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1 Lagrangian modelling of large deformation induced by progressive failure of  
2 sensitive clays with elastoviscoplasticity

3  
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12 Abstract

13 This paper presents a Lagrangian formulation of elastoviscoplasticity, based on the Particle  
14 Finite Element Method, for progressive failure analysis of sensitive clays. The sensitive clay  
15 is represented by an elastoviscoplastic model which is a mixture of the Bingham model, for  
16 describing rheological behaviour, and the Tresca model with strain softening for capturing  
17 the progressive failure behaviour. The finite element formulation for the incremental  
18 elastoviscoplastic analysis is reformulated, through the application of the Hellinger-Reissner  
19 variational theorem, as an equivalent optimization program that can be solved efficiently  
20 using modern algorithms such as the interior-point method. The recast formulation is then  
21 incorporated into the framework of the Particle Finite Element Method for investigating  
22 progressive failure problems related to sensitive clays, such as the collapse of a sensitive clay  
23 column and the retrogressive failure of a slope in sensitive clays, where extremely large  
24 deformation occurs.

25 **Keywords:** Sensitive clays; Progressive failure; Elastoviscoplasticity; Strain softening;  
26 PFEM; Mathematical programming; SOCP

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29        1. Introduction

30 Sensitive clay is characterized by a decrease in its shear strength when experiencing plastic  
31 deformation. A highly sensitive clay may possess sensitivity, defined as a ratio of the  
32 undisturbed shear strength and the remoulded shear strength, of the order of magnitude of a  
33 hundred. For example, the reported values of the sensitivity of the clay involved in the 1893  
34 Verdalen landslide and the 2012 Byneset landslide are 300 and 120, respectively [1]. Due to  
35 the strong strain-softening behaviour, geostructures built on a layer of sensitive clay often fail  
36 in a progressive manner. Moreover, unexpectedly catastrophic failure of the geostructure  
37 might also be induced by a small perturbation. Typical examples are the multiple  
38 retrogressive slides and spreads in sensitive clays observed in Canada and Scandinavia [2],  
39 which occurred suddenly, covered large areas (more than 1 hectare) and were caused by an  
40 initially small slope failure.

41

42 Reliable prediction of the progressive failure behaviour of sensitive clays is of critical  
43 importance. It can provide guidelines for relevant engineering practice, for example  
44 construction on sensitive clays, and also assist in minimizing the degree of destruction caused  
45 by potential geohazards (such as the fore-mentioned large landslides). Although numerical  
46 simulation is a powerful tool for analyzing complex geotechnical problems, robust modelling  
47 of the large deformations induced by progressive failure in sensitive clays is still a formidable  
48 task. Indeed, a major challenge is the complex behaviour that is typically exhibited by  
49 sensitive clays. An undisturbed sensitive clay usually behaves like a solid body, but may  
50 change to be a semi-liquid material after being remolded [1]. The transformation between  
51 these two states is caused by strain softening. Suitable constitutive models must be capable of  
52 describing the rheological behaviour of a sensitive clay, since this is crucial for estimating the  
53 run-out distance of landslides [3-5] as well as capturing the strain-softening behaviour that

54 contributes to the phenomenon of progressive failure [2, 6, 7]. Sensitive clays typically  
55 undergo extremely large deformation along localized shear zones due to strain-softening.  
56 This feature can cause severe mesh distortion when the traditional finite element method is  
57 adopted and result in computational difficulties. Additionally, the free-surface evolution  
58 induced by extreme deformation also challenges the use of the traditional FEM because of its  
59 use of a fixed mesh topology. Recently, some alternative numerical approaches have been  
60 proposed for modelling the progressive failure of sensitive clays involving large deformation.  
61 Wang *et. al* [8] studied retrogressive and progressive slope failure in sensitive clays using the  
62 material point method. Dey *et. al* [9-11] analyzed the spread in sensitive clay slopes due to  
63 progressive failure by implementing a strain-softening model into the ABAQUS Coupled  
64 Eulerian Lagrangian approach. Although these procedures reproduced the pronounced  
65 progressive failure behaviour of sensitive clays, it is notable that classical rate-independent  
66 models were utilised. However, ignoring the rheology of sensitive clays may lead to the  
67 inaccurate predictions. Analytical approaches, such as shear band propagation approaches  
68 [12-16], have also been used to study the progressive failure process in catastrophic  
69 landslides in nature. Recent developments in the shear band propagation approach for  
70 analyzing catastrophic and progressive failure are summarized in [17].

71

72 This paper provides an alternative Lagrangian computational approach for the analysis of  
73 progressive failure of sensitive clays involving extremely large deformation. An advanced  
74 elastoviscoplastic constitutive relationship, which is a combination of the Bingham model  
75 and the Tresca model with strain softening, is adopted for describing their complex behaviour.  
76 To solve the resulting elastoviscoplastic problem with strain softening, a generalized  
77 incremental Hellinger-Reissner variational theorem [18] is proposed which recasts the  
78 associated governing equations into an equivalent min-max program. After finite element

79 discretisation, the resulting problem can be converted into a standard second-order cone  
80 programming problem which may be solved efficiently using modern optimization  
81 algorithms (for example, the primal-dual interior point method [19]). Typical advantages of  
82 such a solution strategy include the possibility of analyzing the existence, uniqueness,  
83 sensitivity and stability of the solution [20], the natural treatment of the singularities in the  
84 Mohr-Coulomb and Drucker-Prager yield criteria [21-23], the straightforward extension from  
85 single-surface plasticity to multi-surface plasticity [21], and the straightforward  
86 implementation of contact between deformable and rigid bodies [24, 25]. The proposed  
87 solution algorithm is incorporated into the framework of the Particle Finite Element Method  
88 (PFEM) [26-28] for handling large deformation. The PFEM is a novel continuum approach  
89 suitable for simulating problems involving both solid-like and fluid-like behaviour [27, 29,  
90 30]. It makes use of particles to represent the material, as in meshfree particle methods, but  
91 solves the governing equations via a standard finite element procedure. Consequently, the  
92 PFEM inherits both the solid mathematical foundation of the traditional FEM as well as the  
93 capability of meshfree particle methods for handling extremely large deformation and free-  
94 surface evolution.

95

96 The paper is organized as follows. Section 2 presents the governing equations for dynamic  
97 analysis of an elastoviscoplastic problem. An incremental mixed variational principle is then  
98 proposed in Section 3 for recasting the governing equations into a min-max problem. Finite  
99 element discretisation is performed in Section 4, and the Particle Finite Element Method is  
100 described briefly in Section 5. Numerical examples are given in Section 6, before conclusions  
101 are drawn in Section 7.

102

103       2. Governing equations for elastoviscoplasticity

104 Consider a medium with volume  $\Omega$  and surface  $\Gamma = \Gamma_u \cup \Gamma_t$ , where  $\Gamma_u$  and  $\Gamma_t$  are the  
 105 kinematic and traction boundaries, respectively. The partition of the surface obeys the  
 106 constraint  $\Gamma_u \cap \Gamma_t = \emptyset$  where  $\emptyset$  is a null set. The momentum conservation equation, the  
 107 kinematic equations for displacement gradients, and the corresponding boundary conditions  
 108 read

$$109 \quad \nabla^T \boldsymbol{\sigma} + \mathbf{b} = \rho \ddot{\mathbf{u}} \quad \text{in } \Omega \quad (1)$$

$$110 \quad \boldsymbol{\varepsilon} = \nabla^T \mathbf{u} \quad \text{in } \Omega \quad (2)$$

$$111 \quad \mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \Gamma_u \quad (3)$$

$$112 \quad \mathbf{N}^T \boldsymbol{\sigma} = \bar{\mathbf{t}} \quad \text{on } \Gamma_t \quad (4)$$

113 where  $\boldsymbol{\sigma}$  and  $\boldsymbol{\varepsilon}$  are the Cauchy stress and the strain,  $\mathbf{b}$  is the body force,  $\mathbf{u}$  is the  
 114 displacement,  $\bar{\mathbf{u}}$  and  $\bar{\mathbf{t}}$  are the prescribed displacements and external tractions,  $\mathbf{N}$  consists  
 115 of components of the outward normal to the boundary  $\Gamma_t$ , and  $\nabla$  is the usual linear operator  
 116 taking the form of

$$117 \quad \nabla^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \quad (5)$$

118 in a plane-strain case. A superposed dot represents differentiation with respect to time.

119

120 Assuming the material is elastoviscoplastic, the total strain rate  $\dot{\boldsymbol{\varepsilon}}$  can then be split into an  
 121 elastic strain rate  $\dot{\boldsymbol{\varepsilon}}^e$  and a viscoplastic strain rate  $\dot{\boldsymbol{\varepsilon}}^{vp}$

$$122 \quad \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^{vp} \quad (6)$$

123 The elastic strain rate is determined through Hook's law as

$$124 \quad \dot{\boldsymbol{\varepsilon}}^e = \mathbb{C} \dot{\boldsymbol{\sigma}} \quad (7)$$

125 where  $\mathbb{C}$  is the elastic compliance matrix. The material is elastic if the stress state is inside  
 126 the yield domain, namely

$$127 \quad F(\boldsymbol{\sigma}) < 0 \Rightarrow \dot{\boldsymbol{\epsilon}}^{vp} = \mathbf{0} \quad (8)$$

128 where  $F$  is the yield function. In contrast, stress states satisfying  $F(\boldsymbol{\sigma}) \geq 0$  lead to a  
 129 viscoplastic strain rate. The classical Bingham model is utilized in this paper for describing  
 130 the rheological properties of the sensitive clay. Despite its simple form, it performs well for  
 131 approximating the plastic flow behaviour of these soils, especially Canadian clays [5]. The  
 132 total stress thus is rewritten as

$$133 \quad \boldsymbol{\sigma} = \boldsymbol{\tau} + \eta \dot{\boldsymbol{\epsilon}}^{vp} \quad (9)$$

134 where  $\eta$  is the viscosity coefficient,  $\boldsymbol{\tau}$  is the stress lying on the boundary of  $F$  so that  
 135  $F(\boldsymbol{\tau}) = 0$ , and the quantity  $\boldsymbol{\sigma} - \boldsymbol{\tau}$  is called the overstress. The viscoplastic strain rate is also  
 136 normal to the yield surface at  $\boldsymbol{\tau}$ :

$$137 \quad \dot{\boldsymbol{\epsilon}}^{vp} = \dot{\lambda} \nabla_{\boldsymbol{\tau}} F(\boldsymbol{\tau}) \quad (10)$$

138 where  $\dot{\lambda}$  is the rate of the non-negative plastic multiplier and  $\nabla_{\boldsymbol{\tau}}$  is the gradient operator. It is  
 139 clear that the above elastoviscoplastic model reduces to the classical elastoplastic model in  
 140 the limiting case of  $\eta = 0$ .

