log(\beta)}{\log \lambda_{\max}(\beta) \log(1-\pi)} \log(\epsilon).$$

We show in Appendix D.4 that to have $\lambda_{\max}(\beta) < 1$ we should have $\beta = \alpha\kappa$ with $\alpha < \frac{1}{3(1+2\kappa)}$.

To reason in terms of rate, if the rate of convergence is r then we should have $T_{\text{tot}} \geq \frac{\log(\epsilon)}{\log(1-r)}$. So identifying the rate of convergence of the algorithm yields $r = 1 - \exp\left(\frac{\log(1-\pi)\log(\lambda_{\max}(\beta))}{\log(\beta)}\right)$. If α and κ are not too large, we can get a simplified expression for the rate, characterized as follows.

Corollary 4. *Let $\Delta^\star_{t\,T_{\text{in}}+s} := \Delta^s_t + \Gamma_t$. If $\kappa < \frac{1}{2}$ and $\alpha = \frac{1}{12}$, if $T_{\text{in}} \geq \frac{\log(\alpha\kappa)}{\log(1-\pi)}$, then, there exist a constant $C' > 0$ such that after a total of s inner updates, we have*

$$\mathbb{E}[\Delta^\star_s] \leq C'\Big(1 - \frac{\kappa\pi}{2\log(12/\kappa)}\Big)^s.$$

6.2 Convergence results with SDCA

Given the structure of D_ρ, if the functions f_c^* in (7) are strongly convex, a good candidate for $\mathcal{A}$ is stochastic dual coordinate ascent (SDCA). Indeed, the results of Shalev-Shwartz and Zhang (2016) show that

Proposition 1. *If $\mathcal{A}$ is SDCA, let $|\mathcal{C}|$ be the total number of cliques, σ_c the strong convexity constant of f_c^*, and L_c the Lipschitz constant of $\mu_c \mapsto r(\mu)$, then $\mathcal{A}$ is linearly convergent with rate $\pi = \min_{c\in\mathcal{C}} \frac{\sigma_c}{|\mathcal{C}|(\sigma_c+L_c)}$.*

Moreover SDCA allows us to bound the duality gap by the increase of D_ρ, which yields linear convergence in the primal.

Proposition 2. *Let $\hat{w}^t = w(\hat{\mu}^t)$. If $\mathcal{A}$ is SDCA, then*

$$\mathbb{E}[P(\hat{w}^t) - P(w^\star)] \leq \frac{1}{\pi}\mathbb{E}[\hat{\Delta}_t] + \mathbb{E}[\Gamma_t].$$

For the sake of the natural surrogates for the entropy (like the Gini-OTRW entropy proposed in Appendix C), individual functions f_c^* are not strongly convex, although $-D_\rho$ is strongly convex, because the entropy surrogate is strongly concave on $\mathcal{L}$ and the term $\|A\mu\|^2$ is strongly convex on $\text{Ker}(A)^\perp$. In that case another decomposition is relevant: if σ is the strong convexity constant of $-D_\rho$, then let $\tilde{f}_c^*(\mu_c) = \iota_{\triangle_c}(\mu_c) + \sigma\|\mu_c\|_2^2$ and $\tilde{r}(\mu) = -H_{\text{Approx}}(\mu) + r(\mu) - \sigma\|\mu_c\|_2^2$. We again have $D_\rho(\mu) = -\sum_{c\in\mathcal{C}} \tilde{f}_c^*(\mu_c) - \tilde{r}(\mu)$, with $\tilde{f}_c^*$ strongly convex and $\tilde{r}$ convex and smooth. SDCA and its theory are here applicable again and guarantees that Proposition 1 and following hold. However, for the convergence in the primal a slightly different argument is needed.

Proposition 3. *Let $w^{t,s} = w(\mu^{t,s})$. If $\mathcal{A}$ is a linearly convergent algorithm and the function $\mu \mapsto -H_{\text{approx}} + \frac{1}{2\rho}\|A\mu\|_2^2$ is strongly convex, then $P(w^{t,s}) - P(w^\star)$ converges to 0 linearly.*

6.3 Discussion

Optimization with inexact gradients (Devolder et al., 2014) and inexact proximal operators (Schmidt et al., 2011) have been shown to yield the same convergence rate as their exact counterparts, provided that errors decrease at a certain rate. Linear convergence of an inexact augmented Lagrangian method in which both inner and outer optimizations use Nesterov's accelerated gradient descent is shown in Lan and Monteiro (2016). We use the same ideas, except that we leverage the large finite sum structure of the dual problem to use randomized algorithms. The use of warm-start is also similar to its use in the meta-algorithm proposed by Lin et al. (2017), who use inexact gradient descent on the Moreau-Yosida regularization of a non-smooth objective. In our context, this approach would actually be applicable by working on $P_\rho(w,\xi)$ instead of working in the dual. An investigation in this direction is of interest but beyond the scope of this paper.

7 Experiments

We evaluate our algorithm IDAL on three different CRF models including 1) a simulated Gaussian mixture Potts model with grid graph and two clique types (nodes and edges); 2) a semantic segmentation model with planar graph and two clique types (nodes and edges); 3) a multi-label classification model with fully-connected graph and unique clique type for all cliques.

We compare with algorithms using only clique-wise oracles for solving $\min_\xi \max_\mu D_\rho(\mu,\xi)$, namely, the soft-constrained block-coordinate Frank-Wolfe algorithm (SoftBCFW) by Meshi et al. (2015b) and the greedy direction method of multipliers (GDMM) algorithm by Yen et al. (2016). Note that SoftBCFW in fact solves only the special case $\max_\mu D_\rho(\mu, \xi \equiv 0)$, thus it will converge to a different point than IDAL. In addition, we include a third baseline for the special case using SDCA (referred as SoftSDCA). Since SoftBCFW and GDMM have been shown outperforming other baselines such as Lacoste-Julien et al. (2013), Meshi et al. (2010)

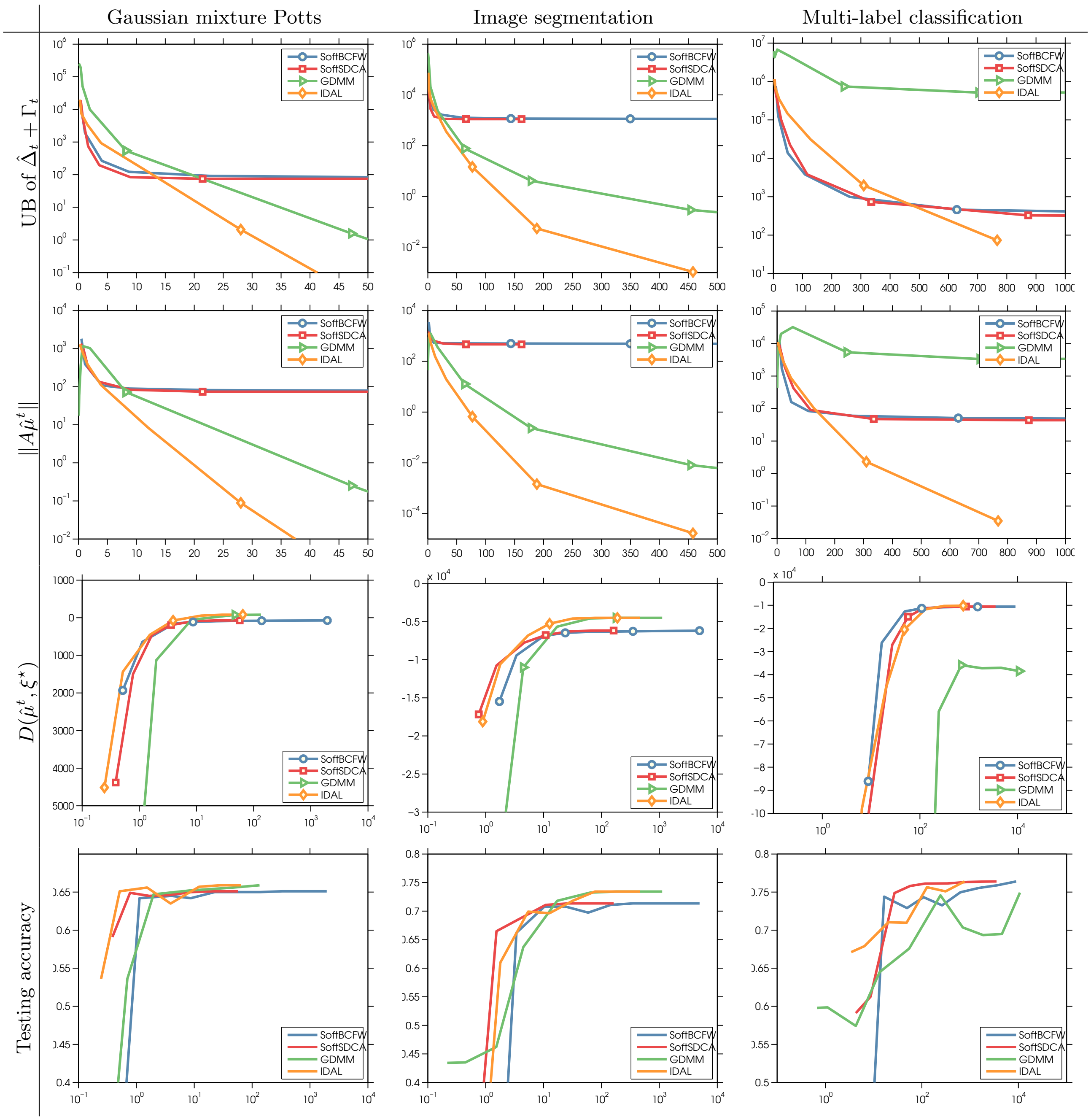

Figure 1: The comparison between IDAL and other baselines. For the choices of hyperparameters in terms of accuracy and speed, we set $(\lambda = 10, \rho = 1, \gamma = 1)$ for Gaussian mixture Potts, $(\lambda = 10, \rho = 1, \gamma = 10)$ for semantic segmentation and $(\lambda = 1, \rho = 0.1, \gamma = 1)$ for multi-label classification. The x-axis is running time in seconds.

and Hazan and Urtasun (2010), we will not make an extensive comparison for all these algorithms.

7.1 Setup

Gaussian mixture Potts models This is an extension of the Potts model given observations, whose conditional density function is defined via Bayes' rule $p(y|x) \propto p(x|y)p(y)$, with $p(y)$ a Potts distribution associated with a grid graph and parameterized by $w_{\text{binary}} \in \mathbb{R}^{k^2}$, and with $p(x|y) = \prod_i p(x_i|y_i)$ assumed to factorize into independent conditional Gaussian distributions with canonical parameters $w_{\text{unary}} \in \mathbb{R}^{2k}$, i.e., $p(x_i|y_i) \propto \exp(\langle w_{\text{unary}}(y_i), [x_i, x_i^2] \rangle)$. We consider a 10×10 grid graph with node cardinality $k = 5$. To generate the data, we first draw the label y from $p(y)$, and then the observation x_i is generated from the conditional Gaussian $p(x_i|y_i)$ for each node. The simulated dataset contains 100 samples and is equally divided for training and testing.

Semantic image segmentation We consider a typical CRF model used in computer vision for labeling image pixels with semantic classes. The graph is built upon clustering pixels into superpixels. Each superpixel defines a node. Two superpixels with a shared boundary define an edge. The CRF model takes the form $p(y|x) \propto \exp\big(\sum_i w_{\text{unary}}^\intercal \psi_i(x, y_i) + \sum_{i,j} w_{\text{binary}}^\intercal \psi_{ij}(x, y_i, y_j)\big)$, where $\psi_i(x, y_i)$ measure the intra-cluster compatibility within the superpixel i, and $\psi_{ij}(x, y_i, y_j)$ measure the inter-cluster compatibility between superpixels i and j. We conduct the experiment on the *MSRC-21* dataset introduced by Shotton et al. (2006), which has 21 classes, 335 training images and 256 testing images.

Multi-label classification The task for this problem is assigning each input vector a set of binary target labels. It is natural to model the inter-label dependencies by CRFs that treat each label as a node in a fully connected label graph. Following Finley and Joachims (2008), we define the CRF density function as $p(y|x) \propto \exp(\sum_i w_i^\intercal \phi_i(x, y_i) + \sum_{i,j} w_{ij}^\intercal \phi_{ij}(y_i, y_j))$, where the feature maps are specified as $\phi_i(x, y_i) = y_i \otimes x$ for each node and $\phi_{ij}(y_i, y_j) = y_i \otimes y_j$ for each edge. We conduct the experiments on the *Yeast* dataset[2], which contains 1500 training samples and 917 testing samples. Each sample has 14 labels and 103 attributes.

