Limit Analysis of Plates

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Synonyms

yield design, ultimate state, collapse load, bearing capacity

Definitions

Limit analysis (or yield design theory) enables to determine the ultimate load that a structure can sustain under the conditions of static equilibrium and fulfillment of a yield (or strength) criterion. It relies on the use of two bounding theorems (resp. static and kinematic approach) which enable to bracket the ultimate load by a lower (resp. upper) bound. In the case of plates in bending, both equilibrium and yield conditions can be formulated in terms of generalized stresses, e.g. bending moments for the case of thin plates.

Introduction

To simplify the presentation, it will be considered that the plate loading depends upon one loading parameter only, which will be taken as a distributed out-of-plane loading $\mathbf{f} = -p(x_\alpha)\mathbf{e}_3$. The generalization to multiple loading parameters is straightforward and more details can be found in Salençon (1983, 2013).

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Introducing a reference load $p_0(x_\alpha)$ and a scalar load multiplier λ such that $p(x_\alpha) = \lambda p_0(x_\alpha)$, limit analysis aims at finding the maximum value of the load multiplier λ^+ corresponding to the limit load $p^+(x_\alpha) = \lambda^+ p_0(x_\alpha)$. The existence of two bounding theorems in limit analysis enables to bracket λ^+ from below using a static lower bound approach and from above using a kinematic upper bound approach.

Lower Bound Static Approach

The static approach simply states that λ^+ is the maximum value of λ so that there exists a distribution of statically admissible fields in equilibrium with $\lambda p_0(x_\alpha)$ which satisfy the strength criterion $G(x_\alpha)$ everywhere:

$$\lambda^+ = \sup_{\lambda} \left\{ \lambda \text{ s.t. } \exists (M_{\alpha\beta}, Q_\alpha) \in SA(\lambda p_0) \text{ and } (M_{\alpha\beta}, Q_\alpha) \in G(x_\alpha) \quad \forall x_\alpha \in \Omega \right\} \tag{1}$$

where the set of statically admissible bending moments and shear forces is defined as:

$$(M_{\alpha\beta}, Q_\alpha) \in SA(\lambda p_0) \Leftrightarrow \begin{cases} M_{\alpha\beta,\beta} - Q_\alpha = 0 & \text{in } \Omega \\ Q_{\alpha,\alpha} - \lambda p_0 = 0 & \text{in } \Omega \\ [\![M_{\alpha\beta}]\!] n_\beta = [\![Q_\alpha]\!] n_\alpha = 0 & \text{through any } \Gamma \\ M_{\alpha\beta} n_\beta = 0 & \text{on } \partial\Omega_{M_\alpha} \\ Q_\alpha n_\beta = 0 & \text{on } \partial\Omega_Q \end{cases} \tag{2}$$

where the third condition represents the field continuity through any potential discontinuity line Γ of unit normal vector n and the last two conditions represent boundary conditions on some part of the plate boundary.

$G(x_\alpha)$ represents the strength criterion in the generalized bending moment-shear force space and is a convex set in this five-dimensional space. In general, it is also bounded and contains the origin.

By considering any distribution of bending moments and shear forces $(M_{\alpha\beta}, Q_\alpha)$ in equilibrium with a given value λ_l of the load multiplier and satisfying the strength criterion everywhere, the static approach of limit analysis ensures that λ_l is a lower bound estimation to the true collapse load λ^+:

$$\lambda_l \leq \lambda^+ \tag{3}$$

This approach can be implemented either analytically by the construction of appropriate generalized forces fields depending on a few optimization parameters

or numerically from finite-element generated fields and convex optimization algorithms.

The Case Of Thin Plates

Since only plates in bending are considered here, the normal forces have not been written which implicitly assumes that the plate is infinitely resisting with respect to these normal forces. This assumption is in general valid for out-of-plane loaded homogeneous plates.