141

142 Laboratory tests show that the undrained shear strength of a sensitive clay decreases with  
 143 increasing plastic shear strain. For materials exhibiting softening/hardening behaviour, the  
 144 yield criterion function is expressed by  $F(\boldsymbol{\tau}, \boldsymbol{\kappa}) = 0$ , where  $\boldsymbol{\kappa}$  is a set of hardening/softening  
 145 variables which relate to the viscoplastic strain in the form of

$$146 \quad \boldsymbol{\kappa} = H(\boldsymbol{\epsilon}^{vp}) \quad (11)$$

147 Specifically, for the Tresca yield criterion, we have

148 
$$F(\boldsymbol{\sigma}, \kappa) = \sqrt{(\sigma_{xx} - \sigma_{yy})^2 + 4\sigma_{xy}^2} - 2c_u(\kappa) \quad (12)$$

149 where cohesion softening is adopted to capture the basic post-failure behaviour. Following  
 150 [31, 32], strain-softening is accounted for by reducing the cohesion  $c_u$  using a bilinear  
 151 function (Figure 1) of the equivalent deviatoric plastic strain,  $\kappa = \int \dot{\kappa} dt$ , where  
 152  $\dot{\kappa} = \sqrt{0.5 \dot{\epsilon}_{ij}^{vp} \dot{\epsilon}_{ij}^{vp}}$  and  $\dot{\epsilon}_{ij}^{vp}$  is the rate of deviatoric viscoplastic strain tensor given by

153 
$$\dot{\epsilon}_{ij}^{vp} = \dot{\epsilon}_{ij}^{vp} - \frac{1}{3} \dot{\epsilon}_{kk}^{vp} \delta_{ij} \quad (13)$$

154 in which  $\delta_{ij}$  is the Kronecker delta.

155

### 156 3. Variational principle

#### 157 3.1 Hellinger-Reissner variational principle

158 The Hellinger-Reissner (HR) variational principle is of a mixed kind. Unlike the principle of  
 159 minimum potential energy, in which displacements are considered as the only master field,  
 160 the Hellinger-Reissner variational principle treats both the displacements and the stresses as  
 161 the master fields [18].

162

163 For an elastostatic boundary-value problem, the Hellinger-Reissner functional [18] may be  
 164 expressed as

165 
$$\Pi(\boldsymbol{\sigma}, \mathbf{u}) = \int_{\Omega} \left( -\frac{1}{2} \boldsymbol{\sigma}^T \mathbf{C} \boldsymbol{\sigma} + \boldsymbol{\sigma}^T \mathbf{S}^T \mathbf{u} \right) d\Omega - \int_{\Omega} \mathbf{b}^T \mathbf{u} d\Omega - \int_{\Gamma_t} \bar{\mathbf{t}}^T \mathbf{u} d\Gamma \quad (14)$$

166 The stationary value for the Hellinger-Reissner functional cannot be shown to be an  
 167 extremum. Instead, the point obtained by  $\delta \Pi(\boldsymbol{\sigma}, \mathbf{u}) = 0$  is a saddle point and, consequently,  
 168 the problem becomes one of a min-max optimisation:



169 
$$\min_{\mathbf{u}} \max_{\boldsymbol{\sigma}} \int_{\Omega} \left( -\frac{1}{2} \boldsymbol{\sigma}^T \mathbb{C} \boldsymbol{\sigma} + \boldsymbol{\sigma}^T \mathbf{S}^T \mathbf{u} \right) d\Omega - \int_{\Omega} \mathbf{b}^T \mathbf{u} d\Omega - \int_{\Gamma_t} \bar{\mathbf{t}}^T \mathbf{u} d\Gamma \quad (15)$$

170 where the internal work is maximised with respect to the stresses and the total potential  
171 energy is minimised with respect to the displacements.

172

### 173 3.2 Generalised Hellinger-Reissner variational principle

174 A generalised Hellinger-Reissner variational principle is proposed for incremental analysis of  
175 elastoviscoplasticity. The governing equations summarised in section 2 are first discretized in  
176 time using the standard  $\theta$ -method. Details of the time discretisation, as well as the resulting  
177 incremental equations, are documented in Appendix A.

178

179 We present here the generalized incremental Hellinger-Reissner variational principle for  
180 incremental elasto-viscoplastic analysis. As for elastostatics, the principle is expressed in the  
181 form of a min-max program:

$$\begin{aligned}
 \min_{\Delta \mathbf{u}} \max_{(\boldsymbol{\sigma}, \boldsymbol{\tau}, \mathbf{r})_{n+1}} & -\frac{1}{2} \int_{\Omega} \Delta \boldsymbol{\sigma}^T \mathbb{C} \Delta \boldsymbol{\sigma} d\Omega + \int_{\Omega} \boldsymbol{\sigma}_{n+1}^T \nabla^T (\Delta \mathbf{u}) d\Omega + \int_{\Omega} \frac{1-\theta_1}{\theta_1} \boldsymbol{\sigma}_n^T \nabla^T (\Delta \mathbf{u}) d\Omega \\
 & -\frac{1}{2} \int_{\Omega} \mathbf{r}_{n+1}^T \frac{\Delta t^2}{\tilde{\rho}} \mathbf{r}_{n+1} d\Omega + \int_{\Omega} \mathbf{r}_{n+1}^T \Delta \mathbf{u} d\Omega \\
 182 & -\frac{1}{2} \int_{\Omega} (\Delta \boldsymbol{\sigma} - \Delta \boldsymbol{\tau})^T \frac{\theta_3 \Delta t}{\eta} (\Delta \boldsymbol{\sigma} - \Delta \boldsymbol{\tau}) d\Omega - \int_{\Omega} \Delta \boldsymbol{\sigma}^T \frac{\Delta t}{\eta} (\boldsymbol{\sigma}_n - \boldsymbol{\tau}_n) d\Omega \\
 & + \int_{\Omega} (\boldsymbol{\sigma}_n - \boldsymbol{\tau}_n)^T \frac{\Delta t}{\eta} \Delta \boldsymbol{\tau} d\Omega - \int_{\Omega} \tilde{\mathbf{b}}^T \Delta \mathbf{u} d\Omega - \int_{\Gamma_t} \tilde{\mathbf{t}}^T \Delta \mathbf{u} d\Gamma \\
 \text{subject to} & F(\boldsymbol{\tau}_{n+1}) \leq 0
 \end{aligned} \quad (16)$$

183 where  $\mathbf{r}$  is a set of variables that can be interpreted as dynamic forces. To illustrate the  
184 equivalence between the program (16) and the incremental form of the governing equations  
185 presented in Appendix A, the Karush-Kuhn-Tucker (KKT) optimality conditions associated  
186 with (16) are now derived. Following [23, 33], the inequality constraint is first converted into  
187 an equality by adding a positively-restricted variable  $s_{n+1}$ . Then, the inequality on  $s_{n+1}$  is

188 represented by introducing a penalty term in the objective function:

$$\begin{aligned}
& \min_{\Delta \mathbf{u}} \max_{(\boldsymbol{\sigma}, \boldsymbol{\tau}, \mathbf{r})_{n+1}} -\frac{1}{2} \int_{\Omega} \Delta \boldsymbol{\sigma}^T \mathbb{C} \Delta \boldsymbol{\sigma} d\Omega + \int_{\Omega} \boldsymbol{\sigma}_{n+1}^T \nabla^T (\Delta \mathbf{u}) d\Omega + \int_{\Omega} \frac{1-\theta_1}{\theta_1} \boldsymbol{\sigma}_n^T \nabla^T (\Delta \mathbf{u}) d\Omega \\
& -\frac{1}{2} \int_{\Omega} \mathbf{r}_{n+1}^T \frac{\Delta t^2}{\tilde{\rho}} \mathbf{r}_{n+1} d\Omega + \int_{\Omega} \mathbf{r}_{n+1}^T \Delta \mathbf{u} d\Omega \\
189 & -\frac{1}{2} \int_{\Omega} (\Delta \boldsymbol{\sigma} - \Delta \boldsymbol{\tau})^T \frac{\theta_3 \Delta t}{\eta} (\Delta \boldsymbol{\sigma} - \Delta \boldsymbol{\tau}) d\Omega - \int_{\Omega} \Delta \boldsymbol{\sigma}^T \frac{\Delta t}{\eta} (\boldsymbol{\sigma}_n - \boldsymbol{\tau}_n) d\Omega \\
& + \int_{\Omega} (\boldsymbol{\sigma}_n - \boldsymbol{\tau}_n)^T \frac{\Delta t}{\eta} \Delta \boldsymbol{\tau} d\Omega - \int_{\Omega} \tilde{\mathbf{b}}^T \Delta \mathbf{u} d\Omega - \int_{\Gamma_t} \tilde{\mathbf{t}}^T \Delta \mathbf{u} d\Gamma + \int_{\Omega} \mu \ln s_{n+1} d\Omega \\
& \text{subject to } F(\boldsymbol{\tau}_{n+1}) + s_{n+1} = 0
\end{aligned} \tag{17}$$

190 where  $\mu$  is a sufficiently small positive constant. The penalty term  $\mu \ln s_{n+1}$  in the objective  
191 function imposes the non-negativity requirement on  $s_{n+1}$  naturally, and is known as a  
192 logarithmic barrier function. The Lagrangian associated with program (17) is

$$\begin{aligned}
& \mathcal{L}(\Delta \mathbf{u}, \boldsymbol{\sigma}_{n+1}, \boldsymbol{\tau}_{n+1}, \mathbf{r}_{n+1}, \Delta \lambda, s_{n+1}) \\
& = -\frac{1}{2} \int_{\Omega} \Delta \boldsymbol{\sigma}^T \mathbb{C} \Delta \boldsymbol{\sigma} d\Omega + \int_{\Omega} \boldsymbol{\sigma}_{n+1}^T \nabla^T (\Delta \mathbf{u}) d\Omega + \int_{\Omega} \frac{1-\theta_1}{\theta_1} \boldsymbol{\sigma}_n^T \nabla^T (\Delta \mathbf{u}) d\Omega \\
193 & -\frac{1}{2} \int_{\Omega} \mathbf{r}_{n+1}^T \frac{\Delta t^2}{\tilde{\rho}} \mathbf{r}_{n+1} d\Omega + \int_{\Omega} \mathbf{r}_{n+1}^T \Delta \mathbf{u} d\Omega - \frac{1}{2} \int_{\Omega} (\Delta \boldsymbol{\sigma} - \Delta \boldsymbol{\tau})^T \frac{\theta_3 \Delta t}{\eta} (\Delta \boldsymbol{\sigma} - \Delta \boldsymbol{\tau}) d\Omega \\
& - \int_{\Omega} \Delta \boldsymbol{\sigma}^T \frac{\Delta t}{\eta} (\boldsymbol{\sigma}_n - \boldsymbol{\tau}_n) d\Omega + \int_{\Omega} (\boldsymbol{\sigma}_n - \boldsymbol{\tau}_n)^T \frac{\Delta t}{\eta} \Delta \boldsymbol{\tau} d\Omega - \int_{\Omega} \tilde{\mathbf{b}}^T \Delta \mathbf{u} d\Omega - \int_{\Gamma_t} \tilde{\mathbf{t}}^T \Delta \mathbf{u} d\Gamma \\
& + \int_{\Omega} \mu \ln s_{n+1} d\Omega - \int_{\Omega} \Delta \lambda (F(\boldsymbol{\tau}_{n+1}) + s_{n+1}) d\Omega
\end{aligned} \tag{18}$$