Hyperparameters In theory, T_{in} could be very large depending on the choice of α and the condition number π. We find that in practice only a relatively small T_{in} is needed. We empirically choose $T_{\text{in}} = \frac{1}{2}|\mathcal{C}|$. We set the number of outer iterations $T_{\text{ex}} = 3000$ and the stopping threshold $\epsilon = 10^{-3}$. The ranges of λ is pre-defined as $\{10, 1.0, 0.01, 0.001\}$ and the range of γ is $\{100.0, 10.0, 1.0, 0.001\}$. For each experiment, we choose the best λ and γ in terms of the validation accuracy and a reasonable running time (not all experiments finished in 3000 outer iterations). We set $\rho = 1.0$ or $\rho = 0.1$ as in Meshi et al. (2015b).

7.2 Results

To compare IDAL with GDMM, we use the criterion $P_\rho(\hat{w}^t, \hat{\delta}^t, \xi^t) - D_\rho(\hat{\mu}^t, \xi^t) + P_\rho(\hat{w}^t, \hat{\delta}^t, \xi^t) - D_\rho(\bar{\mu}^{T_{\text{ex}}}, \xi^{T_{\text{ex}}})$, which is an upper bound of the theoretical quantity $\hat{\Delta}_t + \Gamma_t$ that we analyzed. To compare IDAL with SoftBCFW, since $\xi = 0$ for SoftBCFW, we use the criterion $D_\rho(\hat{\mu}^t, \xi^\star)$, in which $\xi^\star$ is obtained from running IDAL to convergence. Besides, we also use the criteria $\|A\hat{\mu}^t\|^2$ (it measures the convergence of $d(\xi)$, since $\nabla d(\xi^t) \simeq A\hat{\mu}^t$) and the testing accuracy, which are applicable for all three algorithms. The results are shown in Figure 1.

There are several interesting points that we can say based on the results: 1) by tightening the marginalization constraints $A\mu = 0$, it does help to gain a better testing accuracy (e.g., IDAL gains small improvements over SoftBCFW); 2) based on the curves of $D_\rho(\mu, \xi^\star)$, we can see that it is key to approach $\mu^\star$ by first obtaining $\xi^\star$, which again shows the importance of enforcing exactness of the local consistency polytope; 3) IDAL is shown to be a faster algorithm than GDMM. One possible reason is that GDMM is in fact an active-set algorithm, which means the number of updated cliques at very beginning is insufficient comparing to IDAL. Based on our analysis, we have shown that the quality of the approximate gradient $A\hat{\mu}_t$ depends on T_{in}. Therefore, it is very likely that GDMM suffers from a slow convergence because of the poor gradients.

8 Conclusion

We proposed a relaxed dual augmented Lagrangian formulation for CRF learning, in which, thanks to dual decomposition, SDCA can be used to partially optimize over mean parameters in order to yield a sufficiently good approximation of the multiplier gradient. Our theoretical analysis shows that if warm-starts are leveraged and multiplier gradients are approximated with a linearly convergent algorithm, global linear convergence can be obtained. If SDCA is used, linear convergence is obtained both in the primal and for the convergence of the dual Lagrangian method.

Comparing to other baselines such as GDMM and SoftBCFW, our algorithm is faster in terms of the distance to the optimal objective function value (i.e. $\hat{\Delta}_t + \Gamma_t$) and the feasibility of the constraints $\|A\mu\|_2^2$.

It would be of interest to investigate the use of the same dual augmented Lagrangian formulation for both inference and learning, since according to Wainwright (2006), this should improve the performance.

In future work, we intend to investigate applications to other problems in machine learning, the use of Nesterov acceleration or quasi-Newton methods for multiplier updates, or the connection to other approaches based on Moreau-Yosida regularization.

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[2] http://sourceforge.net/projects/mulan/files/datasets/yeast.rar

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Supplementary Material for

SDCA-Powered Inexact Dual Augmented Lagrangian Method for Fast CRF Learning

A Loss-Augmented CRF

In order to extend our learning formulation so as to encompass as well max-margin structured learning (i.e., structured SVM) in additional to maximum likelihood learning, we show in this section that our formulation can be generalized to cover the loss-augmented CRF learning introduced by Pletscher et al. (2010) and Hazan and Urtasun (2010).

The loss-augmented CRF $p_\gamma(y \mid y^\star, x)$ is an extension of the standard CRF with additional user-defined loss functions $\ell_c(y_c^\star, y_c)$ for all cliques and an extra temperature hyperparameter $\gamma \in (0, +\infty)$. We introduce a modified natural parameter $\eta_\ell(w) := \eta(w) + \ell$ (similarly we have θ_ℓ) that includes the loss term $\ell = \big[[\ell_c(y_c^\star, y_c) \colon y_c \in \mathcal{Y}_c] \colon c \in \mathcal{C}\big]$. The density function of the loss-augmented CRF then takes the form

$$p_\gamma(y \mid y^\star, x; w) = \exp\Big(\langle \eta_\ell(w)/\gamma, T(y)\rangle - F(\eta_\ell(w)/\gamma)\Big). \tag{1}$$

A justification for the form of the loss-augmented CRF is based on a rationale that distinguishes the label to predict y (which is essentially true unknown label) from the label provided by the annotation $y^\star$. The assumption made is then that, given y_c, the annotation $y_c^\star$ is independent of x and $y_{c'}$ for $c' \neq c$. This entails that $p(y, y^\star|x) = p_\gamma(y \mid y^\star, x) \propto p(y \mid y^\star) p_\gamma(y \mid x)$, which yields the above form for $p_\gamma(y \mid y^\star, x; w)$ by Bayes' rule for $p(y \mid y^\star) \propto \exp(\sum_{c\in\mathcal{C}} \ell_c(y_c^\star, y_c))$.

For learning, we use a rescaled maximum likelihood objective (i.e., multiplied by γ) of the form

$$\min_w \gamma F\Big(\tfrac{1}{\gamma}\theta_\ell(w)\Big) + \frac{\lambda}{2}\|w\|_2^2, \tag{2}$$

with which we can see γ only affects the entropy term in the variational representation of F, thus it plays a role to determine the learning regime. When $\gamma \to 0$, we retrieve a max-margin formulation for structured output learning, since the corresponding variational problem based on Fenchel duality is

$$\min_w \max_{\mu\in\mathcal{M}} \langle \mu, \theta_\ell\rangle + \frac{\lambda}{2}\|w\|_2^2. \tag{3}$$

Note that this is identical to the linear programming relaxation of the structured SVM formulation studied by Meshi et al. (2010).

It is also possible to retrieve the maximum likelihood regime by making a change of variable: $w' = w/\gamma$. Then, (2) becomes

$$\min_{w'} F\Big(\theta(w') + \frac{1}{\gamma}\ell\Big) + \frac{\lambda\gamma}{2}\|w'\|_2^2. \tag{4}$$

Increasing γ decreases the effect of the loss term and simultaneously increases the effect of the regularization. The maximum likelihood regime is thus retrieved by letting $\gamma \to +\infty$ and $\lambda \to 0$.

B Derivations of dual, and relaxed primal and dual objectives

In this section, we derive the dual objective $D(\mu)$ of $P(w)$. Given the augmented Lagrangian $D_\rho(\mu,\xi)$, we first introduce a relaxed primal $\tilde{P}_\rho(w,\delta,\xi)$ involving a new primal variable δ whose components can be interpreted as messages exchanged between cliques in the context of marginal inference via message-passing algorithms. The partial minimization with respect to δ then yields the corresponding primal of $D_\rho(\mu,\xi)$ with respect to μ for a fixed ξ: $P_\rho(w,\xi) := \min_\delta \tilde{P}_\rho(w,\delta,\xi)$, which can be interpreted as a Moreau-Yoshida smoothing of the original objective $P_\rho(w)$.

B.1 Derivation of the dual objective $D(\mu)$

Given that $\theta_\ell(w) = \Psi^\intercal w + \ell$ and introducing the Fenchel conjugate of $F_{\mathcal{L}}$, we have

$$\begin{aligned}
P(w) &= \gamma F_{\mathcal{L}}\Big(\frac{1}{\gamma}\theta_\ell(w)\Big) + \frac{\lambda}{2}\|w\|_2^2 \\
&= \max_{\mu\in\mathcal{L}} \Big[\langle \Psi^\intercal w + \ell, \mu\rangle - \gamma F^*_{\mathcal{L}}(\mu)\Big] + \frac{\lambda}{2}\|w\|_2^2.
\end{aligned}$$

Given that the local polytope constraints are defined by linear inequalities, weak Slater constraint qualification are satisfied, so that strong duality holds and an equivalent dual problem in μ is obtained by switching the order of $\min_w$ and $\max_\mu$:

$$\begin{aligned}
D(\mu) &= \langle \ell,\mu\rangle - \gamma F^*_{\mathcal{L}}(\mu) + \min_w \Big[\langle \Psi^\intercal w, \mu\rangle + \frac{\lambda}{2}\|w\|_2^2\Big] \\
&= \langle \ell,\mu\rangle - \gamma F^*_{\mathcal{L}}(\mu) - \lambda \max_w \Big[-\frac{1}{\lambda}\langle \Psi\mu, w\rangle - \frac{1}{2}\|w\|_2^2\Big] \\
&= \langle \ell,\mu\rangle - \gamma F^*_{\mathcal{L}}(\mu) - \frac{1}{2\lambda}\|\Psi\mu\|_2^2.
\end{aligned}$$

B.2 Derivation of an extended primal $\tilde{P}_\rho(w,\delta,\xi)$

Proposition 4. *For a fixed ξ, the primal objective function of $D_\rho(\mu,\xi)$ takes the form*

$$P_\rho(w,\xi) := \min_\delta \Big[\tilde{P}_\rho(w,\delta,\xi) := \gamma F_{\mathcal{I}}\Big(\frac{\theta(w)+A^\intercal\delta}{\gamma}\Big) + \frac{\lambda}{2}\|w\|_2^2 + \frac{\rho}{2}\|\delta-\xi\|^2\Big].$$

Proof. Clearly, we have $D(\mu) = \min_\xi D_\rho(\mu,\xi)$. For a fixed value of ξ, consider the Lagrangian

$$L_{\rho,\xi}(\mu,\nu,\nu',w,\delta) = \langle \ell,\mu\rangle - \gamma F^*_{\mathcal{I}}(\mu) - \frac{1}{2\lambda}\|\nu\|^2 - \frac{1}{2\rho}\|\nu'\|^2 + \langle \xi,\nu'\rangle + \langle w, \Psi\mu - \nu\rangle + \langle \delta, A\mu - \nu'\rangle;$$

Then clearly $\min_{w,\delta} L_{\rho,\xi}(\mu,\nu,\nu';w,\delta) = D_\rho(\mu,\xi)$. We compute the associated primal as

$$\begin{aligned}
\tilde{P}_\rho(w,\delta,\xi) &= \max_{\mu,\nu,\nu'} L_{\rho,\xi}(\mu,\nu,\nu',w,\delta) \\
&= \max_u \Big[\langle \mu, \ell + \Psi^\intercal w + A^\intercal\delta\rangle - \gamma F^*_{\mathcal{I}}(\mu)\Big] + \max_\nu \Big[\langle \nu, -w\rangle - \frac{1}{2\lambda}\|\nu\|^2\Big] + \max_{\nu'}\Big[\langle \nu', \xi-\delta\rangle - \frac{1}{2\rho}\|\nu'\|^2\Big],
\end{aligned}$$

which yields the desired form of $P_\rho(w,\xi) = \min_\delta \tilde{P}_\rho(w,\delta,\xi)$ upon expliciting Fenchel conjugates. □

Proposition 5.

$$\min_\delta F_{\mathcal{I}}\big(\tfrac{1}{\gamma}(\theta(w) + A^\intercal\delta)\big) = F_{\mathcal{L}}\big(\tfrac{1}{\gamma}\theta(w)\big) \qquad \textit{and} \qquad \min_{\xi,\delta} \tilde{P}_\rho(w,\delta,\xi) = P(w).$$

Proof. We have

$$\begin{aligned}\min_{\delta} F_{\mathcal{I}}\big(\tfrac{1}{\gamma}(\theta(w)+A^{\intercal}\delta)\big) &= \min_{\delta}\max_{\mu}\Big(\frac{1}{\gamma}\langle\theta(w)+A^{\intercal}\delta,\mu\rangle + H_{\text{Approx}}(\mu) - \iota_{\mathcal{I}}(\mu)\Big)\\ &= \max_{\mu}\Big(\frac{1}{\gamma}\langle\theta(w),\mu\rangle + H_{\text{Approx}}(\mu) - \iota_{\mathcal{I}}(\mu) - \iota_{\{A\mu=0\}}\Big)\\ &= F_{\mathcal{L}}\big(\tfrac{1}{\gamma}\theta(w)\big),\end{aligned}$$

where the second equality follows by exchanging minimization and maximization (strong duality holds by Slater's conditions) and minimizing with respect to δ.