Besides, the limit analysis of plates is generally restricted to the case of thin plates. In this context, the thin-plate hypothesis amounts to assuming that the plate is infinitely resisting with respect to shear forces so that the strength criterion is written solely on the bending moments:

$$(M_{\alpha\beta}, Q_\alpha) \in G \Leftrightarrow M_{\alpha\beta} \in \widetilde{G} \tag{4}$$

where $\widetilde{G}$ is a convex set in the three-dimensional space of bending moments. Indeed, the bending moment strength can be estimated as $|M_{\alpha\beta}| \sim \sigma_0 h^2$ whereas the shear strength behaves as $|Q_\alpha| \sim \sigma_0 h$, where σ_0 is the material strength. For a plate of in-plane typical length L, the ratio between the bending moments and shear forces is $|M_{\alpha\beta}|/|Q_\alpha L| \sim h/L \ll 1$ so that the plate shear strength can be assumed to be much higher than its bending strength.

Since Q_α and, hence, the bending moment divergence can now become infinite due to unlimited strength, the appropriate functional space for $M_{\alpha\beta}$ becomes larger and the conditions for static admissibility (2) should become those of the Love-Kirchhoff model (see Chapter *Direct derivation of plate theories*).

Upper Bound Kinematic Approach

The upper bound kinematic approach is the dual characterization of the static approach (1) by means of the virtual work principle. Indeed, considering any kinematically admissible (K.A.) virtual motion consisting of an out-of-plane velocity $\widehat{U}_3$ and material rotation $\widehat{\phi}_\alpha$, the principle of virtual work states that:

$$(M_{\alpha\beta}, Q_\alpha) \in SA(\lambda p_0) \Leftrightarrow W_{int}(\widehat{U}_3, \widehat{\phi}_\alpha) + W_{ext}(\widehat{U}_3, \widehat{\phi}_\alpha) = 0 \quad \forall (\widehat{U}_3, \widehat{\phi}_\alpha) \text{ K.A.} \tag{5}$$

where the virtual work of internal and external forces are given by:

$$W_{int}(\widehat{U}_3,\widehat{\phi}_\alpha) = -\int_\Omega \left(M_{\alpha\beta}\widehat{\chi}_{\alpha\beta} + Q_\alpha(\widehat{U}_{3,\alpha}+\widehat{\phi}_\alpha)\right)\mathrm{d}\Omega \tag{6}$$

$$W_{ext}(\widehat{U}_3,\widehat{\phi}_\alpha) = \lambda\int_\Omega(-\widehat{U}_3)p_0\mathrm{d}\Omega = \lambda W^0_{ext}(\widehat{U}_3,\widehat{\phi}_\alpha) \tag{7}$$

where $\widehat{\chi}_{\alpha\beta} = (\widehat{\phi}_{\beta,\alpha}+\widehat{\phi}_{\alpha,\beta})/2$. In the case of discontinuous virtual velocity fields through a line Γ the work of internal forces has to be completed by the following additional line integral:

$$W^{[\![\cdot]\!]}_{int}(\widehat{U}_3,\widehat{\phi}_\alpha) = -\int_\Gamma \left((M_{\alpha\beta}n_\beta)[\![\widehat{\phi}_\alpha]\!] + Q_\alpha n_\alpha [\![\widehat{U}_3]\!]\right)\mathrm{d}l \tag{8}$$

Hence, using the virtual work principle:

$$\begin{aligned}&(M_{\alpha\beta},Q_\alpha)\in SA(\lambda p_0) \text{ and } (M_{\alpha\beta},Q_\alpha)\in G(x_\alpha) \Leftrightarrow\\ &W_{ext}(\widehat{U}_3,\widehat{\phi}_\alpha) = -W_{int}(\widehat{U}_3,\widehat{\phi}_\alpha)\leq W_{rm}(\widehat{U}_3,\widehat{\phi}_\alpha)\quad \forall(\widehat{U}_3,\widehat{\phi}_\alpha)\text{ K.A.}\end{aligned} \tag{9}$$

where the maximum resisting work has been introduced as:

$$W_{rm}(\widehat{U}_3,\widehat{\phi}_\alpha) = \int_\Omega \pi(\widehat{\chi}_{\alpha\beta},\widehat{U}_{3,\alpha}+\widehat{\phi}_\alpha)\mathrm{d}\Omega + \int_\Gamma \Pi(n_\alpha;[\![\widehat{\phi}_\alpha]\!],[\![\widehat{U}_3]\!])\mathrm{d}l \tag{10}$$

with the following support functions (plastic dissipation in a perfectly plastic framework):

$$\pi(\widehat{\chi}_{\alpha\beta},\widehat{U}_{3,\alpha}+\widehat{\phi}_\alpha) = \sup_{(M_{\alpha\beta},Q_\alpha)\in G}\left\{M_{\alpha\beta}\widehat{\chi}_{\alpha\beta}+Q_\alpha(\widehat{U}_{3,\alpha}+\widehat{\phi}_\alpha)\right\} \tag{11}$$

$$\Pi(n_\alpha;[\![\widehat{\phi}_\alpha]\!],[\![\widehat{U}_3]\!]) = \sup_{(M_{\alpha\beta},Q_\alpha)\in G}\left\{(M_{\alpha\beta}n_\beta)[\![\widehat{\phi}_\alpha]\!]+Q_\alpha n_\alpha[\![\widehat{U}_3]\!]\right\} \tag{12}$$

Comparing (9) with (1) yields the following necessary condition for a loading factor to remain less or equal to its ultimate value of λ^+:

$$\lambda\leq\lambda^+ \Leftrightarrow W_{ext}(\widehat{U}_3,\widehat{\phi}_\alpha)\leq W_{rm}(\widehat{U}_3,\widehat{\phi}_\alpha)\quad\forall(\widehat{U}_3,\widehat{\phi}_\alpha)\text{ K.A.} \tag{13}$$

An equivalent formulation in the form of a minimum principle also reads as:

$$\begin{aligned}\lambda^+ = \inf_{(\widehat{U}_3,\widehat{\phi}_\alpha)\text{ K.A.}}\ & W_{rm}(\widehat{U}_3,\widehat{\phi}_\alpha)\\ \text{s.t. } & W^0_{ext}(\widehat{U}_3,\widehat{\phi}_\alpha)=1\end{aligned} \tag{14}$$

Similarly to the lower bound static approach, by considering any distribution of kinematically admissible virtual fields $(\widehat{U}_3,\widehat{\phi}_\alpha)$, the following upper bound estimate to the true collapse load is obtained:

$$\lambda^+\leq\lambda^u \tag{15}$$

where $\lambda^u = W_{rm}(\widehat{U}_3, \widehat{\phi}_\alpha)/W_{ext}(\widehat{U}_3, \widehat{\phi}_\alpha)$. In particular, the use of discontinuous velocity fields is an efficient way of obtaining simple upper bound estimates which corresponds to the yield line method (Johansen 1962) in the case of thin plates.

The Case Of Thin Plates

An infinite resistance with respect to shear forces translates into the fact that the support functions (12) are given by:

$$\pi(\widehat{\chi}_{\alpha\beta}, \widehat{U}_{3,\alpha} + \widehat{\phi}_\alpha) = \begin{cases} \sup\limits_{M_{\alpha\beta}\in\widetilde{G}} \left\{ M_{\alpha\beta}\widehat{\chi}_{\alpha\beta} \right\} & \text{if } \widehat{U}_{3,\alpha} + \widehat{\phi}_\alpha = 0 \\ +\infty & \text{otherwise} \end{cases} \tag{16}$$

$$\Pi(n_\alpha; [\![\widehat{\phi}_\alpha]\!], [\![\widehat{U}_3]\!]) = \begin{cases} \sup\limits_{M_{\alpha\beta}\in\widetilde{G}} \left\{ (M_{\alpha\beta}n_\beta)[\![\widehat{\phi}_\alpha]\!] \right\} & \text{if } [\![\widehat{U}_3]\!] = 0 \\ +\infty & \text{otherwise} \end{cases} \tag{17}$$