194 The KKT optimality conditions are found by differentiating the above Lagrangian with  
195 respect to the optimisation variables, namely:

$$\frac{\partial \mathcal{L}}{\partial \Delta \mathbf{u}} = \begin{cases} \nabla^T \boldsymbol{\sigma}_{n+1} + \frac{1-\theta_1}{\theta_1} \nabla^T \boldsymbol{\sigma}_n + \mathbf{r}_{n+1} - \tilde{\mathbf{b}} = \mathbf{0} & \text{in } \Omega \\ \mathbf{N}^T (\boldsymbol{\sigma}_{n+1} + \frac{1-\theta_1}{\theta_1} \boldsymbol{\sigma}_n) = \tilde{\mathbf{t}} & \text{on } \Gamma_t \end{cases} \tag{19}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\sigma}_{n+1}} = \nabla^T (\Delta \mathbf{u}) - \mathbb{C} \Delta \boldsymbol{\sigma} - \frac{\theta_3 \Delta t}{\eta} (\Delta \boldsymbol{\sigma} - \Delta \boldsymbol{\tau}) - \frac{\Delta t}{\eta} (\boldsymbol{\sigma}_n - \boldsymbol{\tau}_n) = \mathbf{0} \quad \text{in } \Omega \tag{20}$$

$$198 \quad \frac{\partial \mathcal{L}}{\partial \boldsymbol{\tau}_{n+1}} = \frac{\theta_3 \Delta t}{\eta} (\Delta \boldsymbol{\sigma} - \Delta \boldsymbol{\tau}) + \frac{\Delta t}{\eta} (\boldsymbol{\sigma}_n - \boldsymbol{\tau}_n) - \Delta \lambda \nabla_G F(\boldsymbol{\tau}_{n+1}) = \mathbf{0} \quad \text{in } \Omega \quad (21)$$

$$199 \quad \frac{\partial \mathcal{L}}{\partial \mathbf{r}_{n+1}} = \frac{\Delta t^2}{\tilde{\rho}} \mathbf{r}_{n+1} - \Delta \mathbf{u} = \mathbf{0} \quad \text{in } \Omega \quad (22)$$

$$200 \quad \frac{\partial \mathcal{L}}{\partial \Delta \lambda} = F(\boldsymbol{\tau}_{n+1}) + s_{n+1} = 0 \quad \text{in } \Omega \quad (23)$$

$$201 \quad \frac{\partial \mathcal{L}}{\partial s_{n+1}} = \mu s_{n+1}^{-1} - \Delta \lambda = 0 \Rightarrow \mu = s_{n+1} \Delta \lambda \quad \text{in } \Omega \quad (24)$$

202 It is apparent that the KKT conditions (19)-(22) are equivalent to the corresponding  
 203 incremental equations presented in Appendix A. The last two conditions recover the yield  
 204 condition and the complementarity condition shown in (50) when  $\mu \rightarrow 0^+$ , given that the  
 205 penalty multiplier  $\Delta \lambda \geq 0$ , and  $s_{n+1} > 0$ . The essential boundary condition (40) is assumed to  
 206 hold *a priori*, and thus is not reflected in the KKT conditions. From condition (22) we can  
 207 also see that the newly introduced variables  $\mathbf{r}$  are dynamic forces.

208

### 209 3.3 Material hardening/softening

210 The variational principle (16) can also be extended to handle more complex models involving  
 211 hardening/softening yield surfaces following [34]. More specifically, the min-max program  
 212 considering material hardening/softening is expressed as:

$$\begin{aligned}
& \min_{\Delta \mathbf{u}} \max_{(\boldsymbol{\sigma}, \boldsymbol{\tau}, \mathbf{r})_{n+1}} -\frac{1}{2} \int_{\Omega} \Delta \boldsymbol{\sigma}^T \mathbb{C} \Delta \boldsymbol{\sigma} d\Omega + \int_{\Omega} \boldsymbol{\sigma}_{n+1}^T \nabla^T (\Delta \mathbf{u}) d\Omega + \int_{\Omega} \frac{1-\theta_1}{\theta_1} \boldsymbol{\sigma}_n^T \nabla^T (\Delta \mathbf{u}) d\Omega \\
& -\frac{1}{2} \int_{\Omega} \mathbf{r}_{n+1}^T \frac{\Delta t^2}{\tilde{\rho}} \mathbf{r}_{n+1} d\Omega + \int_{\Omega} \mathbf{r}_{n+1}^T \Delta \mathbf{u} d\Omega \\
& -\frac{1}{2} \int_{\Omega} (\Delta \boldsymbol{\sigma} - \Delta \boldsymbol{\tau})^T \frac{\theta_3 \Delta t}{\eta} (\Delta \boldsymbol{\sigma} - \Delta \boldsymbol{\tau}) d\Omega - \int_{\Omega} \Delta \boldsymbol{\sigma}^T \frac{\Delta t}{\eta} (\boldsymbol{\sigma}_n - \boldsymbol{\tau}_n) d\Omega \\
& + \int_{\Omega} (\boldsymbol{\sigma}_n - \boldsymbol{\tau}_n)^T \frac{\Delta t}{\eta} \Delta \boldsymbol{\tau} d\Omega - \frac{1}{2} \int_{\Omega} \mathcal{H}_t^{-1} \Delta \kappa^2 d\Omega - \int_{\Omega} \tilde{\mathbf{b}}^T \Delta \mathbf{u} d\Omega - \int_{\Gamma_t} \tilde{\mathbf{t}}^T \Delta \mathbf{u} d\Gamma \\
& \text{subject to } \underline{F(\boldsymbol{\tau}_{n+1}, \kappa_{n+1})} \leq 0
\end{aligned} \tag{25}$$

214 The underlined term is the newly introduced one with  $\mathcal{H}_t$  being a new constitutive modulus  
215 associated with hardening/softening. The according KKT condition related to the variable  $\kappa$   
216 is

$$\frac{\partial \mathcal{L}}{\partial \kappa_{n+1}} = -\mathcal{H}_t^{-1} \Delta \kappa - \Delta \lambda \nabla_{\kappa} F(\boldsymbol{\tau}_{n+1}, \kappa_{n+1}) = 0 \Rightarrow \Delta \kappa = -\Delta \lambda \mathcal{H}_t \nabla_{\kappa} F(\boldsymbol{\tau}_{n+1}, \kappa_{n+1}) \tag{26}$$

218 which is the hardening/softening law, i.e. the evolution law, for the variable  $\kappa$ . The  
219 constitutive modulus,  $\mathcal{H}_t$ , can be derived by first expanding Eq. (11) using a Taylor series

$$\kappa_{n+1} = \kappa_n + \frac{dH(\boldsymbol{\epsilon}_n^{\text{vp}})}{d\boldsymbol{\epsilon}^{\text{vp}}} \Delta \boldsymbol{\epsilon}^{\text{vp}} \Rightarrow \Delta \kappa = \frac{dH(\boldsymbol{\epsilon}_n^{\text{vp}})}{d\boldsymbol{\epsilon}^{\text{vp}}} \Delta \boldsymbol{\epsilon}^{\text{vp}} \tag{27}$$

221 Since Eq. (26) cannot be brought to be equal to the actual hardening/softening law (27) using  
222 a constant modulus, we therefore use the following tangent modulus as in [34]

$$\mathcal{H}_t = -\frac{dH(\boldsymbol{\epsilon}_n^{\text{vp}})}{d\boldsymbol{\epsilon}^{\text{vp}}} \frac{\nabla_{\boldsymbol{\tau}} F(\boldsymbol{\tau}_n, \kappa_n)}{\nabla_{\kappa} F(\boldsymbol{\tau}_n, \kappa_n)} \tag{28}$$

224 which is updated at the beginning of each time step. Such a treatment of material  
225 hardening/softening behaviour in mathematical programming has been used successfully for  
226 approximating the hardening/softening behaviour in the Cam clay model [34].

227

228 4. Finite element formulation

229 The min-max program (25) can now be discretized using finite elements. For the sake of  
 230 convenience, an intermediate variable  $\boldsymbol{\sigma}^e = \boldsymbol{\sigma} - \boldsymbol{\tau}$  (overstress) is introduced, which enables  
 231 the optimization problem (25) to be expressed as

$$\begin{aligned}
 \min_{\Delta \mathbf{u}} \max_{(\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\sigma}^e, \mathbf{r})_{n+1}} & -\frac{1}{2} \int_{\Omega} \Delta \boldsymbol{\sigma}^T \mathbb{C} \Delta \boldsymbol{\sigma} d\Omega + \int_{\Omega} \boldsymbol{\sigma}_{n+1}^T \nabla^T (\Delta \mathbf{u}) d\Omega + \int_{\Omega} \frac{1-\theta_1}{\theta_1} \boldsymbol{\sigma}_n^T \nabla^T (\Delta \mathbf{u}) d\Omega \\
 & -\frac{1}{2} \int_{\Omega} \mathbf{r}_{n+1}^T \frac{\Delta t^2}{\tilde{\rho}} \mathbf{r}_{n+1} d\Omega + \int_{\Omega} \mathbf{r}_{n+1}^T \Delta \mathbf{u} d\Omega \\
 232 & -\frac{1}{2} \int_{\Omega} \Delta \boldsymbol{\sigma}^{eT} \frac{\theta_3 \Delta t}{\eta} \Delta \boldsymbol{\sigma}^e d\Omega - \int_{\Omega} \Delta \boldsymbol{\sigma}^{eT} \frac{\Delta t}{\eta} \boldsymbol{\sigma}_n^e d\Omega \\
 & -\frac{1}{2} \int_{\Omega} \mathcal{H}_i^{-1} \Delta \kappa^2 d\Omega - \int_{\Omega} \tilde{\mathbf{b}}^T \Delta \mathbf{u} d\Omega - \int_{\Gamma_i} \tilde{\mathbf{t}}^T \Delta \mathbf{u} d\Gamma \\
 \text{subject to} & \Delta \boldsymbol{\sigma}^e = \Delta \boldsymbol{\sigma} - \Delta \boldsymbol{\tau} \\
 & F(\boldsymbol{\tau}_{n+1}, \kappa_{n+1}) \leq 0
 \end{aligned} \tag{29}$$