To show that $\min_{\xi,\delta}\tilde{P}_{\rho}(w,\delta,\xi) = P(w)$, it is easy to minimize over ξ first, which cancels out the term $\frac{\rho}{2}\|\delta-\xi\|^2$ by setting $\xi=\delta$. Then, δ only appears in $F_{\mathcal{I}}$ and the result follows from the first result. □

B.3 Interpretation as Moreau-Yosida smoothing

To understand the structure of $P_{\rho}(w,\xi)$, we shall look at $\tilde{P}_{\rho}(w,\delta,\xi)$. One may be interested in where does δ comes from? In fact, forming the Lagrangian of $\min_w P(w)$ with Lagrangian multiplier δ corresponding to the marginalization constraint $A\mu=0$, we see that

$$L(w,\delta,\mu) := \langle\theta_{\ell}(w),\mu\rangle - \gamma F_{\mathcal{I}}^{*}(\mu) + \frac{\lambda}{2}\|w\|_2^2 + \langle\delta, A\mu\rangle.$$

Recall that the Moreau-Yosida regularization of a function f is defined as the infimal convolution

$$M_{\rho f}(x) = \min_{z}\Big[f(z) + \frac{\rho}{2}\|z-x\|^2\Big].$$

Both $P_{\rho}(w,\xi)$ and $D_{\rho}(\mu,\xi)$ have a nice interpretation in terms of the Lagrangian L and Moreau-Yosida regularization. Note that the Moreau-Yosida regularization admits the same optimum as the original function, and that it is smooth even when the original function is not. It is furthermore $\frac{\gamma\rho}{\gamma+\rho}$-strongly convex if the original function is γ-strongly convex.

Proposition 6. *$P_{\rho}(w,\xi)$ and $D_{\rho}(\mu,\xi)$ are respectively the Moreau-Yosida regularizations of $L_{\mu^{\star}}: w,\delta\mapsto \max_{\mu} L(w,\delta,\mu)$ and $L_{w^{\star}}: \mu,\delta\mapsto\min_w L(w,\delta,\mu)$ about δ. that is*

$$\begin{aligned}P_{\rho}(w,\xi) &= M_{\rho L_{\mu^{\star}}}(w,\xi) = \min_{\delta}\Big[\max_{\mu} L(w,\delta,\mu) + \frac{\rho}{2}\|\delta-\xi\|_2^2\Big]\\ D_{\rho}(\mu,\xi) &= M_{\rho L_{w^{\star}}}(\mu,\xi) = \min_{\delta}\Big[\min_{w} L(w,\delta,\mu) + \frac{\rho}{2}\|\delta-\xi\|_2^2\Big].\end{aligned}$$

Proof. For $P_{\rho}(w,\xi)$, note that $\max_{\mu} L(w,\delta,\mu) \equiv \gamma F_{\mathcal{I}}\Big(\frac{\theta(w)+A^{\intercal}\delta}{\gamma}\Big) + \frac{\lambda}{2}\|w\|_2^2$. The equivalent form is immediately derived from Proposition 4.

For $D_{\rho}(\mu,\xi)$, note that $\min_w L(w,\delta,\mu) \equiv \langle\theta,\mu\rangle - \gamma F_{\mathcal{I}}^{*}(\mu) - \frac{1}{2\lambda}\|\Psi\mu\|^2 + \langle\delta,A\mu\rangle$, and $\min_{\delta}\langle\delta,A\mu\rangle + \frac{\rho}{2}\|\delta-\xi\|^2 \equiv \langle\xi,A\mu\rangle - \frac{1}{2\rho}\|A\mu\|_2^2$. Thus, the equivalence holds.

□

Note that the penalty formulation corresponds to a special case of $\tilde{P}_{\rho}(w,\delta,\xi)$ and $D_{\rho}(\mu,\xi)$ with $\xi=0$. It introduces an additional term $\frac{\rho}{2}\|\delta\|^2$, thus making the primal strongly convex with respect to δ and the dual smoother in μ. This effect is similar to that of using Moreau-Yosida smoothing. However, the additional term $\frac{\rho}{2}\|\delta\|^2$ will never vanish, so $A\mu=0$ will never be satisfied. The more $A\mu=0$ is violated, the less the structure of CRF will be perserved.

B.4 Duality gaps and representer theorem

Besides, if we define $\mathrm{gap}(w, \delta, \mu, \xi) := \tilde{P}_\rho(w, \delta, \xi) - D_\rho(\mu, \xi)$ as an upper-bound estimate of the duality gap $P_\rho(w, \xi) - D_\rho(\mu, \xi)$, specifically

$$\begin{aligned}\mathrm{gap}(w, \delta, \mu, \xi) = &\Big[\gamma F_{\mathcal{I}}\big(\tfrac{1}{\gamma}\theta_\ell(w) + A^\intercal\delta\big) + \gamma F_{\mathcal{I}}^*(\mu) - \langle \theta_\ell(w) + A^\intercal\delta, \mu\rangle\Big] \\ &+ \Big[\frac{\lambda}{2}\|w\|^2 + \frac{1}{2\lambda}\|\Psi\mu\|^2 - \langle -w, \Psi\mu\rangle\Big] + \Big[\frac{\rho}{2}\|\xi - \delta\|^2 + \frac{1}{2\rho}\|A\mu\|^2 - \langle \xi - \delta, A\mu\rangle\Big],\end{aligned}$$

we can see that the recovered w and δ by the optimality condition make the 2nd and 3rd term of $\mathrm{gap}(w, \delta, \mu, \xi)$ disappear. We will see later this is important in designing the algorithm to solve $\max_\mu D_\rho(\mu, \xi)$.

Finally, we give a rough picture of all the quantities that we introduced in this section, which can be easily derived from Proposition 6.

Corollary 5. *The relations between D, D_ρ, P and P_ρ could be summarized as*

$$\begin{aligned}&D(\mu) \le D_\rho(\mu, \xi) \le P_\rho(w, \xi); \\ &D_\rho(\mu) \le P(w) \le P_\rho(w, \xi) \le \tilde{P}_\rho(w, \delta, \xi); \\ &\max_\mu \min_\xi D_\rho(\mu, \xi) \le \min_w P(w),\end{aligned}$$

with equalities hold for the saddle point $(\mu^\star, w^\star, \xi^\star)$. Moreover, the first-order optimality conditions are given as

$$w^\star = -\frac{1}{\lambda}\Psi\mu^\star, \quad \delta^\star = \xi^\star - \frac{1}{\rho}A\mu^\star \tag{5}$$

Proof. By constructions, $D(\mu) = \min_\xi D_\rho(\mu, \xi) \le D_\rho(\mu, \xi)$ and $P_\rho(w, \xi) = \min_\delta \tilde{P}_\rho(w, \delta, \xi) \le \tilde{P}_\rho(w, \delta, \xi)$. Other inequalities are the consequences of Proposition 6 and the min-max inequality. Since the strong duality holds (Slater conditions satisfied and the problem is convex), we know that the equalities will hold at the saddle point.

Given the saddle point $(\mu^\star, w^\star, \xi^\star)$, to derive $w^\star, \delta^\star$ from $\mu^\star$, we know that $w^\star, \delta^\star = \arg\min_{w,\delta} \tilde{P}_\rho(w, \delta, \xi^\star)$. The result follows after computing $\nabla_w \tilde{P}_\rho(w, \delta, \xi^\star) = 0$ and $\nabla_\delta \tilde{P}_\rho(w, \delta, \xi^\star) = 0$. ☐

So our strategy for CRF learning is $\min_\xi \max_\mu D_\rho(\mu, \xi)$, since we know that

$$D_\rho(\mu^\star, \xi^\star) \equiv L(w^\star, \delta^\star, \mu^\star) \equiv P(w^\star).$$

Since we work on the space of μ and ξ, to compute the primal objectives or the duality gap, we can use the mapping specified by the optimality condition (5). More precisely, we define

$$w(\mu^{t,s}) = -\frac{1}{\lambda}\Psi\mu^{t,s}, \quad \delta(\mu^{t,s}, \xi^t) = \xi^t - \frac{1}{\rho}A\mu^{t,s},$$

which is equivalent to the representer theorem. The above condition is also useful to recover intermediate $w^{t,s}$ from $\mu^{t,s}$, which allows us to test on the validation set or decide if we should stop the learning earlier.

B.5 Comparison with State-of-the-Art Structured Learning Methods

A number of recent works for CRF learning can be viewed as optimizing formulations which are exactly or fairly close to one of $P(w), D(\mu), P_\rho(w, \delta), \tilde{P}_\rho(w, \delta, \xi)$ or $D_\rho(\mu, \xi)$. In the following table, we compare these approaches, in terms of the optimization formulation, the convergence rate (respectively in the primal or in the dual), and the inference oracle used for computing the gradients (or blockwise gradients).

Table 1: The Comparison of Structured Learning Methods

Method	Learning Regime	Primal/Dual	Convergence	Inference Oracle
Meshi et al. (2010)	SSVM	Primal (w, δ)	Sublinear	graphwise MAP (inexact)
Hazan and Urtasun (2010)	LossAugCRF	Primal (w, δ)	Sublinear	graphwise marginal (inexact)
Lacoste-Julien et al. (2013)	SSVM	Dual (μ)	Sublinear	graphwise MAP
Schmidt et al. (2015)	CRF	Primal (w)	Linear	graphwise marginal
Tang et al. (2016)	CRF	Dual (μ)	Sublinear	graphwise MAP
Meshi et al. (2015)	SSVM (soft)	Dual $(\mu, \xi = 0)$	Sublinear	cliquewise MAP
Yen et al. (2016)	SSVM	Dual (μ, ξ)	Linear	cliquewise MAP
IDAL	LossAugCRF	Dual (μ, ξ)	Linear	cliquewise marginal

C Gini Oriented Tree-Reweighted Entropy

The Bethe entropy (Yedidia et al., 2005) is generally non-concave. Its concave counterparts, such as the tree-reweighted entropy (Wainwright et al., 2005) or the region-based entropy (London et al., 2015; Yedidia et al., 2005), are only concave on the local consistency polytope, but non-concave on $\mathcal{I}\backslash\mathcal{L}$ (i.e., when $A\mu \neq 0$). Indeed, the Bethe entropy and its concave variants are of the form $H_{\text{Bethe}}(\mu) = \sum_{i\in\mathcal{V}} c_i H_i(\mu_i) + \sum_{\{i,j\}\in\mathcal{E}} c_{ij} H_{ij}(\mu_{ij})$, where c_i and c_{ij} are counting numbers. Even when H_{Bethe} is concave on $\mathcal{L}$, some of the c_i or c_{ij} can be negative.

The construction of the oriented tree-reweighted entropy stems from the expression of the entropy of a directed tree as the sum of of the entropy of the root and the conditional entropies of the variable at each node given their parent variable. Precisely, for an oriented tree T with the root i_0, the joint entropy can be computed as

$$H_T(Y) := H(Y_{i_0}) + \sum_{j\to i\in T} H(Y_i \mid Y_j). \tag{6}$$

On a general graph, if T is a (directed) spanning tree of the graph, then

$$H_T(Y) := H(Y_{i_0}) + \sum_{t\to i\in T} H(Y_i \mid Y_j) \geq H(Y_{i_0}) + \sum_{k=1}^{m} H(Y_{i_k} \mid Y_{i_{k-1}}, \dots, Y_{i_0}) =: H_{\text{Shannon}}(Y). \tag{7}$$

Thus, for any probability distribution over the set of valid directed spanning trees, in which tree T has probability ρ_T, the inequality above entails that $H_{\text{Shannon}}(Y) \leq \sum_T \rho_T H_T(Y) =: H_{\text{OTRW}}(Y)$, where $\rho_T \geq 0$ and $\sum_T \rho_T = 1$.