meaning that the maximum resisting work takes a finite value only if the Love-Kirchhoff kinematics $\widehat{U}_{3,\alpha} + \widehat{\phi}_\alpha = 0$ and the continuity of the virtual out-of-plane velocity field $[\![\widehat{U}_3]\!] = 0$ are verified. These conditions are hence interpreted as a relevance condition associated with the infinite shear strength hypothesis. In this case, the virtual strain curvature rate is equal to $\widehat{\chi}_{\alpha\beta} = -\widehat{U}_{3,\alpha\beta} = \widehat{K}_{\alpha\beta}$ and the material rotation jump $[\![\widehat{\phi}_\alpha]\!]$ can only have a non-zero jump along the normal component through the line Γ because of Hadamard's lemma:

$$[\![\widehat{\phi}_\alpha]\!] = -[\![\widehat{U}_{3,\alpha}]\!] = -[\![\widehat{U}_{3,\beta}n_\beta]\!]n_\alpha = -[\![\partial_n\widehat{U}_3]\!]n_\alpha \tag{18}$$

Assuming that these two conditions are satisfied, the upper bound kinematic approach for thin plates reads as:

$$\lambda \leq \lambda^+ \Leftrightarrow W_{ext}(\widehat{U}_3) \leq W_{rm}(\widehat{U}_3) \quad \forall \widehat{U}_3 \text{ K.A.} \tag{19}$$

with:

$$W_{rm}(\widehat{U}_3) = \int_\Omega \widetilde{\pi}(-\widehat{U}_{3,\alpha\beta})\mathrm{d}\Omega + \int_\Gamma \widetilde{\Pi}(n_\alpha; -[\![\partial_n\widehat{U}_3]\!])\mathrm{d}l \tag{20}$$

$$\widetilde{\pi}(\widehat{\chi}_{\alpha\beta}) = \sup_{M_{\alpha\beta}\in\widetilde{G}} \left\{ M_{\alpha\beta}\widehat{\chi}_{\alpha\beta} \right\} \tag{21}$$

$$\widetilde{\Pi}(n_\alpha; \widehat{\theta}) = \sup_{M_{\alpha\beta}\in\widetilde{G}} \left\{ (n_\alpha M_{\alpha\beta} n_\beta)\widehat{\theta} \right\} = \sup_{M_{\alpha\beta}\in\widetilde{G}} \left\{ M_{nn}\widehat{\theta} \right\} \tag{22}$$

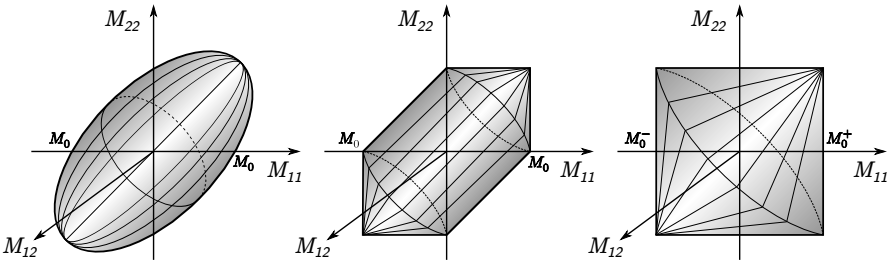

Fig. 1 Usual strength criteria for isotropic plates : von Mises (left), Tresca (middle), Johansen (right).

Usual Strength Criteria For Thin Plates

It can be shown that thin plate strength criteria assuming infinite strength with respect to membrane and shear forces can be easily derived from the strength criterion of the local constitutive material. Indeed, in the case of a homogeneous plate of thickness h made of a material obeying a plane-stress strength criterion $\mathscr{G}$, the bending strength criterion is given as $\widetilde{G} = \dfrac{h^2}{4}\mathscr{G}$ if $\mathscr{G}$ is symmetric with respect to the origin: $\sigma_{\alpha\beta} \in \mathscr{G} \Leftrightarrow -\sigma_{\alpha\beta} \in \mathscr{G}$. Hence, the bending strength criterion has the same form as its plane-stress counterpart $\mathscr{G}$. Here are three commonly used strength criteria for isotropic thin plates (see Figure 1).