233 Using standard finite element notations, we have

$$\begin{aligned}
 234 & \boldsymbol{\sigma}(\mathbf{x}) \approx \mathbf{N}_{\boldsymbol{\sigma}} \hat{\boldsymbol{\sigma}}, \quad \boldsymbol{\sigma}^e(\mathbf{x}) \approx \mathbf{N}_{\boldsymbol{\sigma}^e} \hat{\boldsymbol{\sigma}}^e, \quad \boldsymbol{\tau}(\mathbf{x}) \approx \mathbf{N}_{\boldsymbol{\tau}} \hat{\boldsymbol{\tau}}, \\
 & \mathbf{r}(\mathbf{x}) \approx \mathbf{N}_{\mathbf{r}} \hat{\mathbf{r}}, \quad \mathbf{u}(\mathbf{x}) \approx \mathbf{N}_{\mathbf{u}} \hat{\mathbf{u}}, \quad \nabla^T \mathbf{u} \approx \mathbf{B}_{\mathbf{u}} \hat{\mathbf{u}}, \\
 & \kappa(\mathbf{x}) \approx \mathbf{N}_{\kappa} \hat{\kappa}
 \end{aligned} \tag{30}$$

235 where  $\hat{\boldsymbol{\sigma}}$ ,  $\hat{\boldsymbol{\sigma}}^e$ ,  $\hat{\boldsymbol{\tau}}$ ,  $\hat{\mathbf{r}}$ ,  $\hat{\mathbf{u}}$ , and  $\hat{\kappa}$  are vectors containing the values of the corresponding field  
 236 variables at interpolation points,  $\mathbf{N}$  is a matrix consisting of shape functions, and  $\mathbf{B}_{\mathbf{u}} = \nabla^T \mathbf{N}_{\mathbf{u}}$   
 237 . The mixed finite element shown in Figure 2 is adopted in this study, where the distribution  
 238 of the interpolation points for the different variables is depicted. Substituting the above  
 239 equations into the program (29) leads to

$$\begin{aligned}
& \min_{\Delta \hat{\mathbf{u}}} \max_{(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\sigma}}^e, \hat{\mathbf{r}}, \hat{\mathbf{k}})_{n+1}} -\frac{1}{2} \Delta \hat{\boldsymbol{\sigma}}^T \mathbf{C} \Delta \hat{\boldsymbol{\sigma}} + \Delta \hat{\mathbf{u}}^T \mathbf{B}^T \hat{\boldsymbol{\sigma}}_{n+1} + \Delta \hat{\mathbf{u}}^T \frac{1-\theta_1}{\theta_1} \mathbf{B}^T \hat{\boldsymbol{\sigma}}_n \\
& -\frac{1}{2} \hat{\mathbf{r}}_{n+1}^T \mathbf{D} \hat{\mathbf{r}}_{n+1} + \Delta \hat{\mathbf{u}}^T \mathbf{A}^T \hat{\mathbf{r}}_{n+1} - \frac{1}{2} \Delta \hat{\boldsymbol{\sigma}}^{eT} \mathbf{M} \Delta \hat{\boldsymbol{\sigma}}^e \\
& -\Delta \hat{\boldsymbol{\sigma}}^{eT} \mathbf{f}^c - \frac{1}{2} \Delta \hat{\mathbf{k}}^T \mathbf{H} \Delta \hat{\mathbf{k}} - \Delta \hat{\mathbf{u}}^T \mathbf{f}^e
\end{aligned} \tag{31}$$

$$\text{subject to } \Delta \hat{\boldsymbol{\sigma}}^e = \Delta \hat{\boldsymbol{\sigma}} - \Delta \hat{\boldsymbol{\tau}}$$

$$F_j(\hat{\boldsymbol{\tau}}_{n+1}, \hat{\mathbf{k}}_{n+1}) \leq 0, \quad j=1, 2, \dots, N_G$$

241 where

$$\begin{aligned}
\mathbf{C} &= \int_{\Omega} \mathbf{N}_{\boldsymbol{\sigma}}^T \mathbf{C} \mathbf{N}_{\boldsymbol{\sigma}} d\Omega, \quad \mathbf{B}^T = \int_{\Omega} \mathbf{B}_{\mathbf{u}}^T \mathbf{N}_{\boldsymbol{\sigma}} d\Omega, \\
\mathbf{D} &= \int_{\Omega} \mathbf{N}_{\mathbf{r}}^T \frac{\Delta t^2}{\tilde{\rho}} \mathbf{N}_{\mathbf{r}} d\Omega, \quad \mathbf{A}^T = \int_{\Omega} \mathbf{N}_{\mathbf{u}}^T \mathbf{N}_{\mathbf{r}} d\Omega, \\
\mathbf{M} &= \int_{\Omega} \mathbf{N}_{\boldsymbol{\sigma}^e}^T \frac{\theta_3 \Delta t}{\eta} \mathbf{N}_{\boldsymbol{\sigma}^e} d\Omega, \quad \mathbf{H} = \int_{\Omega} \mathbf{N}_{\mathbf{k}}^T \frac{1}{\mathcal{H}_t} \mathbf{N}_{\mathbf{k}} d\Omega, \\
\mathbf{f}^e &= \int_{\Omega} \mathbf{N}_{\mathbf{u}}^T \tilde{\mathbf{b}} d\Omega + \int_{\Gamma_t} \mathbf{N}_{\mathbf{u}}^T \tilde{\mathbf{t}} d\Gamma, \quad \mathbf{f}^c = \int_{\Omega} \mathbf{N}_{\boldsymbol{\sigma}}^T \frac{\Delta t}{\eta} \boldsymbol{\sigma}_n^e d\Omega
\end{aligned} \tag{32}$$

243 The yield conditions are enforced at all Gauss integration points, with  $N_G$  being the total  
244 number of such points. After solving the minimization part over  $\Delta u$  of program (31), and  
245 transforming the maximum into a minimum with an opposite sign, we obtain

$$\begin{aligned}
& \min_{(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\sigma}}^e, \hat{\mathbf{r}}, \hat{\mathbf{k}})_{n+1}} \frac{1}{2} \Delta \hat{\boldsymbol{\sigma}}^T \mathbf{C} \Delta \hat{\boldsymbol{\sigma}} + \frac{1}{2} \hat{\mathbf{r}}_{n+1}^T \mathbf{D} \hat{\mathbf{r}}_{n+1} + \frac{1}{2} \Delta \hat{\boldsymbol{\sigma}}^{eT} \mathbf{M} \Delta \hat{\boldsymbol{\sigma}}^e \\
& + \frac{1}{2} \Delta \hat{\mathbf{k}}^T \mathbf{H} \Delta \hat{\mathbf{k}} + \Delta \hat{\boldsymbol{\sigma}}^{eT} \mathbf{f}^c \\
& \text{subject to } \mathbf{B}^T \hat{\boldsymbol{\sigma}}_{n+1} + \frac{1-\theta_1}{\theta_1} \mathbf{B}^T \hat{\boldsymbol{\sigma}}_n + \mathbf{A}^T \hat{\mathbf{r}}_{n+1} - \mathbf{f}^c = \mathbf{0} \\
& \Delta \hat{\boldsymbol{\sigma}}^e = \Delta \hat{\boldsymbol{\sigma}} - \Delta \hat{\boldsymbol{\tau}} \\
& F_j(\hat{\boldsymbol{\tau}}_{n+1}, \hat{\mathbf{k}}_{n+1}) \leq 0, \quad j=1, 2, \dots, N_G
\end{aligned} \tag{33}$$

247 The natural boundary condition (44) has been included through the terms  $\mathbf{f}^e$ , whereas the  
248 imposition of essential boundary conditions for the displacements requires the introduction  
249 of a new variable  $\hat{\mathbf{r}}_{n+1}^u$  since the displacement increment  $\Delta \mathbf{u}$  is a field variable for the dual

250 problem of program (33). More specifically, the program turns out to be

$$\begin{aligned}
& \min_{(\hat{\mathbf{g}}, \hat{\mathbf{t}}, \hat{\mathbf{g}}^e, \hat{\mathbf{r}}, \hat{\mathbf{k}}, \hat{\mathbf{r}}^u)_{n+1}} \frac{1}{2} \Delta \hat{\mathbf{g}}^T \mathbf{C} \Delta \hat{\mathbf{g}} + \frac{1}{2} \hat{\mathbf{r}}_{n+1}^T \mathbf{D} \hat{\mathbf{r}}_{n+1} + \frac{1}{2} \Delta \hat{\mathbf{g}}^{eT} \mathbf{M} \Delta \hat{\mathbf{g}}^e \\
& \quad + \frac{1}{2} \Delta \hat{\mathbf{k}}^T \mathbf{H} \Delta \hat{\mathbf{k}} + \Delta \hat{\mathbf{g}}^{eT} \mathbf{f}^c - \underline{(\mathbf{E} \mathbf{U}^d)^T \hat{\mathbf{r}}_{n+1}^u} \\
251 \quad \text{subject to} \quad & \mathbf{B}^T \hat{\mathbf{g}}_{n+1} + \mathbf{A}^T \hat{\mathbf{r}}_{n+1} - \underline{\mathbf{E} \hat{\mathbf{r}}_{n+1}^u} = \mathbf{f}^e - \frac{1-\theta_1}{\theta_1} \mathbf{B}^T \hat{\mathbf{g}}_n \quad (34) \\
& \Delta \hat{\mathbf{g}}^e = \Delta \hat{\mathbf{g}} - \Delta \hat{\mathbf{t}} \\
& F_j(\hat{\mathbf{t}}_{n+1}, \hat{\mathbf{k}}_{n+1}) \leq 0, \quad j=1, 2, \dots, N_G
\end{aligned}$$

252 where the essential boundary condition (44) are enforced,  $\mathbf{E}$  is an index matrix consisting of  
253 entries equal to 0 and 1,  $\mathbf{U}^d$  is a vector consisting of the prescribed displacements at mesh  
254 nodes, and the newly introduced  $\hat{\mathbf{r}}_{n+1}^u$  represents the nodal reaction force. As shown, both the  
255 objective function and the constraints of program (34) are altered (the underlined terms) due  
256 to the imposition of the essential boundary conditions. The validity of the above can be  
257 checked by differentiating the Lagrangian associated with the program (34) with respect to  
258  $\hat{\mathbf{r}}_{n+1}^u$ , resulting in

$$259 \quad \frac{\partial \mathcal{L}}{\partial \hat{\mathbf{r}}_{n+1}^u} = \mathbf{E} \mathbf{U}^d - \mathbf{E} \Delta \hat{\mathbf{u}}_{n+1} = \mathbf{0} \quad (35)$$