$H_T(Y)$ is concave since it is a sum of concave functions, and so is $H_{\text{OTRW}}(Y)$ (who is a convex combination of $H_T(Y)$). To see that, we need to prove the following fact.

Fact 1 (Concavity of the conditional entropy)**.** *The conditional entropy $H(Y_j \mid Y_i)$ is in fact a function of μ_{ij}, namely $H(Y_j \mid Y_i) = H(\mu_{ij}) - H(A_i\mu_{ij})$. Moreover, $H(Y_j \mid Y_i)$ is a concave function of μ_{ij}.*

Proof. By definition,

$$H(Y_j \mid Y_i) = \sum_{y_j, y_i} \mu_{ij}(y_j, y_i) \log \frac{\sum_{y_j} \mu_{ij}(y_j, y_i)}{\mu_{ij}(y_j, y_i)} = H(\mu_{ij}) - H(A_i\mu_{ij}).$$

To show $H(Y_j \mid Y_i)$ is concave, we compute its Hessian:

$$\begin{aligned}
\frac{\partial^2 H(Y_j \mid Y_i)}{\partial \mu_{ij}^2} &= -\text{diag}\Big(\mathbf{1} \oslash \mu_{ij}\Big) + A^\intercal \text{diag}\Big(\mathbf{1} \oslash A\mu\Big) A \\
&= -\text{diag}\Big(\mathbf{1} \oslash \mu_{ij}\Big) + \text{diag}\Big(\{\frac{1}{\tilde{\mu}_i(y_i)} \mathbf{1}\mathbf{1}^\intercal\}_{y_i=1}^{k_i}\Big)
\end{aligned}$$

where $\tilde{\mu}_i = A_i \mu_{ij}$, and $\oslash$ denotes entrywise division. Let's focus on the i-th block of the negative Hessian. To show that the i-th block is positive semidefinite, that is, that

$$\operatorname{diag}\Big(\Big\{\frac{1}{\mu_{ij}(y_i, y_j)}\Big\}_{1 \le y_j \le k_j}\Big) - \frac{1}{\tilde{\mu}_i(y_i)} \mathbf{1}\mathbf{1}^\intercal \succeq 0, \tag{8}$$

we can use the Schur complement condition for positive semidefiniteness. Let $U = \tilde{\mu}_i(y_i)$. Since $\tilde{\mu}_i(y_i) \succ 0$,

$$L - B^\intercal U^{-1} B \succeq 0 \quad \text{iff} \quad \begin{bmatrix} U & B \\ B^\intercal & L \end{bmatrix} = \begin{bmatrix} \tilde{\mu}_i(y_i) & \mathbf{1}^\intercal \\ \mathbf{1} & \operatorname{diag}\Big(\{\frac{1}{\mu_{ij}(y_i,y_j)}\}_{1 \le y_j \le k_j}\Big) \end{bmatrix} \succeq 0.$$

We also have $L = \operatorname{diag}\Big(\Big\{\frac{1}{\mu_{ij}(y_i,y_j)}\Big\}_{y_j=1}^{k_j}\Big) \succ 0$, then

$$\begin{bmatrix} U & B \\ B^\intercal & L \end{bmatrix} \succeq 0 \quad \text{iff} \quad U - BL^{-1}B^\intercal = \tilde{\mu}_i(y_i) - \mathbf{1}^\intercal \operatorname{diag}\Big(\{\mu_{ij}(y_i, y_j)\}_{y_j=1}^{k_j}\Big)\mathbf{1} = \tilde{\mu}_i(y_i) - \tilde{\mu}_i(y_i) \succeq 0.$$

Because the last inequality holds, we know (8) must be true, which implies that the Hessian of $H(Y_j \mid Y_i)$ is negative semidefinite, thus $H(Y_j \mid Y_i)$ is concave. $\square$

Note that $H_{\text{OTRW}}(\mu)$ is concave on the entire set $\mathcal{I}$, unlike many Bethe entropy variants who are only concave in the local consistency polytope.

We define $\overrightarrow{\mathcal{E}}$ the directed edge set by expanding each edge from $\mathcal{E}$ with two directed edges, ρ_i and $\rho_{i|j}$ respectively as the probabilities of i (as the root) and $i \to t$ appearing in an oriented spanning tree when the latter is drawn with probability ρ_T. Then the oriented tree-reweighted entropy takes the form

$$\begin{aligned} H_{\text{OTRW}}(\mu) := \sum_{\{i,j\} \in \mathcal{E}} & \rho_{j|i}\big[H_e(\mu_{ij}) - H_i(A_i \mu_{ij})\big] \\ & + \rho_{i|j}\big[H_e(\mu_{ij}) - H_j(A_j \mu_{ij})\big] + \sum_{i \in \mathcal{V}} \rho_i H_i(\mu_i), \end{aligned} \tag{9}$$

where $H_i(\mu_i) = -\sum_{y_i} \mu_i(y_i) \log \mu_i(y_i)$, $H_e(\mu_{ij}) = -\sum_{y_i, y_j} \mu_i(y_i, y_j) \log \mu_i(y_i, y_j)$ and $\rho_i, \rho_{i|j}, \rho_{j|i}$ are node/edge appearance probabilities in $[0, 1]$. H_{OTRW} is concave, since H_i is concave and it can be checked that so is $\mu_{ij} \mapsto H_e(\mu_{ij}) - H_i(A_i \mu_{ij})$ (although not strongly concave). It is easy to precompute the appearance probabilities ρ_i and $\rho_{i|j}$ via a variant of the directed matrix-tree theorem. See Koo et al. (2007) for more details.

A generic difficulty with entropies, is that H_i and H_e do not have Lipschitz gradients, which prevents the direct application of proximal methods with usual quadratic proximity terms. We thus propose to replace H_i and H_e by their second-order Taylor approximation around the uniform distribution. This yields a surrogate of the form

$$\begin{aligned} H_{\text{GTRW}}(\mu) := \sum_{\{i,j\} \in \mathcal{E}} & \varepsilon \Big[k_i \rho_{j|i} \|A_i \mu_{ij}\|^2 + k_j \rho_{i|j} \|A_j \mu_{ij}\|^2\Big] \\ & - k_i k_j (\rho_{i|j} + \rho_{j|i}) \|\mu_{ij}\|^2 + \sum_{i \in \mathcal{V}} k_i \rho_i (1 - \|\mu_i\|^2), \end{aligned} \tag{10}$$

where $\varepsilon = 1$. Since this function is not strongly convex w.r.t. μ_{ij} because $k_j I_{k_i} - A_i^\intercal A_i$ has a non-trivial kernel, so we also consider variants with $\varepsilon < 1$ and denote them $H_{\text{GTRW},\varepsilon}$. We call this approximation the *Gini OTRW entropy*, since it is consistent with the definition of Gini conditional entropy of Furuichi (2006).

D Proof of Lemma 1 and associated lemma

To prove Lemma 1, we first need to show $d(\xi)$ is a smooth function, and then we build up the associated lemmas which will be used in the proof of Lemma 1. Finally, in the end of this section, we prove Corollary 2 as a result to show the linear convergence in the primal.

D.1 Smoothness of $d(\xi)$

Lemma D.1. *(Hong and Luo, 2017, Lemma 2.3) $d(\xi)$ is convex and L_d-smooth, where $L_d \leq \rho$.*

Proof. By definition we have

$$D_\rho(\mu, 0) = -\langle \mu, \ell \rangle + \gamma F_{\mathcal{I}}^*(\mu) + \frac{1}{2\lambda}\|\Psi\mu\|^2 + \frac{1}{2\rho}\|A\mu\|^2.$$

We then have $d(\xi) = \max_\mu D_\rho(\mu, \xi) = \max_\mu \langle \mu, A^\intercal \xi \rangle - D_\rho(\mu, 0)$ so that if $J(\mu) := D_\rho(\mu, 0)$, then $d(\xi) = J^*(A^\intercal \xi)$ and d is a convex function by Fenchel conjugacy.

For any ξ_1 and ξ_2, denote by μ_1 and μ_2 the minimizers of $D_\rho(\cdot, \xi_1)$ and $D_\rho(\cdot, \xi_2)$ respectively. By convexity of $d(\xi)$ and the definition of subgradient, there exists $s_1 \in \partial F_{\mathcal{I}}^*(\mu_1)$ and $s_2 \in \partial F_{\mathcal{I}}^*(\mu_2)$ such that

$$A^\intercal \xi_1 + \ell - \gamma s_1 - \frac{1}{\lambda}\Psi^\intercal \Psi \mu_1 - \frac{1}{\rho} A^\intercal A \mu_1 = 0$$
$$A^\intercal \xi_2 + \ell - \gamma s_2 - \frac{1}{\lambda}\Psi^\intercal \Psi \mu_2 - \frac{1}{\rho} A^\intercal A \mu_2 = 0$$

By convexity of $F_{\mathcal{I}}^*(\mu)$, we have

$$\langle s_1 - s_2, \mu_1 - \mu_2 \rangle \geq 0,$$

which together with the equations above yields

$$\langle A^\intercal(\xi_1 - \xi_2) - \frac{1}{\lambda}\Psi^\intercal\Psi(\mu_1 - \mu_2) - \frac{1}{\rho}A^\intercal A(\mu_1 - \mu_2), \mu_1 - \mu_2 \rangle \geq 0.$$

Hence,

$$\langle \xi_1 - \xi_2, A(\mu_1 - \mu_2) \rangle \geq \frac{1}{\lambda}\|\Psi(\mu_1 - \mu_2)\|^2 + \frac{1}{\rho}\|A(\mu_1 - \mu_2)\|^2 \geq \frac{1}{\rho}\|A(\mu_1 - \mu_2)\|^2.$$

Now substituting $\nabla d(\xi_1) - \nabla d(\xi_2) = A(\mu_1 - \mu_2)$ into the above inequality and using the Cauchy-Schwarz inequality yields

$$\|\nabla d(\xi_1) - \nabla d(\xi_2)\| \leq \rho \|\xi_1 - \xi_2\|.$$

That completes the proof. ∎

D.2 Associated lemmas for Lemma 1

We first quantify in the next two lemmas how much $D(\mu, \xi^t)$ should be minimized in μ to provide a sufficiently accurate approximate gradient that it guarantees descent on d.

Lemma D.2 (Error on the gradient)**.** *Denote $\bar{\mu}^t := \mu^\star(\xi^t) = \operatorname{argmin}_\mu D(\mu, \xi^t)$; $g_t := \nabla d(\xi^t) = A\mu^\star(\xi^t)$ and $\hat{g}_t := A\hat{\mu}^t$. Let $\hat{\Delta}_t := D_\rho(\bar{\mu}^t, \xi^t) - D_\rho(\hat{\mu}^t, \xi^t)$. We have $\frac{1}{2L_d}\|\hat{g}_t - g_t\|^2 \leq \hat{\Delta}_t$, where L_d is the smoothness constant of d.*

Proof. Let $d^*(y) = \max_\xi \langle \xi, y \rangle - d(\xi)$. Then, it can easily be checked by using the definition of d and exchanging the order of maximization and minimization that $d^*(y) = \min_\mu D_\rho(\mu, 0) + \iota_{\{A\mu = y\}}$,

Since d is convex, we have $d(\xi) = \max_y \langle \xi, y \rangle - d^*(y)$, so that if $y^\star(\xi)$ is a maximizer for fixed ξ we have

$$0 \in \xi - \partial d^*(y^\star(\xi)) \Rightarrow \xi \in \partial d^*(y^\star(\xi)).$$

The strong convexity of $d^*(y)$ implies that, for all y,

$$d^*(y) - d^*(y^\star(\xi)) - \langle \xi, y - y^\star(\xi) \rangle \geq \frac{1}{2L_d}\|y - y^\star(\xi)\|^2.$$

But for any μ, we have $D_\rho(\mu, \xi) = \langle A\mu, \xi\rangle - D_\rho(\mu, 0) \le \langle A\mu, \xi\rangle - d^*(A\mu)$, and, for $\mu^\star(\xi)$, this inequality is an equality, since we have $D_\rho(\mu^\star(\xi), \xi) = \langle y^\star(\xi), \xi\rangle - d^*(y^\star(\xi))$ and $y^\star(\xi) = A\mu^\star(\xi)$. As a consequence, setting $y = A\mu$, we have

$$D_\rho(\mu^\star(\xi), \xi) - D_\rho(\mu, \xi) \ge \frac{1}{2L_d}\|A\mu - A\mu^\star(\xi)\|^2$$

by definition of $D_\rho(\mu, \xi)$. We conclude the proof by substituting μ with $\hat{\mu}^t$ and ξ with ξ^t. $\square$

Lemma D.3 (Guaranteed decrease on d)**.** *If we take inexact gradient on ξ with a fixed step size $\frac{1}{L_d}$, namely $\xi^{t+1} = \xi^t - \frac{1}{L_d}\hat{g}_t$, then*

$$d(\xi^t) - d(\xi^{t+1}) \ge \frac{\tau}{L_d}\Gamma_t - \hat{\Delta}_t, \tag{11}$$

where $\tau \in (0, L_d)$ satisfying $\frac{1}{2\tau}\|g_t\|^2 \ge \Gamma_t$.