von Mises Criterion

The von Mises criterion is thus given by:

$$M_{\alpha\beta} \in \widetilde{G}_{\text{von Mises}} \Leftrightarrow \sqrt{M_{11}^2 + M_{22}^2 - M_{11}M_{22} + 3M_{12}^2} \leq M_0 \tag{23}$$

where $M_0 = \sigma_0 h^2/4$, σ_0 being the material uniaxial tension/compression strength. Its support functions are given by:

$$\widetilde{\pi}(\widehat{\chi}_{\alpha\beta}) = \frac{2}{\sqrt{3}} M_0 \sqrt{\widehat{\chi}_{11}^2 + \widehat{\chi}_{22}^2 + \widehat{\chi}_{11}\widehat{\chi}_{22} + \widehat{\chi}_{12}^2} \tag{24}$$

$$\widetilde{\Pi}(n_\alpha; \widehat{\theta}) = \frac{2}{\sqrt{3}} M_0 |\widehat{\theta}| \tag{25}$$

Tresca Criterion

Similarly, the Tresca criterion is given by:

$$M_{\alpha\beta} \in \widetilde{G}_{\text{Tresca}} \Leftrightarrow \max\{|M_I|, |M_{II}|, |M_I - M_{II}|\} \leq M_0 \tag{26}$$

where M_I, M_{II} are the principal bending moments. Its support functions are given by:

$$\widetilde{\pi}(\widehat{\chi}_{\alpha\beta}) = M_0 \max\{|\widehat{\chi}_I|, |\widehat{\chi}_{II}|, |\widehat{\chi}_I + \widehat{\chi}_{II}|\} \tag{27}$$

$$\widetilde{\Pi}(n_\alpha; \widehat{\theta}) = M_0 |\widehat{\theta}| \tag{28}$$

Johansen Square Criterion

Johansen's square criterion is widely used to model the bending strength of isotropic reinforced concrete slabs (Johansen 1962). It reads as:

$$M_{\alpha\beta} \in \widetilde{G}_{\text{Johansen}} \Leftrightarrow -M_0^- \leq M_I, M_{II} \leq M_0^+ \tag{29}$$

where M_0^+ (resp. M_0^-) are the positive (resp. negative) bending moment limits. Its support functions are given by:

$$\widetilde{\pi}(\widehat{\chi}_{\alpha\beta}) = \max\left\{-M_0^- \widehat{\chi}_I, M_0^+ \widehat{\chi}_I\right\} + \max\left\{-M_0^- \widehat{\chi}_{II}, M_0^+ \widehat{\chi}_{II}\right\} \tag{30}$$

$$\widetilde{\Pi}(n_\alpha; \widehat{\theta}) = \max\left\{-M_0^- \widehat{\theta}, M_0^+ \widehat{\theta}\right\} \tag{31}$$

Illustrative Examples

Circular Plate Under Uniform Loading

This example considers a circular plate of radius R simply supported on its boundary and subjected to a uniformly distributed loading of intensity p. A Tresca strength criterion of uniform bending resistance M_0 is first considered.

Lower Bound Static Approach

The following bending moment field:

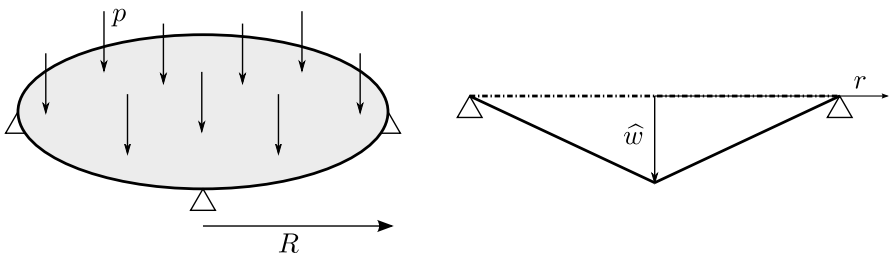

Fig. 2 Simply supported circular plate problem (left) and conic collapse mechanism used for the upper bound approach (right).