260 This is obviously the discretised form of the displacement boundary conditions (45).

261

262 Interaction between a deformable body and a rigid surface can be achieved in a  
263 straightforward manner in the above program according to [35]. The classical Coulomb  
264 model for frictional contact is adopted in this study, which is

$$265 \quad \begin{aligned}
& g_N \geq 0, \quad p \geq 0, \quad p g_N = 0, \\
& |q| - \mu p \leq 0
\end{aligned} \quad (36)$$

266 As shown in Figure 3,  $g_N$  is the gap between the material and the rigid surface,  $p$  is the

267 contact pressure which is positive corresponding to compression,  $q$  is the tangential stress,  
 268 and  $\mu$  is the friction coefficient between the material and the surface. After enforcing the  
 269 conditions in (36) on finite element nodes, the principle reads

$$\begin{aligned}
 & \min_{(\hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\sigma}}^e, \hat{\mathbf{r}}, \hat{\mathbf{k}}, \hat{\mathbf{r}}^u)_{n+1}} \frac{1}{2} \Delta \hat{\boldsymbol{\sigma}}^T \mathbf{C} \Delta \hat{\boldsymbol{\sigma}} + \frac{1}{2} \hat{\mathbf{r}}_{n+1}^T \mathbf{D} \hat{\mathbf{r}}_{n+1} + \frac{1}{2} \Delta \hat{\boldsymbol{\sigma}}^e T \mathbf{M} \Delta \hat{\boldsymbol{\sigma}}^e \\
 & \quad + \frac{1}{2} \Delta \hat{\mathbf{k}}^T \mathbf{H} \Delta \hat{\mathbf{k}} + \Delta \hat{\boldsymbol{\sigma}}^e T \mathbf{f}^c - (\mathbf{E} \mathbf{U}^d)^T \hat{\mathbf{r}}_{n+1}^u + \sum_{j=1}^{n_c} g_{0j} p_j \\
 \text{subject to} \quad & \mathbf{B}^T \hat{\boldsymbol{\sigma}}_{n+1} + \mathbf{A}^T \hat{\mathbf{r}}_{n+1} - \mathbf{E} \hat{\mathbf{r}}_{n+1}^u + \mathbf{E}^c \boldsymbol{\rho} = \mathbf{f}^e - \frac{1-\theta_1}{\theta_1} \mathbf{B}^T \hat{\boldsymbol{\sigma}}_n \\
 270 \quad & \Delta \hat{\boldsymbol{\sigma}}^e = \Delta \hat{\boldsymbol{\sigma}} - \Delta \hat{\boldsymbol{\tau}} \tag{37} \\
 & p_k = -\mathbf{n}^T \boldsymbol{\rho}_k, \quad k = 1, \dots, n_c \\
 & q_k = -\hat{\mathbf{n}}^T \boldsymbol{\rho}_k, \quad |q_k| - \mu p_k \leq 0 \\
 & F_j^*(\hat{\boldsymbol{\tau}}_{n+1}, \hat{\mathbf{k}}_{n+1}) \leq 0, \quad j = 1, 2, \dots, N_G
 \end{aligned}$$

271 where  $\boldsymbol{\rho} = (\rho_1, \rho_2)^T$  are the nodal forces,  $\mathbf{n} = (n_1, n_2)^T$  and  $\hat{\mathbf{n}} = (-n_2, n_1)^T$  are the normal and  
 272 the tangent to the rigid boundary,  $\mathbf{E}^c$  is an index matrix of zeros and ones, and  $n_c$  is the  
 273 number of potential contacts. The above program is the final optimization problem to be  
 274 solved. While it may be solved in a number of ways using either general or specialized  
 275 methods, it is transformed here into a second-order cone program (SOCP) and then resolved  
 276 using the high performance optimization solver MOSEK [4]. The transformation of  
 277 programs of the same type as (37) into a SOCP is straightforward, and has been  
 278 documented in detail in [23, 34]. The main operation is to recast the quadratic terms in the  
 279 objective function to linear ones, subject to a quadratic constraint, and to reform the yield  
 280 function as a cone. Due to the attractive advantages presented in the introduction, a variety  
 281 of mechanics problems have been formulated and solved in such a manner, including  
 282 computational limit analysis of solids and plates [36-38], static/dynamic analysis of



283 elastoplastic frames and solids [21, 35, 39, 40], analysis of steady-state non-Newtonian  
284 fluid flows [41], consolidation analysis [23], and the analysis of granular contact dynamics  
285 [42-44].

286

## 287 5. Particle Finite Element Method

288 The Particle Finite Element Method (PFEM) is a Lagrangian approach capable of handling  
289 general large deformation problems without any real limitation on the magnitude of the  
290 deformation [27, 29, 45, 46]. Its major characteristic is to treat mesh nodes as ‘particles’ that  
291 can move freely, and even separate from, the computational domain to which they originally  
292 belong. The basic steps of the utilized PFEM are summarized (see also Figure 4) in the  
293 following, with more details given in [35]:

- 294 (1) Suppose that we have a cloud of particles,  $C^n$ , at time  $t_n$  ;
- 295 (2) Identify the computational domain using the  $\alpha$ -shape method [47] on the basis of  $C^n$  ;
- 296 (3) Create a finite element mesh,  $M^n$ , through a triangulation of the recognized domain  
297 and discretize governing equations on  $M^n$  ;
- 298 (4) Map the state variables such as stresses, strains, velocities, etc. from the old mesh,  
299  $M^{n-1}$ , to the new mesh,  $M^n$  ;
- 300 (5) Solve the discrete governing equations on the new mesh,  $M^n$ , through a standard  
301 finite element procedure;
- 302 (6) Update the position of mesh nodes to arrive at  $C^{n+1}$  and repeat.

303 To date, a number of challenging problems involving large deformation and free-surface  
304 evolution have been tackled by the PFEM. These include the modelling of granular flows [24,  
305 25, 35, 48, 49], landslides [29, 50], landslide-generated waves [30, 46], multi-fluid flows [51-  
306 53], fluid-structure interaction [27, 54, 55], soil-structure interaction [35, 40], bubble

307 dynamics [56], the melting and spreading of polymers [57], industrial forming processes, and  
308 the flow of fresh cement [58]. In this paper, the solution algorithm for elastoviscoplastic  
309 analysis with strain softening is incorporated into the PFEM for progressive failure analysis  
310 of sensitive clays. It is notable that the governing equations proposed are on the basis of the  
311 infinitesimal strain theory which may lead to several errors for large deformation analysis.  
312 The most serious one is the generation of strains as a result of rigid body motion. However, it  
313 has been shown in [35, 59] that this and related errors are relatively minor when the time  
314 steps used are small. As such, the price to pay for the convenience of being able to operate  
315 with usual infinitesimal strain theory appears to be very small. Indeed, such a strategy has  
316 been verified against analytical solutions for penetration problems [60] and validated  
317 qualitatively as well as quantitatively against both quasi-static and dynamic collapse of a  
318 granular column [24, 25] and the penetration of shallowly embedded pipelines [61].  
319 Furthermore, it succeeds in reproducing a real-world flow-like landslide [29].

320

## 321 6. Numerical Examples

322 This section discusses numerical results for progressive failure analysis of sensitive clays  
323 using the proposed approach. Note that finite element analysis of strain-softening materials  
324 encounters issues of mesh sensitivity when using rate-independent models because the field  
325 equations that describe the motion of the body may lose hyperbolicity. Indeed, the  
326 corresponding boundary-value problem becomes ill-posed, with pathologically mesh-  
327 dependent solutions in which the width of the shear bands depends on the mesh size. The  
328 application of rate-dependent models is an effective way to circumvent this problem. It has  
329 been shown that viscous terms introduce a length scale effect into the initial boundary-value  
330 problem, even the rate-dependent model does not explicitly contain a parameter with the  
331 dimension of length [62, 63]. Consequently, viscoplastic models result in solutions where the

332 shear bands have a finite width when strain localization occurs. It should be noted, however,  
333 that the main objective of this work is to capture the entire failure process in sensitive clays  
334 involving large deformation, rather than to predict the thickness of localized shear bands. As  
335 noted by Moore [64], the typical thickness of a shear band in clay at failure is between 0.01  
336 and 2 cm, and thus it is impractical to predict both the microscopic and macroscopic soil  
337 response using a purely continuum model where a large earth structure is considered. One  
338 possible way of accounting for the responses on both the macro and micro levels is through  
339 the multiscale computational modelling technique [65-68].

340

### 341 6.1 One-dimensional elasto-viscoplastic problem

342 To verify the proposed variational principle, we consider an axial bar subject to a prescribed  
343 load (Figure 5(a)). The material is represented by a one-dimensional elastoviscoplastic model  
344 (Figure 5(b)). If the mass of the bar is sufficiently small, so that any induced inertial forces  
345 are negligible, the load produces a uniform stress and strain along the bar and an analytical  
346 solution is available. In the following, the material parameters of the bar are assumed to be:  
347 Young's modulus  $E = 5 \times 10^4$  Pa, the initial yield stress  $\sigma_{Y_0} = 100$  Pa, and the viscosity  
348 coefficient  $\eta = 1000$  Pa·s.

349 The ability of the proposed formulation to capture the strain-rate dependence of the stress  
350 response and stress relaxation behaviour is examined first. To this end, we set the prescribed  
351 strain increase at a constant rate  $\alpha$  until time  $t^* = 0.4$  s and then hold the strain constant,  
352 leading to stress relaxation. The analytical solution of this problem is available [69] and we  
353 consider three different load rates, namely  $\alpha = 0.2, 0.4,$  and  $0.6$ , respectively, to produce a  
354 rate-dependent response (Figure 6(a)). The yield stress is set to be constant ( $\sigma_Y = \sigma_{Y_0}$ ) in this  
355 case (Figure 6(b)) and the time increment is  $\Delta t = 2 \times 10^{-3}$  s in all simulations. Figure 7

356 illustrates the simulated stress response for different load rates as well as the corresponding  
357 analytical solutions. For all cases, the resulting stresses increase in a stable manner until their  
358 maximum values are reached. A higher load rate results in a larger maximum stress reflecting  
359 the effect of viscosity. At the time  $t=t^*$ , the stresses for all three cases drop sharply  
360 representing stress relaxation behaviour. Eventually, the residual stresses for all cases  
361 asymptote towards the initial stress strength of the material,  $\sigma_{y_0}$ . All the simulated results  
362 agree with the analytical solutions, which verifies the proposed variational formulation and  
363 finite element implementation.