Proof. Since $d(\xi)$ is L_d-smooth, we have

$$d(\xi^{t+1}) - d(\xi^t) \le \langle \nabla d(\xi^t), \xi^{t+1} - \xi^t\rangle + \frac{L_d}{2}\|\xi^{t+1} - \xi^t\|^2$$

Using the gradient step and $\nabla d(\xi^t) = g_t$, the above inequality can be simplified as

$$\begin{aligned} d(\xi^{t+1}) - d(\xi^t) &\le \langle g_t, -1/L_d\hat{g}_t\rangle + \frac{L_d}{2}\|1/L_d\hat{g}_t\|^2 \\ &= \frac{1}{2L_d}\Big(\|\hat{g}_t - g_t\|^2 - \|g_t\|^2\Big). \end{aligned} \tag{12}$$

We notice that the error bound given by the Lemma 2.3 of Hong and Luo (2017) holds for $d(\xi)$. Specifically,

$$\exists \tau' > 0, \text{ such that } \|\nabla d(\xi)\| \ge \tau'\|\xi - \xi^\star\|.$$

Since $d(\xi)$ is L_d-smooth and $\nabla d(\xi^\star) = 0$, we have

$$d(\xi) - d(\xi^\star) \le \frac{L_d}{2}\|\xi - \xi^\star\|^2 \le \frac{L_d}{2\tau'}\|\nabla d(\xi)\|^2,$$

which implies

$$\frac{1}{2\tau}\|g_t\|^2 \ge \Gamma_t,$$

where $\tau = \frac{\tau'}{L_d}$. By using (12) and the above inequality on $\|g_t\|^2$, we obtain

$$d(\xi^t) - d(\xi^{t+1}) \ge \frac{1}{2L_d}\Big(\|g_t\|^2 - \|\hat{g}_t - g_t\|^2\Big) \ge \frac{\tau}{L_d}\Gamma_t - \hat{\Delta}_t.$$

$\square$

Since for each value of ξ^t the value and gradient of $d(\xi^t)$ need to be computed approximately by minimizing the augmented Lagrangian $D_\rho(\cdot, \xi^t)$, and since the difference between two consecutive strongly convex objectives is $D_\rho(\mu, \xi^t) - D_\rho(\mu, \xi^{t-1}) = \langle \xi^{t-1} - \xi^t, A\mu\rangle$, which is a function that converges to zero when if the sequence $\{\xi^t\}_t$ converges, a warm-restart strategy using $\hat{\mu}^t$ as the initial point to the subproblem $\max_\mu D_\rho(\mu, \xi^{t+1})$ is beneficial, as characterized by the following lemma.

Lemma D.4 (Dual gap at warm start)**.** *Denote $\Delta^0_{t+1} := D_\rho(\bar{\mu}^{t+1}, \xi^{t+1}) - D_\rho(\mu^{t+1,0}, \xi^{t+1})$. If we let $\mu^{t+1,0} = \hat{\mu}^t$, then*

$$\Delta^0_{t+1} \le (4 + \frac{2}{\omega})\hat{\Delta}_t + (1 + 2\omega)\Gamma_t, \quad \forall \omega > 0. \tag{13}$$

Proof. By definition, we have $D_\rho(\bar{\mu}^{t+1}, \xi^{t+1}) = D_\rho(\mu^{t+1,*}, \xi^{t+1}) = d(\xi^{t+1})$. The initial gap of μ at iteration t can then be decomposed as

$$\begin{aligned}
\Delta_{t+1}^0 &= D_\rho(\bar{\mu}^{t+1}, \xi^{t+1}) - D_\rho(\mu^{t+1,0}, \xi^{t+1}) + d(\xi^t) - d(\xi^t) - D_\rho(\hat{\mu}^t, \xi^t) + D_\rho(\hat{\mu}^t, \xi^t) \\
&= \Big[d(\xi^t) - D_\rho(\hat{\mu}^t, \xi^t)\Big] + \Big[D_\rho(\hat{\mu}^t, \xi^t) - D_\rho(\mu^{t+1,0}, \xi^{t+1})\Big] + D_\rho(\bar{\mu}^{t+1}, \xi^{t+1}) - d(\xi^t) \\
&= \Big[D_\rho(\bar{\mu}^t, \xi^t) - D_\rho(\hat{\mu}^t, \xi^t)\Big] + \Big[D_\rho(\hat{\mu}^t, \xi^t) - D_\rho(\hat{\mu}^t, \xi^{t+1})\Big] + d(\xi^{t+1}) - d(\xi^t) \\
&= \hat{\Delta}_t + \frac{1}{L_d}\|\hat{g}_t\|^2 + d(\xi^{t+1}) - d(\xi^t)
\end{aligned}$$

Again, we used the gradient step $\xi^{t+1} = \xi^t - \frac{1}{L_d}\hat{g}_t$, and recall that $A\hat{\mu}^t = \hat{g}_t$.

Now, we can bound the term $\|\hat{g}_t\|^2$ from above using the fact that

$$\begin{aligned}
\frac{1}{L_d}\|\hat{g}_t\|^2 &= \frac{1}{L_d}\Big[\|g_t\|^2 + 2\langle g_t, \hat{g}_t - g_t\rangle + \|\hat{g}_t - g_t\|^2\Big] \\
&\leq \frac{1}{L_d}\Big[(1+\omega)\|g_t\|^2 + (1 + 1/\omega)\|\hat{g}_t - g_t\|^2\Big],
\end{aligned}$$

where the last inequality stems from the Cauchy-Schwarz inequality $\langle g_t, \hat{g}_t - g_t\rangle \leq \|g_t\|\,\|\hat{g}_t - g_t\|$ and the fact that for any any $a, b \in \mathbb{R}$ and $\omega > 0$, we have $2ab \leq \omega a^2 + b^2/\omega$.

Combining the upper bound of $d(\xi^{t+1}) - d(\xi^t)$ from (12), we get

$$\Delta_{t+1}^0 \leq \hat{\Delta}_t + \frac{3\omega + 2}{2\omega L_d}\|\hat{g}_t - g_t\|^2 + \frac{2\omega + 1}{2L_d}\|g_t\|^2. \tag{14}$$

Here, we can use again Lemma D.2 and the fact that $\frac{1}{2L_d}\|g_t\|^2 \leq \Gamma_t$, which is due to the smoothness of $d(\xi)$. It follows that

$$\Delta_{t+1}^0 \leq \left(4 + \tfrac{2}{\omega}\right)\hat{\Delta}_t + \left(1 + 2\omega\right)\Gamma_t, \quad \forall \omega > 0.$$

□

D.3 Proof of Lemma 1

Combining Lemma D.3 and D.4, we now show that IDAL enjoys a linear convergence rate if we take a fixed number of inner iterations to estimate the gradient.

Lemma 1 (Linear convergence of the outer iteration)**.** *Suppose we have an algorithm $\mathcal{A}$ to approximately solve $\max_\mu D_\rho(\mu, \xi^t)$ in the sense that*

$$\exists \beta \in (0,1), \qquad \mathbb{E}[\hat{\Delta}_t] \leq \beta\,\mathbb{E}[\Delta_t^0].$$

Then $\exists \kappa \in (0,1)$ characterizing d and $C > 0$ such that, for any $\omega > 0$, after T_{ex} gradient steps on ξ, the suboptimalities $\Delta_{T_{\text{ex}}}$ and $\Gamma_{T_{\text{ex}}}$ are bounded from above:

$$\left\|\begin{matrix}\mathbb{E}[\hat{\Delta}_{T_{\text{ex}}}] \\ \mathbb{E}[\Gamma_{T_{\text{ex}}}]\end{matrix}\right\| \leq C\,\lambda_{\max}(\beta)^{T_{\text{ex}}}\left\|\begin{matrix}\mathbb{E}[\hat{\Delta}_0] \\ \mathbb{E}[\Gamma_0]\end{matrix}\right\|, \quad \text{where} \quad M(\beta) = \begin{bmatrix}\beta(4 + \frac{2}{\omega}) & \beta(1+2\omega) \\ 1 & 1-\kappa\end{bmatrix}, \tag{15}$$

and $\lambda_{\max}(\beta)$ is the largest eigenvalue of $M(\beta)$. Thereby, if β is chosen so that $\lambda_{\max}(\beta) < 1$, Algorithm 1 is linearly convergent with a rate $\lambda_{\max}(\beta)$.

Proof. Note that $\Gamma_{t+1} - \Gamma_t = d(\xi^{t+1}) - d(\xi^t)$. By using Lemma D.3, we have an upper bound on Γ_{t+1} in terms of Γ_t and $\hat{\Delta}_t$, namely

$$\Gamma_{t+1} \leq \hat{\Delta}_t + (1-\kappa)\,\Gamma_t \qquad \text{with} \qquad \kappa = \frac{\tau}{L_d}. \tag{16}$$

On the other hand, we can also derive an upper bound on $\hat{\Delta}_{t+1}$ in terms of Γ_t and $\hat{\Delta}_t$. To achieve that, we relate the inner problem with Γ_t by running the steps on μ until $\mathbb{E}[\hat{\Delta}_{t+1}] \le (1-\pi)^{T_{\text{in}}} \mathbb{E}[\Delta^0_{t+1}] \le \beta\, \mathbb{E}[\Delta^0_{t+1}]$, which means $T_{\text{in}} \ge \frac{\log \beta}{\log(1-\pi)}$. By Lemma D.4, we have

$$\mathbb{E}[\hat{\Delta}_{t+1}] \le \beta\, \mathbb{E}[\Delta^0_{t+1}] \le \beta\big(4 + \tfrac{2}{\omega}\big)\mathbb{E}[\hat{\Delta}_t] + \beta(1+2\omega)\,\mathbb{E}[\Gamma_t]. \tag{17}$$

Combining (17) and (16), and taking expectations on both sides, we get

$$\begin{bmatrix}\mathbb{E}[\hat{\Delta}_{t+1}] \\ \mathbb{E}[\Gamma_{t+1}]\end{bmatrix} \le M \begin{bmatrix}\mathbb{E}[\hat{\Delta}_t] \\ \mathbb{E}[\Gamma_t]\end{bmatrix} \tag{18}$$

Since by definition, all the elements of M are positive, we can telescope a sequence of matrix multiplications to get

$$\begin{bmatrix}\mathbb{E}[\hat{\Delta}_{T_{\text{ex}}}] \\ \mathbb{E}[\Gamma_{T_{\text{ex}}}]\end{bmatrix} \le M \begin{bmatrix}\mathbb{E}[\hat{\Delta}_{T_{\text{ex}}-1}] \\ \mathbb{E}[\Gamma_{T_{\text{ex}}-1}]\end{bmatrix} \le \cdots \le M^{T_{\text{ex}}} \begin{bmatrix}\mathbb{E}[\hat{\Delta}_0] \\ \mathbb{E}[\Gamma_0]\end{bmatrix} \tag{19}$$

Assuming the eigen decomposition of M takes the form $M = PDP^{-1}$, then $M^t = PD^tP^{-1}$. Applying norms on both sides of the vector inequality, we have

$$\left\|\begin{bmatrix}\mathbb{E}[\hat{\Delta}_{T_{\text{ex}}}] \\ \mathbb{E}[\Gamma_{T_{\text{ex}}}]\end{bmatrix}\right\| \le \|P\|_{\text{op}}\, \lambda_{\max}(\beta)^{T_{\text{ex}}} \|P^{-1}\|_{\text{op}} \left\|\begin{bmatrix}\mathbb{E}[\hat{\Delta}_0] \\ \mathbb{E}[\Gamma_0]\end{bmatrix}\right\|. \tag{20}$$