$$M_{rr} = \frac{p}{6}(r^2 - R^2) \tag{32}$$

$$M_{\theta\theta} = -\frac{pR^2}{6} \tag{33}$$

is statically admissible with the prescribed loading and verifies the Tresca strength criterion everywhere (in particular in $r = 0$) provided that:

$$p \leq \frac{6M_0}{R^2} \tag{34}$$

This static approach thus yields a lower bound estimate to the limit load:

$$\frac{6M_0}{R^2} \leq p^+ \tag{35}$$

Upper Bound Kinematic Approach

The following virtual out-of-plane velocity is considered : $\widehat{U}_3 = -\widehat{w}(1 - r/R)$ with $\widehat{w} \geq 0$. In this particular field, the work of external forces is $W_{ext} = \widehat{w}p\pi R^2/3$ and the maximum resisting work is $W_{rm} = 2\pi\widehat{w}M_0$ since there is no rotation discontinuity and $\widehat{\chi}_{\theta\theta} = \widehat{w}/(Rr)$, others $\widehat{\chi}_{\alpha\beta} = 0$. The upper bound kinematic approach necessitates that for all $p \leq p^+$:

$$W_{ext} = \widehat{w}p\pi R^2/3 \leq 2\pi\widehat{w}M_0 = W_{rm} \quad \forall\widehat{w} \tag{36}$$

yielding the following upper bound:

$$p^+ \leq \frac{6M_0}{R^2} \tag{37}$$

The comparison of both approaches shows that $p^+ = \dfrac{6M_0}{R^2}$ is the exact limit load for this problem.

von Mises Criterion

Assuming now that the plate obeys a von Mises strength criterion, the previous lower bound field still satisfies the von Mises criterion provided that $p \leq \dfrac{6M_0}{R^2}$. Thus $\dfrac{6M_0}{R^2}$ is also a lower bound to p^+ in this case.

Using the same collapse mechanism as before, the value of the maximum resisting work is now $W_{rm} = 4\pi\widehat{w}M_0/\sqrt{3}$ which yields the following upper bound:

$$p^+ \leq \frac{12M_0}{\sqrt{3}R^2} \tag{38}$$

Combining both approaches, the following bracketing of p^+ is obtained in the case of a von Mises plate:

$$\frac{6M_0}{R^2} \leq p^+ \leq \frac{12M_0}{\sqrt{3}R^2} \approx \frac{6.52M_0}{R^2} \tag{39}$$

Square Plate Obeying Johansen's Criterion

This second example considers the case of a simply supported square plate of side length a, uniformly loaded, and obeying Johansen's criterion with $M_0^+ = M_0^- = M_0$.

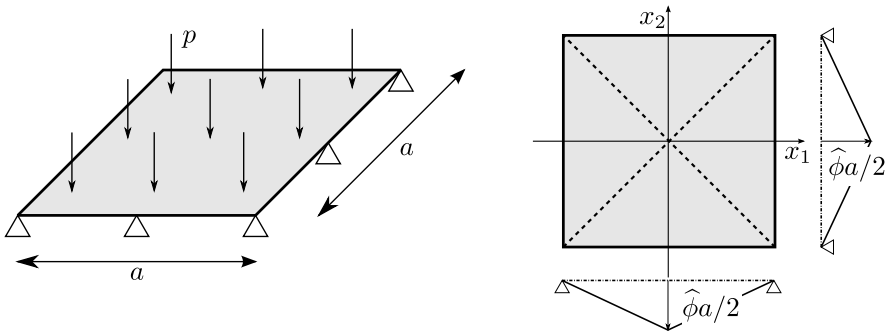

Fig. 3 Simply supported square plate problem (left) and yield-line mechanism used for the upper bound approach (right).