364

365 We now consider the details of strain-softening behaviour. The prescribed strain in this case  
366 increases with a constant rate  $\alpha=0.2$  (Figure 8(a)); however the yield stress strength  $\sigma_y$ ,  
367 which equals  $\sigma_{y_0}=100\text{Pa}$  at the beginning, reduces to its residual value  $\sigma_{yR}=30\%\sigma_{y_0}$  when  
368 the accumulated plastic strain reaches 7% (Figure 8(b)). Such a phenomenon of reduction has  
369 been widely observed for materials undergoing plastic deformation. The initial and residual  
370 yield stress strengths can be interpreted as strengths of a material at undisturbed and  
371 remoulded states. The simulation is conducted using a total of 20, 30, and 40 time increments,  
372 respectively, and again the agreement between the numerical and analytical solutions is  
373 satisfactory (Figure 9).

374

## 375 6.2 Collapse of a sensitive clay column

376 As the second example, we consider the collapse of a sensitive clay column (Figure 10) in a  
377 container which is 50 cm wide and 100 cm high. The container is lifted up quickly leading to  
378 the spread of the sensitive clay. Such an experimental test has been widely used for  
379 investigating the behaviour of granular matter [70-74], but has also been adopted for studying

380 the quickness of sensitive clays [1]. Here, the problem is considered to deform under plane-  
381 strain conditions and only half of the geometry is modelled due to the symmetry. The  
382 material parameters are as follows: Young's modulus  $E = 5 \times 10^6$  Pa, Poisson's ratio  $\nu = 0.49$ ,  
383 density  $\rho = 1.8 \times 10^3$  kg/m<sup>3</sup>, viscosity coefficient  $\eta = 100$  Pa·s, undisturbed shear strength  
384  $c_{up} = 5$  kPa, remoulded shear strength  $c_{ur} = 1$  kPa, and  $\bar{\kappa} = 25\%$ . The frictional coefficient  
385 between the clay and the rigid surface is taken as 0.3 and the gravitational acceleration  
386  $g = -9.8$  m/s<sup>2</sup>. The column is discretized using 7,962 6-node triangular elements with 16,199  
387 nodes, and the time step utilized is  $\Delta t = 0.01$  s.

388

389 The collapse procedure of the column obtained from the simulation is illustrated in Figure 11,  
390 in which the colour is proportional to the accumulated equivalent plastic strain. The

391 normalized time  $\bar{t}$  refers to  $\frac{t}{\sqrt{2h_0/g}}$  with  $h_0$  being the initial height of the column. For

392 initially undisturbed sensitive clays, lifting the container results in two shear bands dividing  
393 the column into three parts (Figure 11(a)). The upper part moves downward while the middle

394 part, which is in the shape of a triangle, is pushed out horizontally. After a considerable  
395 movement of the middle part, a shear band is formed in the lower part (Figure 11(b)) and then

396 one more shear band appears in the upper part (Figure 11(c)). The second shear band in the  
397 upper part deforms another layer of sensitive clay and the lower part, which was intact, is

398 disturbed significantly because of the shear band formed (Figure 11(d)). Further collapse of  
399 the column leads to two more layers being squeezed out (Figure 11(e) and (f)). Localized

400 shear bands can be observed clearly in the final deposit, with some parts of the column  
401 remaining undisturbed throughout the failure process. The collapse of remoulded sensitive

402 clay is also simulated for comparison (Figure 11). As shown, the collapse mechanism for this  
403 case is quite different to the previous case of an undisturbed sample. Rather than fail

404 progressively, nearly all the material experiences plastic deformation with the material near  
405 the bottom possessing the maximum equivalent plastic strain. Figure 12 shows the curves of  
406 the front location and centre height against normalized time for columns of both initially  
407 undisturbed and remoulded sensitive clays. The collapse of the column of remoulded clay  
408 results in a final deposit with a much smaller height and considerably larger length. Both the  
409 sensitive and remoulded clay columns reach their maximum run-out distance at around  
410  $\bar{t} = 2.15$ . However, the final centre height for the remoulded case is obtained earlier ( $\bar{t} = 1.4$ )  
411 than that for the initially undisturbed case ( $\bar{t} = 2.15$ ). Notably, the final centre height and  
412 length are obtained simultaneously for the undisturbed sample, which differs from that for the  
413 remoulded sample. The final mesh topology for both cases, illustrated in Figure 13, verifies  
414 that the proposed approach can handle the extreme mesh distortion that accompanies failure  
415 for this problem. A video of the collapse of both the remoulded and undisturbed sensitive  
416 clay columns is provided in the supplementary materials.

417

418 To estimate the mesh sensitivity, the collapse of the initially undisturbed clay was also re-  
419 analyzed using three different mesh sizes, where the length of the element edge was set to  
420  $h = 1.0$  cm (1,934 triangles),  $0.75$  cm (3,528 triangles) and  $0.5$  cm (7,962 triangles). The  
421 cases are referred to as coarse, medium, and fine meshes. As shown in Figure 14, the curves  
422 of locations against time for all three tests agree well with each other. Moreover, all three  
423 simulations result in very similar final deposits and shear bands (Figure 15), which proves  
424 that a further decrease in the mesh size will not alter the form of the predicted failure mode.

425

### 426 6.3 Retrogressive collapse of a slope in sensitive clay

427 As observed in Scandinavia and eastern Canada, a fast and significant retrogressive collapse  
428 of a slope in sensitive clay may be triggered by a small initial slide [2]. To illustrate the

429 ability of the proposed approach for modelling such a geohazard, we consider the sensitive  
430 clay deposit shown in Figure 16. Here, collapse is triggered by removing a rigid triangular  
431 block at the toe of the slope (which may be caused by erosion or excavation). The material  
432 parameters of the sensitive clay are as follows: Young's modulus  $E = 5 \times 10^6$  Pa, Poisson's  
433 ratio  $\nu = 0.33$ , density  $\rho = 1.8 \times 10^3$  kg/m<sup>3</sup>, viscosity coefficient  $\eta = 100$  Pa·s, undisturbed  
434 shear strength  $c_{uu} = 22$  kPa, remoulded shear strength  $c_{ur} = 1.2$  kPa, and  $\bar{\kappa} = 25\%$ . The  
435 frictional coefficient between the sensitive clay and the rigid bottom surface is set to 0.1 and  
436 the gravitational acceleration  $g = -9.8$  m/s<sup>2</sup>. A total of 18,420 6-noded triangular elements  
437 (37,355 mesh nodes) is used to discretize the initial computational domain. The time step is  
438  $\Delta t = 0.025$  s and the simulation proceeds until the final deposit is obtained.

439

440 The retrogressive failure process from the simulation is illustrated in Figure 17, with the  
441 colour being proportional to the accumulated equivalent plastic strain. As illustrated, the  
442 erosion leads to the first retrogressive collapse C1 in the slope (Figure 17(b)). Two shear  
443 bands initiate from the bottom and propagate towards the top surface and the front inclined  
444 surface, respectively, resulting in a graben. During the sliding, one more shear band is  
445 generated in the graben dividing it into two elastic parts (Figure 17(c) and (d)). As the  
446 disturbed mass due to collapse C1 moves far away from the new slip surface, the second  
447 retrogressive collapse C2 occurs (Figure 17(e) and (f)). This mimics the first criterion for the  
448 occurrence of retrogressive failure, that the slide debris should be able to flow out of the slide  
449 area [1]. The same as that in C1, two plastic shear bands, also originating from the slope base,  
450 are formed in C2 which results in a graben and a horst. The mass in front of the new slope  
451 surface continues to move forward leading to the third retrogressive failure of the slope, C3  
452 (Figure 17(g)). After this, a considerable amount of mass is deposited in front of the new  
453 slope surface, resisting further collapse (Figure 17(h)). The final configuration of the slope,

454 shown in Figure 17(i), indicates that most of the clay involved in C2 and C3 has been  
455 remoulded. This reflects the other criterion for the occurrence of retrogressive failure, which  
456 states that the slide debris should be completely remoulded [1]. Eventually, the retrogressive  
457 failure results in a deposit with a run-out distance of 28.71 m and a retrogression distance of  
458 14.76 m (Figure 17(i)). A video of the entire failure process of the slope is available in the  
459 supplementary materials.

460

461 The velocity of the sliding front and the maximum velocity were also recorded and are  
462 depicted in Figure 18. As illustrated, the maximum velocity is not always located at the  
463 sliding front. This can be explained by examining the velocity contour (Figure 19). The first  
464 retrogressive failure results in the transformation of a part of the gravitational potential  
465 energy of the mass into kinetic energy, with the remaining energy being dissipated by plastic  
466 shearing (Figure 19(a)). The sliding front thus possesses the maximum velocity due to this  
467 transformation. The second retrogressive collapse further releases potential gravitational  
468 energy (Figure 19(b) and pushes the materials in front of it, consequently increasing the  
469 velocity of the corresponding mass. In contrast, the velocity of the sliding front decreases  
470 because of the friction along the basal surface and the effects of plastic dissipation. After a  
471 while, the mass at the middle moves faster than the sliding front does as shown in Figure  
472 19(c). The third retrogressive collapse further releases potential gravitational energy (Figure  
473 19(d)). The velocity of the involved mass in collapse C3 is relatively low, however, because a  
474 considerable body of clay with low velocity is located in front of the new slope surface. Note  
475 that the sliding front already ceased at  $t = 11.4$  s (Figure 19(e)). However, the clay at the  
476 middle of the sliding mass continues to be pushed and moves forward, which eventually  
477 disturbs the sliding front (Figure 19(f) and (g)).

478



479 The value of viscosity, back-calculated from various subaerial and submarine slides by  
480 Edgers and Karlsrud [75] and Johnson and Rodine [76], is in the range of 100 to 1499 Pa·s .  
481 We here investigate the effect of the viscosity on the retrogressive failure by analysing the  
482 problem with varying viscosity coefficients of  $\eta = 1 \text{ Pa}\cdot\text{s}$ ,  $10 \text{ Pa}\cdot\text{s}$ ,  $100 \text{ Pa}\cdot\text{s}$ , and  $1000$   
483  $\text{Pa}\cdot\text{s}$ . Other material parameters for these simulations are the same as in the previous case.  
484 As shown in Figure 20, retrogressive failure occurs four times for both  $\eta = 1 \text{ Pa}\cdot\text{s}$  and  $10$   
485  $\text{Pa}\cdot\text{s}$ . Thus, the retrogression distances for these two cases, 20.25 m and 20.13 m, are very  
486 close (Figure 21), although the run-out distance for  $\eta = 1 \text{ Pa}\cdot\text{s}$  (38.18 m) is slightly larger  
487 than that for  $\eta = 10 \text{ Pa}\cdot\text{s}$  (36.73 m). When  $\eta$  is increased further, fewer retrogressive  
488 failures are induced: for example three times for  $\eta = 100 \text{ Pa}\cdot\text{s}$  and only twice for  $\eta = 1000$   
489  $\text{Pa}\cdot\text{s}$  (Figure 20). This means that an increase of  $\eta$  ( $\eta > 10 \text{ Pa}\cdot\text{s}$ ) causes a decrease in the  
490 run-out and retrogressive distance, as illustrated in Figure 21, because a higher viscosity  
491 results in much more plastic dissipation and, consequently, less potential gravitational energy  
492 is converted into kinetic energy. Such a decrease in energy transformation not only leads to a  
493 smaller run-out distance, but also causes more clay to be deposited in front of the newly  
494 formed slope surface which resists the occurrence of further retrogressive failure.