Note that $C = \|P\|_{\text{op}}\|P^{-1}\|_{\text{op}}$ is a constant. □

Corollary 2. *Let σ denote the strong convexity constant of $\mu \mapsto D_\rho(\mu, \xi)$ and L_d the smoothness constant of d. Assume that $(\|\xi_t\|_2)_{t\in\mathbb{N}}$ is almost surely bounded by a constant B. Then the squared residuals to the constraint $A\mu = 0$ satisfy*

$$\frac{1}{2}\|A\hat{\mu}^t\|_2^2 \le 2L_d\Gamma_t + \frac{2}{\sigma}\|A\|^2_{\text{op}}\hat{\Delta}_t.$$

Furthermore, if we let $D_\infty(\mu) := \langle \ell, \mu\rangle - \gamma F^*_{\mathcal{I}}(\mu) - \frac{1}{2\lambda}\|\Psi\mu\|_2^2$, so that we have $D(\mu) = D_\infty(\mu) - \iota_{\{A\mu=0\}}$, then (given that $\mu^t \in \mathcal{I}$ throughout the algorithm) the gap between the smooth part of the objective in $\hat{\mu}^t$ and at the optimum can be bounded as follows

$$|D_\infty(\hat{\mu}^t) - D_\infty(\mu^\star)| \le B\sqrt{2L_d\,\Gamma_t} + B\frac{\|A\|_{\text{op}}}{\sqrt{\sigma}}\sqrt{2\hat{\Delta}_t} + \Big(1 + 2\frac{L_d}{\rho}\Big)\Gamma_t + \Big(1 + 2\frac{\|A\|^2_{\text{op}}}{\rho\sigma}\Big)\hat{\Delta}_t.$$

Finally, if Γ_t and $\hat{\Delta}_t$ converge to 0 linearly then both the residuals $\|A\hat{\mu}^t\|_2^2$ and the gap in objective value $|D_\infty(\hat{\mu}^t) - D_\infty(\mu^\star)|$ converge to 0 linearly.

Proof. For the first inequality, by Fact D.1 we know that d is an L_d-smooth function. It is then a standard result (see e.g. Nesterov, 2013, Thm 2.1.5) that we therefore have $\|\nabla d(\xi^t)\|_2^2 \le 2L_d(d(\xi^t) - d(\xi^\star)) = 2L_d\Gamma_t$. But since $\nabla d(\xi^t) = A\bar{\mu}^t$, and using the strong convexity of $\mu \mapsto D_\rho(\mu, \xi)$, we have

$$\frac{1}{2}\|A\hat{\mu}^t\|_2^2 \le \|A\bar{\mu}^t\|_2^2 + \|A\|^2_{\text{op}}\|\bar{\mu}^t - \hat{\mu}^t\|_2^2 \le 2L_d\Gamma_t + 2\frac{\|A\|^2_{\text{op}}}{\sigma}\hat{\Delta}_t.$$

For the second inequality, by definition of D_∞, we have $D_\rho(\mu, \xi) := D_\infty(\mu) + \langle \xi, A\mu\rangle - \frac{1}{2\rho}\|A\mu\|_2^2$, and $D_\infty(\mu^\star) = D(\mu^\star)$.

But then

$$\begin{aligned}|D_\infty(\hat{\mu}^t) - D_\infty(\mu^\star)| &= |D_\infty(\hat{\mu}^t) - D_\rho(\hat{\mu}^t, \xi^t)| + |D_\rho(\hat{\mu}^t, \xi^t) - D_\rho(\bar{\mu}^t, \xi^t)| + |D_\rho(\bar{\mu}^t, \xi^t) - D_\infty(\mu^\star)| \\ &\le |\langle \xi^t, A\hat{\mu}^t\rangle| + \frac{1}{2\rho}\|A\hat{\mu}^t\|_2^2 + \hat{\Delta}_t + \Gamma_t.\end{aligned}$$

but we then have $|\langle \xi^t, A\hat{\mu}^t\rangle| + \frac{1}{2\rho}\|A\hat{\mu}^t\|_2^2 \leq B\|A\bar{\mu}^t\| + B\|A\|_{\text{op}}\|\hat{\mu}^t - \bar{\mu}^t\|_2 + \frac{1}{\rho}(\|A\hat{\mu}^t\|_2^2 + \|A\|_{\text{op}}^2\|\hat{\mu}^t - \bar{\mu}^t\|_2^2)$, which yields the result using the same inequalities as before.

Finally, to show the implications of linear convergence, by Lemma 2.3 of Hong and Luo (2017), there exists $\tau' > 0$ such that $\|\nabla d(\xi)\| \geq \tau'\|\xi - \xi^\star\|$. So that, since $\|\nabla d(\xi^t)\|_2^2 \leq 2L_d\Gamma_t$, we have that if the sequence $(\Gamma_t)_{t\in\mathbb{N}}$ is bounded then so is $(\xi^t)_{t\in\mathbb{N}}$. Letting B be a bound on $\|\xi^t\|$ the previous statements shows the results. □

D.4 Proofs of Corollaries 3 and 4 (Total number of iterations)

Corollary 3. *To ensure that $\mathbb{E}\hat{\Delta}_t \leq \epsilon$ and $\mathbb{E}\hat{\Gamma}_t \leq \epsilon$ it is enough to run the algorithm for a total number of inner iteration $T_{\text{tot}} := T_{\text{in}}T_{\text{ex}}$ such that*

$$T_{\text{tot}} \geq \frac{\log(\beta)}{\log \lambda_{\max}(\beta)\log(1-\pi)}\log(\epsilon)$$

Proof. To guarantee that $(1-\pi)^T_{\text{in}} < \beta$ requires that $T_{\text{in}} \geq \frac{\log(1-\pi)}{\log(\beta)}$ and to guaranteed that $\lambda_{\max}(\beta)^{T_{\text{ex}}} < \epsilon$ requires similarly that $T_{\text{ex}} \geq \frac{\log(\epsilon)}{\log(\lambda(\beta))}$. Taking the product of these inequalities yields the result. □

Corollary 4. *Let $\Delta^\star_{t\,T_{\text{in}}+s} := \Delta_t^s + \Gamma_t$. If $\kappa < \frac{1}{2}$ and $\alpha = \frac{1}{12}$, if $T_{\text{in}} \geq \frac{\log(\alpha\kappa)}{\log(1-\pi)}$, then, there exist a constant $C' > 0$ such that after a total of i clique updates, we have*

$$\mathbb{E}[\Delta_s^\star] \leq C'\Big(1 - \frac{\kappa\pi}{2\log(12/\kappa)}\Big)^s.$$

Proof. Using solving the quadratic formula for the largest eigenvalue of a two-by-two matrix yields

$$\lambda_{\max}(\beta) = (1-\kappa+6\beta) + \sqrt{(1-\kappa-6\beta)^2 + 12\beta}.$$

It is immediate to verify that $\lambda_{\max}(\beta) < 1$ if and only if $\beta < \frac{1}{3}\frac{\kappa}{1+2\kappa}$. This shows that we need to choose $\beta = \alpha\kappa$ with $\alpha < \frac{1}{3(1+2\kappa)}$. So in particular, if $\alpha < \frac{1}{9}$, then the previous inequality is satisfied for any $0 < \kappa < 1$.

Moreover, if $\kappa \leq \frac{1}{2}$ and $\alpha < \frac{1}{6}$, we have $\lambda_{\max}(\beta) = \lambda_{\max}(\alpha\kappa) < 1 - \kappa(1-6\alpha)$. Indeed, letting $x = 3\beta$, and $\alpha' = 3\alpha$, we have

$$\begin{aligned}
2\lambda_{\max}(\beta) &= (1-\kappa+2x) + \sqrt{(1-\kappa-2x)^2 + 4x} \\
&= 1-\kappa+2x + \sqrt{(1-\kappa)^2 + 4x\kappa + 4x^2} \\
&= 1-\kappa+2\alpha'\kappa + \sqrt{(1-\kappa)^2 + 4\alpha'\kappa^2 + 4\alpha'^2\kappa^2} \\
&\leq 1-\kappa+2\alpha'\kappa + \sqrt{(1-\kappa)^2 + 4\alpha'\kappa(1-\kappa) + 4\alpha'^2\kappa^2} \\
&\leq 2\,(1-\kappa+6\alpha\kappa).
\end{aligned}$$

Setting $\alpha = \frac{1}{12}$, given that the rate r is $r = 1 - \exp\big(\frac{\log(1-\pi)\log(\lambda_{\max}(\beta))}{\log(\beta)}\big)$, we have

$$r \geq 1 - (1-\pi)^{\frac{\log(1-\frac{\kappa}{2})}{\log(\frac{\kappa}{12})}} \geq \frac{\log(1-\frac{\kappa}{2})}{\log(\frac{\kappa}{12})}\pi \geq \frac{\kappa}{-2\log(\frac{\kappa}{12})}\pi,$$

where, for the second and the third inequality, we used the fact that $\log(1-z) \geq z$ respectively for $z = \pi$ and for $z = -\frac{\kappa}{2}$.

□

E Details of Algorithm $\mathcal{A}$ and convergence proofs for SDCA

In this section, we specify the detailed form of $D_\rho(\mu, \xi)$, and show how to apply the proof scheme of Shalev-Shwartz and Zhang (2016) to SDCA for the maximization of $D_\rho(\mu, \xi)$ w.r.t. μ in order to prove Proposition 1. We first write a fully decomposed expression of $D_\rho(\mu, \xi)$. We have:

$$D_\rho(\mu, \xi) = \sum_{c \in \mathcal{C}} \langle \ell_c, \mu_c \rangle - f_c^*(\mu_c) - \frac{1}{2\lambda} \sum_{\tau \in \mathcal{T}} \Big\| \sum_{c \in \mathcal{C}_\tau} \Psi_c \mu_c \Big\|^2 - \frac{1}{2\rho} \sum_{\substack{e \in \mathcal{E} \\ i \in e}} \|\mu_i - A_i \mu_e\|^2 + \sum_{\substack{e \in \mathcal{E} \\ i \in e}} \langle \xi_{ei}, \mu_i - A_i \mu_e \rangle, \tag{21}$$

where $-f_c^*(\mu_c) = \gamma h_c(\mu_c) - \iota_{\triangle_c}(\mu_c)$.

We assume here that the entropy surrogate used is such that h_c is σ_c-strongly concave w.r.t. μ_c.

In particular this corresponds to two possible choices:

- The naive Gini entropy, for which $h_c(\mu_c) = (1 - \|\mu_c\|_2^2)$.
- The Gini-OTRW entropy (see Appendix C) for which, given positive numbers $\rho_i, \rho_{i|j}$ and $\rho_{j|i}$ for all nodes and edges, we have
  - $h_i(\mu_i) = \rho_i k_i (1 - \|\mu_i\|_2^2)$ for $i \in \mathcal{V}$
  - $h_{ij}(\mu_{ij}) = h_{i|j}(\mu_{ij}) + h_{j|i}(\mu_{ij})$ for $\{i,j\} \in \mathcal{E}$ with $h_{i|j}(\mu_{ij}) = k_i \rho_{i|j}(\varepsilon \|A_j \mu_{ij}\|_2^2 - k_j \|\mu_{ij}\|_2^2)$

  for $\varepsilon < 1$ which is σ_c-strongly concave in μ_c with $\sigma_i = 2k_i\rho_i$ if $i \in \mathcal{V}$ else $\sigma_{\{i,j\}} = 2(1-\varepsilon)k_i k_j(\rho_{i|j} + \rho_{j|i})$. (For $\varepsilon = 1$, the surrogate is not strongly concave, and a modification of the decomposition into a separable terms and a smooth term must be used to leverage strong convexity: see the discussion in Section 6.2 after Proposition 2).