A First Lower Bound Approach

Hillerborg's strip method is a simple way of deriving statically admissible bending moments fields by assuming, in different rectangular portions of the plate, twist-less distributions $M_{12} = 0$ carrying a fraction κ of the total load in direction 1 and the remaining in direction 2 so that the equilibrium equations are written as:

$$M_{11,11} = \kappa p \tag{40}$$
$$M_{22,22} = (1-\kappa)p \tag{41}$$

The integration of these equations and the fulfillment of continuity and boundary conditions yields a statically admissible bending moment distribution. For the case of the square plate, sharing equally the load in both directions $(\kappa = 1/2)$ yields the following bending moment field:

$$M_{11} = \frac{p}{4}(x_1^2 - a^2/4) \tag{42}$$
$$M_{22} = \frac{p}{4}(x_2^2 - a^2/4) \tag{43}$$
$$M_{12} = 0 \tag{44}$$

The satisfaction of Johansen's square criterion yields the following lower bound estimate:

$$\frac{16M_0}{a^2} \leq p^+ \tag{45}$$

A Second Lower Bound Approach

The strip method often yields very crude lower bound estimates of the exact limit load. In the present case, a better solution is obtained by considering the following field:

$$M_{11} = \frac{p}{6}(x_1^2 - a^2/4) \tag{46}$$
$$M_{22} = \frac{p}{6}(x_2^2 - a^2/4) \tag{47}$$
$$M_{12} = \frac{p}{6}x_1 x_2 \tag{48}$$

which is also statically admissible and yields:

$$\frac{24M_0}{a^2} \leq p^+ \tag{49}$$

which improves on the first lower bound approach.

Upper Bound Approach

The yield line method is used to build an upper bound estimate for this problem by considering two yield lines along both diagonals. Each parts of the plate is assumed to rotate downwards along its supporting side. The amplitude of this virtual rotation is denoted by $\widehat{\phi}$. It can be easily seen that each four yield line experiences a normal rotation jump of amplitude $\sqrt{2}\widehat{\phi}$. In this mechanism, the work of external forces is $W_{ext} = pa^3\widehat{\phi}/6$. The maximum resisting work consists of the yield line contributions only and is given by $W_{rm} = 4 \cdot (a/\sqrt{2})\sqrt{2}|\widehat{\phi}|M_0 = 4a|\widehat{\phi}|M_0$. This upper bound thus gives for all $p \leq p^+$:

$$pa^3\widehat{\phi}/6 \leq 4a|\widehat{\phi}|M_0^- \quad \forall\widehat{\phi} \tag{50}$$

Hence $p^+ \leq 24M_0/a^2$, which combined to the second lower bound approach gives the exact solution:

$$p^+ = \frac{24M_0}{a^2} \tag{51}$$

The Case Of Clamped Supports

In presence of fully clamped supports, the same yield line mechanism can be used provided the rotation discontinuity along the edges is taken into account in the total maximum resisting work. Their contribution is $4a|\widehat{\phi}|M_0$ so that the corresponding upper bound is $p^+ \leq 48M_0/a^2$. The exact solution has been shown to be $p^+ = 42.851M_0/a^2$, see Fox (1974).

Conclusions

A large number of solutions as well as a discussion about the influence of membrane forces for the case of shells can be found in the reference textbook of Save et al (1997). Although the yield line method has proved to be an efficient design tool for engineers, complex situations still require a computer-aided implementation of the limit analysis approach. Automated yield line methods using optimization algorithms have been proposed (Faccioli and Vitiello 1973; Munro and Da Fonseca 1978; Johnson 1994; Gilbert et al 2014) but the true collapse load cannot always be reached using yield lines only as pointed out by Bræstrup (1971).

Dedicated finite elements have been proposed along with the use of modern optimization solvers for tackling various strength criteria in the limit analysis of thin plates (Krabbenhoft and Damkilde 2002; Le et al 2010; Bleyer and de Buhan 2013). Finite elements for thick plates can be found in Capsoni and Corradi (1999); Bleyer et al (2015) for the upper bound kinematic approach and in Bleyer and de Buhan (2014) for the lower bound static approach.

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