495

## 496 7. Conclusions

497 The progressive failure process of sensitive clays is simulated using the Particle Finite  
498 Element Method with an advanced elastoviscoplastic model which is a combination of the  
499 Bingham model (for describing rheological behaviour) and the Tresca model with strain-  
500 softening (for capturing progressive failure behaviour). The resulting elastoviscoplastic  
501 analysis is reformulated as an optimization problem on the basis of a mixed variational  
502 principle and resolved in mathematical programming.

503

504 The proposed formulation is verified against the analytical solution of a one-dimensional  
505 elastoviscoplastic problem. The capability of the proposed computational approach for  
506 modelling progressive failure is illustrated by simulating the collapse of a column of sensitive  
507 clay. Additionally, the retrogressive failure of a slope in sensitive clay is reproduced  
508 successfully. The simulation results reflect the essential conditions for the occurrence of  
509 retrogressive collapse which are that the slide debris should be fully remoulded and flow  
510 away from the slide area. Furthermore, the effect of the viscosity of a sensitive clay on the  
511 nature of retrogressive collapse is also studied. Numerical results show that an increase of  
512 viscosity leads to a decrease in both the run-out distance and the retrogression distance due to  
513 the dissipation of a large amount of energy.

514

515 Although the problem in this study is simulated under undrained conditions using total  
516 stresses, an effective stress analysis can also be performed. This can be achieved by merging  
517 the SOCP formulation for consolidation analysis of saturated porous media introduced in [23],  
518 where rate-independent models are utilised, with the mixed variational principle presented in  
519 this paper. The resulting formulation then can be incorporated into the particle finite element  
520 method for investigating the hydro-mechanical mechanism in the progressive failure of  
521 sensitive clays.

522

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526

527 **Appendix A.** Time discretization

528 The momentum conservation equation (1) is first discretized in time using the standard  $\theta$ -  
 529 method as:

$$530 \quad \nabla^T [\theta_1 \boldsymbol{\sigma}_{n+1} + (1 - \theta_1) \boldsymbol{\sigma}_n] + \mathbf{b} = \rho \frac{\mathbf{v}_{n+1} - \mathbf{v}_n}{\Delta t} \quad (38)$$

$$531 \quad \theta_2 \mathbf{v}_{n+1} + (1 - \theta_2) \mathbf{v}_n = \frac{\mathbf{u}_{n+1} - \mathbf{u}_n}{\Delta t} \quad (39)$$

532 where  $\mathbf{v}$  are velocities,  $\theta_1$  and  $\theta_2$  are parameters taking values in  $[0, 1]$ , the subscripts  $n$  and  
 533  $n+1$  refer to the known and new, unknown states, and  $\Delta t = t_{n+1} - t_n$  is the time step.

534 Rearranging the above equations leads to

$$535 \quad \nabla^T \boldsymbol{\sigma}_{n+1} + \frac{1 - \theta_1}{\theta_1} \nabla^T \boldsymbol{\sigma}_n + \tilde{\mathbf{b}} = \tilde{\rho} \frac{\Delta \mathbf{u}}{\Delta t^2} \quad (40)$$

$$536 \quad \mathbf{v}_{n+1} = \frac{1}{\theta_2} \left[ \frac{\Delta \mathbf{u}}{\Delta t} - (1 - \theta_2) \mathbf{v}_n \right] \quad (41)$$

537 with the displacement increments  $\Delta \mathbf{u} = \mathbf{u}_{n+1} - \mathbf{u}_n$  and

$$538 \quad \tilde{\rho} = \frac{\rho}{\theta_1 \theta_2} \quad (42)$$

$$539 \quad \tilde{\mathbf{b}} = \frac{1}{\theta_1} \mathbf{b} + \tilde{\rho} \frac{\mathbf{v}_n}{\Delta t} \quad (43)$$

540 The natural boundary condition is approximated in an analogous manner leading to

$$541 \quad \mathbf{N}^T (\boldsymbol{\sigma}_{n+1} + \frac{1 - \theta_1}{\theta_1} \boldsymbol{\sigma}_n) = \tilde{\mathbf{t}} \quad \text{on } \Gamma_t \quad \text{with } \tilde{\mathbf{t}} = \frac{1}{\theta_1} \bar{\mathbf{t}} \quad (44)$$

542 and the discretised essential boundary condition is

$$543 \quad \mathbf{u}_{n+1} = \bar{\mathbf{u}}_{n+1} \quad \text{on } \Gamma_u \quad (45)$$

544 By introducing another parameter  $\theta_3 \in [0, 1]$ , the incremental equations of the  
 545 elastoviscoplastic model (Eqs. (6)-(10)) are

546 
$$\Delta \boldsymbol{\varepsilon} = \Delta \boldsymbol{\varepsilon}^e + \Delta \boldsymbol{\varepsilon}^{vp} \quad (46)$$

547 
$$\Delta \boldsymbol{\varepsilon}^e = \mathbf{C} \Delta \boldsymbol{\sigma} \quad (47)$$

548 
$$(\boldsymbol{\sigma}_n + \theta_3 \Delta \boldsymbol{\sigma}) - (\boldsymbol{\tau}_n + \theta_3 \Delta \boldsymbol{\tau}) = \eta \frac{\Delta \boldsymbol{\varepsilon}^{vp}}{\Delta t} \Rightarrow (\Delta \boldsymbol{\sigma} - \Delta \boldsymbol{\tau}) + \frac{1}{\theta_3} (\boldsymbol{\sigma}_n - \boldsymbol{\tau}_n) = \frac{\eta}{\theta_3 \Delta t} \Delta \boldsymbol{\varepsilon}^{vp} \quad (48)$$

549 
$$\Delta \boldsymbol{\varepsilon}^{vp} = \Delta \lambda \nabla_G F(\boldsymbol{\tau}_{n+1}) \quad (49)$$

550 
$$F(\boldsymbol{\tau}_{n+1}) \leq 0; \Delta \lambda \geq 0; \Delta \lambda F(\boldsymbol{\tau}_{n+1}) = 0 \quad (50)$$

551 In summary, the governing equations for incremental analysis of elastoviscoplasticity consist  
 552 of Eqs. (40), (41), (44)-(50).

553

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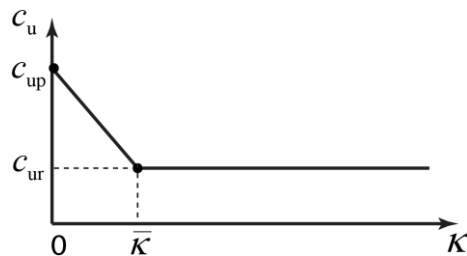
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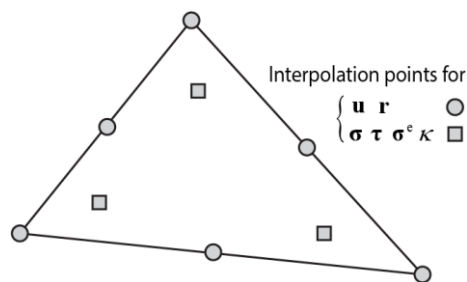


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762 Figure 1. Variation of  $c_u$  with deviatoric plastic strain represented by parameter  $\kappa$

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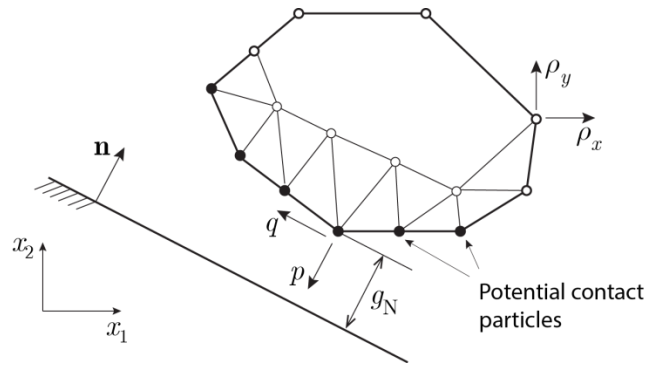


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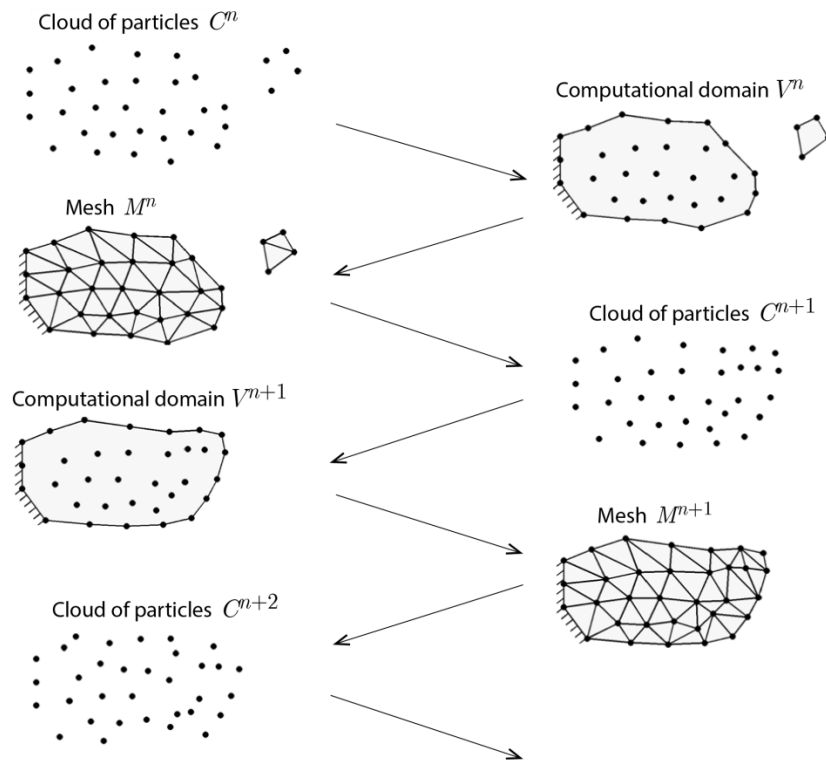
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Figure 2. The mixed triangular element used in the simulation



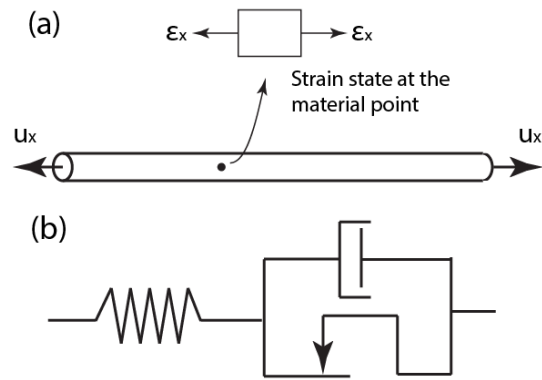
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Figure 3. The contact between a deformable body and a rigid surface



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Figure 4. Steps for the Particle Finite Element Method (after [35])



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782 Figure 5. (a) Schematic diagram of a bar subject to uniaxial loads; and (b) one-dimensional  
 783 elastoviscoplastic model.