The proof of convergence for SDCA is based on showing that the expected increase in dual objective provides an upper bound on a measure of duality gap. For the problem, we are considering the gap of interest is $\text{gap}(w, \delta, \mu, \xi) := \tilde{P}_\rho(w, \delta, \xi) - D_\rho(\mu, \xi)$, which is an upper bound on the duality gap $P_\rho(w, \xi) - D_\rho(\mu, \xi)$. It can be decomposed as follows:

$$\begin{aligned} \text{gap}(w, \delta, \mu, \xi) = \quad & \Big[\gamma F_{\mathcal{I}}(\ell + \Psi^\intercal w + A^\intercal \delta) + \gamma F_{\mathcal{I}}^*(\mu) - \langle \ell + \Psi^\intercal w + A^\intercal \delta, \mu \rangle\Big] \\ & + \Big[\frac{\lambda}{2}\|w\|^2 + \frac{1}{2\lambda}\|\Psi\mu\|^2 - \langle -w, \Psi\mu\rangle\Big] + \Big[\frac{\rho}{2}\|\xi - \delta\|^2 + \frac{1}{2\rho}\|A\mu\|^2 - \langle \xi - \delta, A\mu\rangle\Big] \qquad (22) \\ = \quad & \Big[\sum_{c \in \mathcal{C}} f_c^*\big(\tfrac{1}{\gamma}\tilde{\theta}_c(w, \delta)\big) + f_c^*(\mu_c) - \langle \tilde{\theta}_c(w, \delta), \mu_c \rangle\Big] \\ & + \Big[\sum_{\tau \in \mathcal{T}} \frac{\lambda}{2}\|w_\tau\|^2 + \frac{1}{2\lambda}\Big\|\sum_{c \in \mathcal{C}_\tau} \Psi_c \mu_c\Big\|^2 - \Big\langle -w_\tau, \sum_{c \in \mathcal{C}_\tau} \Psi_c \mu_c \Big\rangle\Big] \\ & + \Big[\sum_{e \in \mathcal{E}} \sum_{i \in e} \frac{\rho}{2}\|\xi_{ei} - \delta_{ei}\|^2 + \frac{1}{2\rho}\|\mu_i - A_i \mu_i\|^2 - \langle \xi_{ei} - \delta_{ei}, \mu_i - A_i \mu_i \rangle\Big], \qquad (23) \end{aligned}$$

where $\tilde{\theta}_c$ is defined by

$$\tilde{\theta}_c(w, \delta) := \begin{cases} \ell_i + \Psi_i^\intercal w_{\tau_i} + \sum_{e \ni s} \delta_{ei} & \text{for } c = i \in \mathcal{V}, \\ \ell_e + \Psi_e^\intercal w_{\tau_e} - \sum_{i \in e} A_i^\intercal \delta_{ei} & \text{for } c = e \in \mathcal{E}. \end{cases} \tag{24}$$

We now proceed to characterize the progress of the algorithm at each iteration, and to that end, we introduce appropriate notations. In particular, since ξ is fixed during the algorithm, we drop the dependance on ξ in different functions: Denote the objective of the subproblem w.r.t. clique c as

$$D_{\rho,c}(\mu_c, \mu_{-c}^s) := -f_c^*(\mu_c) - r(\mu_c, \mu_{-c}^s), \tag{25}$$

with r defined by

$$r(\mu_c, \mu_{-c}^s) := \frac{1}{2\lambda}\Big\| \sum_{b \in \mathcal{C}_{\tau_c} \setminus \{c\}} \Psi_b \mu_b^s + \Psi_c \mu_c \Big\|^2 + \begin{cases} \sum_{e \ni c} \frac{1}{2\rho} \|\mu_i - A_i \mu_e^s\|^2 - \langle \mu_i, \sum_{e \ni i} \xi_{ei} + \ell_i \rangle, & c = i \in \mathcal{V}, \\ \sum_{i \in e} \frac{1}{2\rho} \|\mu_i^s - A_i \mu_e\|^2 - \langle \mu_e, -\sum_{i \in e} A_i^\intercal \xi_{ei} + \ell_e \rangle, & c = e \in \mathcal{E}. \end{cases}$$

It is straightforward to show that r is convex and smooth with cliquewise smoothness constants

$$L_i = \frac{1}{\lambda} \mathrm{eig}_{\max}(\Psi_i^\intercal \Psi_i) + \frac{|\{e : e \ni i\}|}{\rho}, \ i \in \mathcal{V} \quad \text{and} \quad L_e = \frac{1}{\lambda} \mathrm{eig}_{\max}(\Psi_e^\intercal \Psi_e) + \frac{1}{\rho} \sum_{i \in e} \mathrm{eig}_{\max}(A_i^\intercal A_i), \ e \in \mathcal{E}.$$

The proof of convergence hinges on the following key lemma.

Lemma E.1. *Taking one of the following updates on μ_c with μ_{-c} fixed:*

- $\mu_c^{s+1} = \arg\max_{\mu_c} D_{\rho,c}(\mu_c, \mu_{-c}^s)$.
- *or, if $u \in \partial f_c(\tilde{\theta}_c(w^s, \delta^s))$, where f_c is the conjugate function of f_c^*.*

$$\textit{solve} \qquad \hat{\alpha} = \arg\max_{\alpha \in [0,1]} D_{\rho,c}(\mu_c^s + \alpha(u - \mu_c^s); \mu^s), \qquad \textit{and set} \qquad \mu_c^{s+1} = \mu_c^s + \hat{\alpha}(u - \mu_c^s).$$

Then, with $\pi = \min_{c \in \mathcal{C}} \frac{\sigma_c}{|\mathcal{C}|(\sigma_c + L_c)}$, the following inequality holds

$$\mathbb{E}_c[D_\rho(\mu^{s+1}, \xi) - D_\rho(\mu^s, \xi)] \geq \pi \, \mathbb{E}_c[\tilde{P}_\rho(w^s, \delta^s, \xi) - D_\rho(\mu^s, \xi)], \quad \forall \xi,$$

where w^s, δ^s are updated to maintain the optimality conditions:

$$w^s = -\frac{1}{\lambda} \Psi \mu^s, \quad \delta^s = \xi - \frac{1}{\rho} A \mu^s.$$

Proof. Letting $\breve{D}_{\rho,c}$ be defined as,

$$\breve{D}_{\rho,c}(\mu_c; \mu^s) := -f_c^*(\mu_c) - r(\mu^s) - \langle \nabla_{\mu_c} r(\mu^s), \mu_c - \mu_c^s \rangle - \frac{L_c}{2} \|\mu_c - \mu_c^s\|^2,$$

we have $\breve{D}_{\rho,c}(\mu_c; \mu^s) \leq D_{\rho,c}(\mu_c)$, since $\mu_c \mapsto r(\mu_c, \mu_{-c}^s)$ is L_c-smooth.

First, for the update $\mu_c^{s+1} = \arg\max_{\mu_c} D_{\rho,c}(\mu_c, \mu_{-c}^s)$, we have that, for any direction $u - \mu_c^s$ and any step size $\alpha \in [0, 1]$

$$\begin{aligned} D_\rho(\mu^{s+1}, \xi) - D_\rho(\mu^s, \xi) &= D_{\rho,c}(\mu_c^{s+1}, \mu_{-c}^s) - D_{\rho,c}(\mu_c^s, \mu_{-c}^s) \\ &\geq D_{\rho,c}(\mu_c^s + \alpha(u - \mu_c^s), \mu_{-c}^s) - D_{\rho,c}(\mu_c^s, \mu_{-c}^s) \\ &\geq \breve{D}_{\rho,c}(\mu_c^s + \alpha(u - \mu_c^s); \mu^s) - D_{\rho,c}(\mu_c^s, \mu_{-c}^s). \end{aligned} \tag{26}$$

Showing the desired inequality for the second form of update thus implies the inequality for the first type of update. Expliciting $\breve{D}_{\rho,c}(\mu_c^s + \alpha(u - \mu_c^s); \mu^s)$, we have

$$\begin{aligned} \breve{D}_{\rho,c}(\mu_c^s + \alpha(u - \mu_c^s); \mu^s) = &- f_c^*(\mu_c^s + \alpha(u - \mu_c^s)) \\ &- r(\mu^s) - \langle \nabla_{\mu_c} r(\mu^s), \alpha(u - \mu_c^s) \rangle - \frac{\alpha^2 L_c}{2} \|u - \mu_c^s\|^2. \end{aligned} \tag{27}$$

Since $f_c^*(u)$ assumed σ_c-strongly convex, we have

$$f_c^*(\mu_c^s + \alpha(u - \mu_c^s)) \leq \alpha f_c^*(u) + (1 - \alpha) f_c^*(\mu_c^s) - \frac{\sigma_c}{2} \alpha(1 - \alpha) \|u - \mu_c^s\|_2^2. \tag{28}$$

Combining (27) and (28), we obtain

$$\breve{D}_{\rho,c}(\mu_c^s + \alpha(u - \mu_c^s); \mu^s) \geq -\alpha\Big(f_c^*(u) - f_c^*(\mu_c^s) + \langle \nabla_{\mu_c} r(\mu^s), u - \mu_c^s\rangle\Big)$$
$$- f_c^*(\mu_c^s) - r(\mu^s) + \Big(\frac{\sigma_c}{2}\alpha(1-\alpha) - \frac{\alpha^2 L_c}{2}\Big)\|u - \mu_c^s\|^2. \qquad (29)$$

Now, if we choose $u \in \partial f_c\big(-\nabla_{\mu_c} r(\mu^s)\big)$, by Fenchel conjugacy, it follows that

$$f_c\big(-\nabla_{\mu_c} r(\mu^s)\big) = -f_c^*(u) - \langle \nabla_{\mu_c} r(\mu^s), u\rangle.$$

One can easily see that $\tilde{\theta}_c(w^s, \delta^s) = -\nabla_{\mu_c} r(\mu^s)$ by maintaining the optimality conditions

$$\forall c \in \mathcal{C}\colon \quad w_{\tau_c}^s = -\frac{1}{\lambda}\sum_{b \in \mathcal{C}_\tau} \Psi_b \mu_b^s, \qquad \forall e \in \mathcal{E}, i \in e\colon \quad \delta_{ei}^s = \xi_{ei} - \frac{1}{\rho}(\mu_i^s - A_i \mu_e^s).$$

Thus, we can further simplify (29) as

$$\breve{D}_{\rho,c}(\mu_c^s + \alpha(u - \mu_c^s); \mu^s) - D_{\rho,c}(\mu_c^s, \mu_{-c}^s) \geq \alpha\Big(f_c(\tilde{\theta}_c(w^s, \delta^s)) + f_c^*(\mu_c^s) - \langle \tilde{\theta}_c(w^s, \delta^s), \mu_c^s\rangle\Big), \qquad (30)$$

provided that $\frac{\sigma_c}{2}\alpha(1-\alpha) - \frac{\alpha^2 L_c}{2} \geq 0$, that is, $0 \leq \alpha \leq \frac{\sigma_c}{\sigma_c + L_c}$.

The key observation is that

$$\mathrm{gap}(w, \delta, \mu, \xi) = \sum_{c \in \mathcal{C}} f_c(\tilde{\theta}_c(w, \delta)) + f_c^*(\mu_c) - \langle \tilde{\theta}_c(w, \delta), \mu_c\rangle \qquad (31)$$

if we maintain the optimality conditions. By using (31) and taking expectation $\mathbb{E}_c$ w.r.t. a uniform random choice of the clique c on both sides of (30), we guarantee that, for $\alpha \in [0, \frac{\sigma_c}{\sigma_c + L_c}]$,

$$\mathbb{E}_c[D_\rho(\mu^{s+1}, \xi) - D_\rho(\mu^s, \xi)] \geq \mathbb{E}_c\Big[\frac{\alpha}{|\mathcal{C}|}\mathrm{gap}(w^s, \delta^s, \mu^s, \xi)\Big].$$

So, we can choose the maximum value $\frac{\sigma_c}{\sigma_c + L_c}$ for α. It follows that

$$\mathbb{E}_c[D_\rho(\mu^{s+1}, \xi) - D_\rho(\mu^s, \xi)] \geq \Big(\min_{c \in \mathcal{C}} \frac{\sigma_c}{|\mathcal{C}|(\sigma_c + L_c)}\Big)\, \mathbb{E}_c[\,\mathrm{gap}(w^s, \delta^s, \mu^s, \xi)].$$

□

We can now prove Proposition 1.

Proposition 1. *If $\mathcal{A}$ is SDCA, let $|\mathcal{C}|$ be the total number of cliques, σ_c the strong convexity constant of f_c^*, and L_c the Lipschitz constant of $\mu_c \mapsto r(\mu)$, then $\mathcal{A}$ is linearly convergent with rate $\pi = \min_{c \in \mathcal{C}} \frac{\sigma_c}{|\mathcal{C}|(\sigma_c + L_c)}$.*

Proof. Denote $\Delta_t^s := D_\rho(\bar{\mu}^t, \xi^t) - D_\rho(\mu^{t,s}, \xi^t)$. Since we update $\mu^{t,s}$ to $\mu^{t,s+1}$ using SDCA, according to Lemma E.1, we have

$$\begin{aligned}
\mathbb{E}_c[\Delta_t^s - \Delta_t^{s+1}] &= \mathbb{E}_c[D_\rho(\mu^{t,s+1}, \xi^t) - D_\rho(\mu^{t,s}, \xi^t)] \\
&\geq \pi\, \mathbb{E}_c[\tilde{P}_\rho(w(\mu^{t,s}), \delta(\mu^{t,s}, \xi^t), \xi^t) - D_\rho(\mu^{t,s}, \xi^t)] \\
&\geq \pi\, \mathbb{E}_c[D_\rho(\bar{\mu}^t, \xi^t) - D_\rho(\mu^{t,s}, \xi^t)] = \pi \mathbb{E}_c[\Delta_t^s],
\end{aligned}$$

and $\pi = \min_{c \in \mathcal{C}} \frac{\sigma_c}{|\mathcal{C}|(\sigma_c + L_c)}$. The above inequality implies that

$$\mathbb{E}_c[\Delta_t^{s+1}] \leq (1-\pi)\, \mathbb{E}_c[\Delta_t^s] \leq (1-\pi)^{s+1}\, \mathbb{E}_c[\Delta_t^0].$$

The result follows if we set $T_{\mathrm{in}} = s + 1$.

□

E.1 Proof of Propositions 2 and 3 (Linear convergence in the primal)

Proposition 2. *Let $\hat{w}^t = w(\hat{\mu}^t)$. If $\mathcal{A}$ is SDCA, then*

$$\mathbb{E}[P(\hat{w}^t) - P(w^\star)] \leq \mathbb{E}[\frac{1}{\pi}\hat{\Delta}_t + \Gamma_t].$$

Proof. Recall that $P(w^\star) = D(\mu^\star) = D_\rho(\mu^\star, \xi^\star)$ by Corollary 5.

$$\begin{aligned}
P(w^{t,s}) - P(w^\star) &= P(w^{t,s}) - D_\rho(\bar{\mu}^t, \xi^t) + D_\rho(\bar{\mu}^t, \xi^t) - P(w^\star) \\
&= P(w^{t,s}) - D_\rho(\bar{\mu}^t, \xi^t) + D_\rho(\bar{\mu}^t, \xi^t) - D_\rho(\mu^\star, \xi^\star) \\
&\leq \tilde{P}(w^{t,s}, \delta^{t,s}, \xi^t) - D_\rho(\bar{\mu}^t, \xi^t) + d(\xi^t) - d(\xi^\star) \\
&\leq \tilde{P}(w^{t,s}, \delta^{t,s}, \xi^t) - D_\rho(\mu^{t,s}, \xi^t) + d(\xi^t) - d(\xi^\star) \\
&= \mathrm{gap}(w^{t,s}, \delta^{t,s}, \mu^{t,s}, \xi^t) + \Gamma_t
\end{aligned}$$

If $\mathcal{A}$ is SDCA, by Lemma E.1, we have

$$\begin{aligned}
\mathbb{E}[P(w^{t,s}) - P(w^\star)] &= \mathbb{E}[\,\mathrm{gap}(w^{t,s}, \delta^{t,s}, \mu^{t,s}, \xi^t) + \Gamma_t] \\
&\leq \mathbb{E}\Big[\frac{1}{\pi}(\Delta_t^s - \Delta_t^{s+1}) + \Gamma_t\Big] \\
&\leq \frac{1}{\pi}\mathbb{E}[\Delta_t^s] + \mathbb{E}[\Gamma_t].
\end{aligned}$$

Given that $\hat{\Delta}_t = \Delta_t^{T_{\mathrm{in}}}$, the result follows by setting $s = T_{\mathrm{in}}$. □

Proposition 3. *Let $w^{t,s} = w(\mu^{t,s})$. If $\mathcal{A}$ is a linearly convergent algorithm and $\mu \mapsto -H_{\mathrm{approx}} + \frac{1}{2\rho}\|A\mu\|_2^2$ is strongly convex then $P(w^{t,s}) - P(w^\star)$ converges to 0 linearly.*

Proof. Note that, if σ is the strong convexity constant of D_ρ w.r.t. μ, then given that $P_\rho(w, \xi) = \min_\delta \tilde{P}_\rho(w, \delta, \xi)$ with

$$\tilde{P}_\rho(w, \delta, \xi) = \gamma F_{\mathcal{I}}\Big(\frac{\theta(w) + A^\intercal\delta}{\gamma}\Big) + \frac{\rho}{2}\|\delta - \xi\|^2 + \frac{\lambda}{2}\|w\|_2^2,$$

we also have

$$P_\rho(w, \xi) = \max_\mu \Big[\,\langle \mu, \Psi^\intercal w\rangle + \gamma H_{\mathrm{approx}}(\mu) + \langle \xi, A\mu\rangle - \frac{1}{2\rho}\|A\mu\|_2^2\Big] + \frac{\lambda}{2}\|w\|_2^2,$$

which shows that $w \mapsto P_\rho(w, \xi)$ is a function with Lipschitz gradient as the sum of $w \mapsto \frac{\lambda}{2}\|w\|_2^2$ and of the Fenchel conjugate of a strongly convex function. Let L_P be its Lipschitz smoothness constant and note that it does not depend on the value of ξ. We thus have

$$P_\rho(w^{t,s}, \xi^t) - P_\rho(\bar{w}^t, \xi^t) \leq L_P\|w^{t,s} - \bar{w}^t\|_2^2.$$

Then given the representer theorem, and by strong convexity of $\mu \mapsto D_\rho(\mu, \xi)$ we have

$$\|w^{t,s} - \bar{w}^t\|_2^2 = \|\Psi(\mu^{t,s} - \bar{\mu}^t)\|_2^2 \leq \frac{1}{\sigma}\|\Psi\|_{\mathrm{op}}^2(D_\rho(\mu^{t,s}, \xi^t) - D_\rho(\bar{\mu}^t, \xi^t))$$

So that, since $P(w^{t,s}) \leq P_\rho(w^{t,s}, \xi^t)$ and $P(w^\star) = P_\rho(w^\star, \xi^\star)$, we have

$$P(w^{t,s}) - P(w^\star) \leq P_\rho(w^{t,s}, \xi^t) - P_\rho(\bar{w}^t, \xi^t) + P_\rho(\bar{w}^t, \xi^t) - P_\rho(w^\star, \xi^\star) \leq \frac{L_P}{\sigma}\|\Psi\|_{\mathrm{op}}^2\,\Delta_t^s + \Gamma_t.$$

Finally, global linear convergence in the primal also follows from the linear convergence of $\hat{\Delta}_t$ and Γ_t. □

F Notation summary

Given the number of notations in the main paper, we summarize some of them in Tables 2,3 and 4. The block matrices Ψ and A are schematically drawn below to illustrate their structure.

$$\Psi = \begin{array}{cc} & \begin{array}{cc} & c \end{array} \\ \tau_c & \begin{bmatrix} & \vdots \\ \dots & \Psi_c \\ & \end{bmatrix} \end{array} \qquad A = \begin{array}{cc} & \begin{array}{ccc} & i & ij \end{array} \\ ij & \begin{bmatrix} & \vdots & \vdots \\ \dots & I_{k_i} & -A_i \\ & & \end{bmatrix} \end{array}$$

Table 2: Notations for sets

Notation	Dimension	Description
$\mathcal{C}$		the set of cliques
$\mathcal{E}$		the set of edges
$\mathcal{V}$		the set of nodes
$\mathcal{Y}_i = \mathcal{S}_k$		$\mathcal{S}_k := \{u \in \{0,1\}^k \mid \|u\|_1 = 1\}$
$\mathcal{Y}_c$	$\prod_{i \in c} k_i$	$\mathcal{Y}_c := \bigtimes_{i \in c} \mathcal{Y}_i$
$\mathcal{Y}$	$\prod_{i \in V} k_i$	$\mathcal{Y} := \bigtimes_{i \in V} \mathcal{Y}_i$
$\mathcal{T}$		the set of clique types
$\mathcal{C}_\tau$		the set of cliques of type τ
$\mathcal{M}$		the marginal polytope
$\mathcal{L}$		the local polytope
$\mathcal{I}$		$\mathcal{I} := \prod_{i \in V} \Delta_{k_i} \times \prod_{e \in E} \Delta_{k_e}$

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Table 3: Notations for variables, parameters and functions

Notation	Domain	Description
k_i	$\mathbb{N}$	the number of values that Y_i can take
k_c	$\mathbb{N}$	$k_c := \lvert\mathcal{Y}_c\rvert = \prod_{i\in c} k_i$
τ	$\mathbb{N}$	the type of a clique
τ_c	$\mathbb{N}$	the type of clique c
w_τ	$\mathbb{R}^{d_\tau}$	the parameter shared by all cliques with type τ
w	$\mathbb{R}^{\sum_\tau d_\tau}$	$w := [w_\tau]_{\tau\in\mathcal{T}}$
$\phi_c(x, y_c)$	$\mathbb{R}^{d_{\tau_c}}$	the feature vector associated with the clique c given $Y_c = y_c$
$Z(x, w)$	$\mathbb{R}_{+*}$	the partition function of $p(y\vert x; w)$
$\ell_c(y_c^{(n)}, y_c)$	$\mathbb{R}_+$	the user defined loss function associated with the clique c
γ	$(0, +\infty)$	the temperature parameter of the loss-augmented CRF
$\Psi_c^{(n)}$	$\mathbb{R}^{d_{\tau_c}\times k_c}$	$\Psi_c^{(n)} := \left[\phi_c(x^{(n)}, y_c) - \phi_c(x^{(n)}, y_c^{(n)})\right]_{y_c\in\mathcal{Y}_c}$
$\Psi^{(n)}$	$\mathbb{R}^{\sum_\tau d_\tau \times \sum_c k_c}$	see the drawing
$\ell_c^{(n)}$	$\mathbb{R}^{k_c}$	$\ell_c^{(n)} := \left[\ell_c(y_c^{(n)}, y_c)\right]_{y_c\in\mathcal{Y}_c}$
$\ell^{(n)}$	$\mathbb{R}^{\sum_c k_c}$	$\ell^{(n)} := [\ell_c^{(n)}]_{c\in C}$
$\theta_c^{(n)}(w)$	$\mathbb{R}^{k_c}$	$\theta_c^{(n)}(w) := \Psi_c^{(n)\intercal} w_{\tau_c} + \ell_c^{(n)}$
$\theta^{(n)}(w)$	$\mathbb{R}^{\sum_c k_c}$	$\theta^{(n)}(w) := [\theta_c^{(n)}(w)]_{c\in\mathcal{C}}$, the natural parameter
F	$\mathbb{R}^{\sum_c k_c} \to \mathbb{R}$	the log partition function of θ
$T(y)$	$\mathbb{R}^{\sum_c k_c}$	the sufficient statistics
μ_c	$\mathbb{R}^{k_c}$	the mean parameter associated with the clique c
μ	$\mathbb{R}^{\sum_c k_c}$	the mean parameter
F^*	$\mathbb{R}^{\sum_c k_c} \to \mathbb{R}$	the Fenchel conjugate of F
ι_C	$\mathbb{R}^{\sum_c k_c} \to \{0, +\infty\}$	the indicator function of set C
λ	$\mathbb{R}_+$	the coefficient of the regularizer
A_i	$\mathbb{R}^{k_i\times k_e}$	the matrix encoding the marginalization constraint for i in e.
A	$\mathbb{R}^{\sum_e \sum_{i\in e} k_i \times \sum_c k_c}$	see the matrix form

Table 4: Notations smoothness, strong convexity constant and related quantities

Notation	Description
L_d	the Lipschitz constant of $\nabla d(\xi)$
τ	the constant of PL inequality for $d(\xi)$
σ_c	the strong convexity constant of $\mu_c \mapsto -H_{\text{Approx}}(\mu)$
L_c	the Lipschitz constant of $\mu_c \mapsto \frac{1}{\lambda}\Psi^\intercal w(\mu) + \frac{1}{\rho}A^\intercal \delta(\mu, \xi^t)$

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