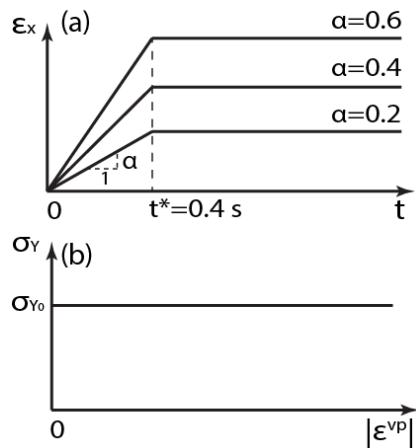
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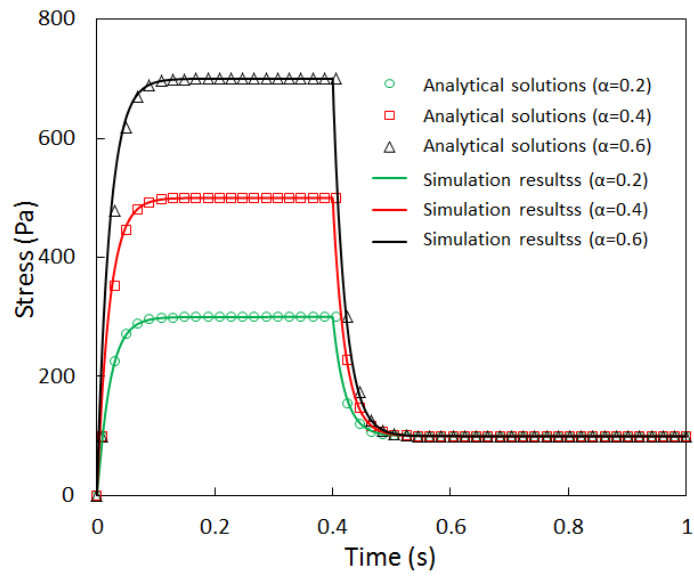
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790 Figure 6. Curves of (a) the applied strain rate and (b) the variation of the yield stress for the  
 791 one-dimensional stress relaxation problem

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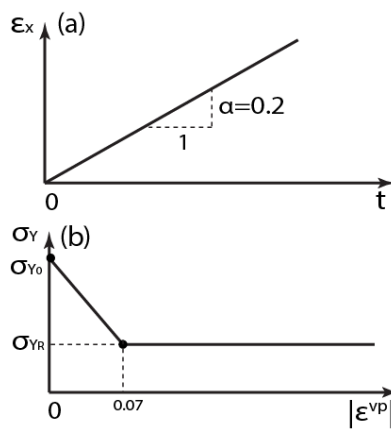
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796 Figure 7. Comparison of numerical and analytical solutions for the one-dimensional stress  
797 relaxation problem



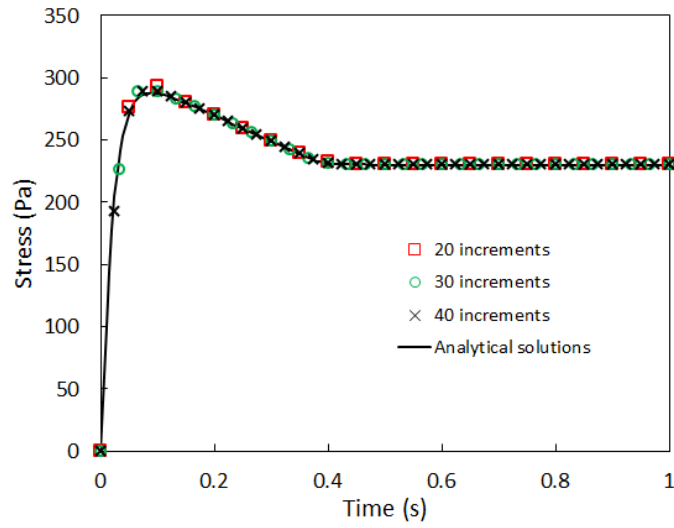
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799 Figure 8. Curves of (a) the applied strain rate and (b) the variation of the yield stress for the  
800 one-dimensional strain-softening problem

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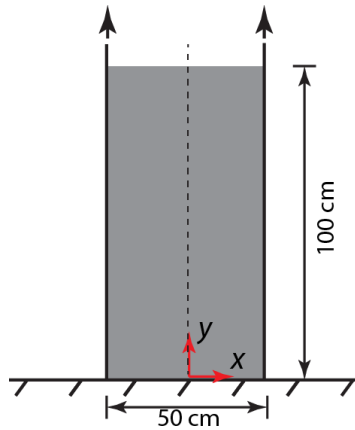


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805 Figure 9. Comparison of numerical and analytical solutions for the one-dimensional  
 806 elastoviscoplastic problem with strain softening

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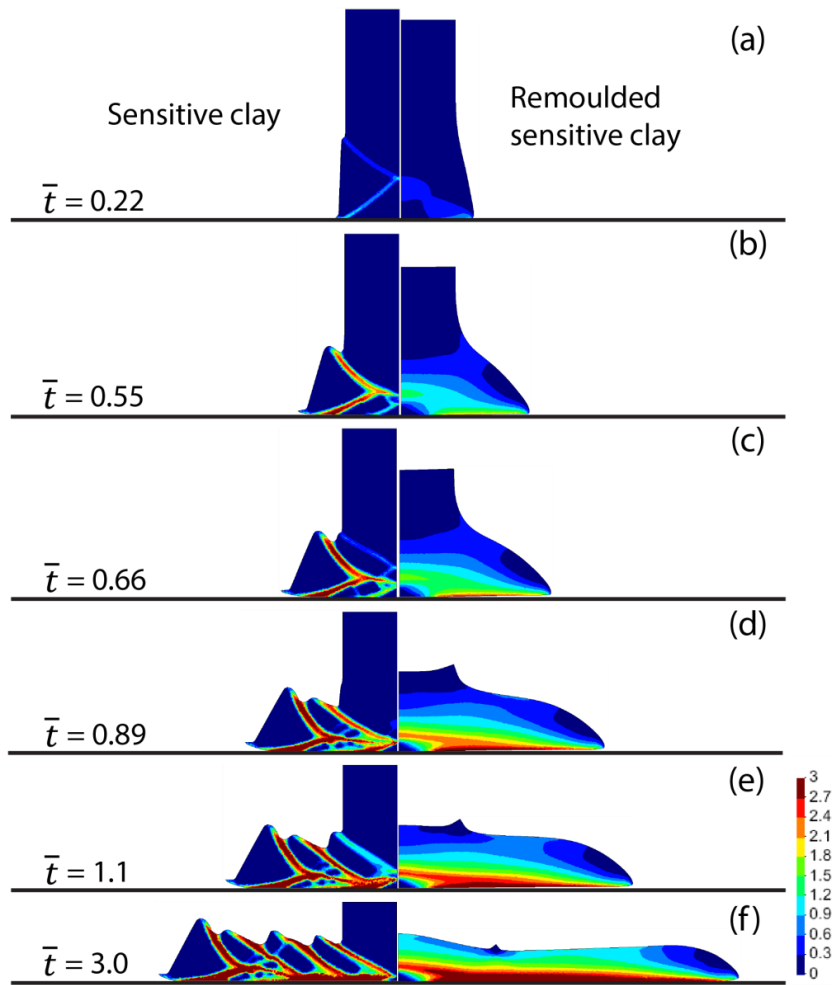


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810 Figure 10. Schematic diagram for the collapse of a column of sensitive clays

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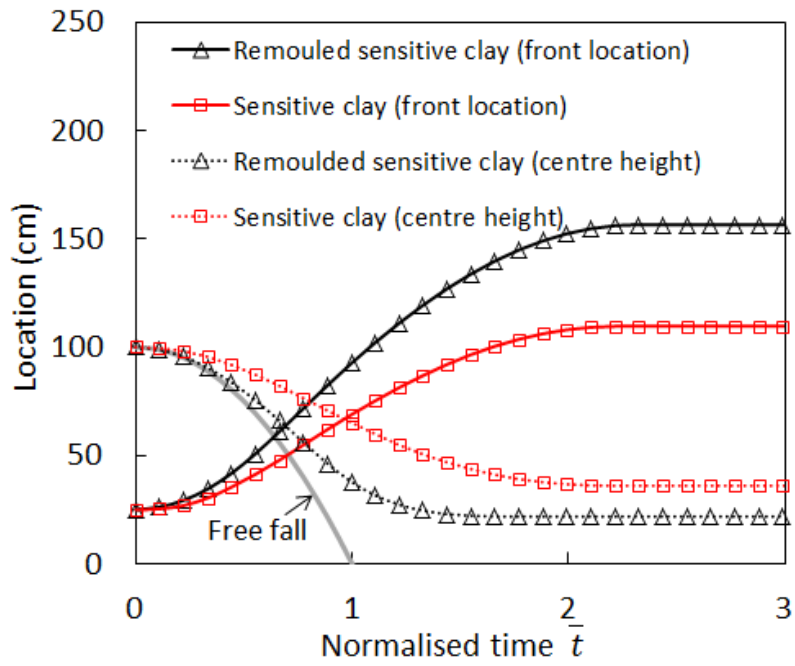
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814 Figure 11. Collapse evolution processes of the column of initially undisturbed sensitive clays  
 815 and remoulded sensitive clays. Colours are proportional to accumulated equivalent plastic  
 816 strain

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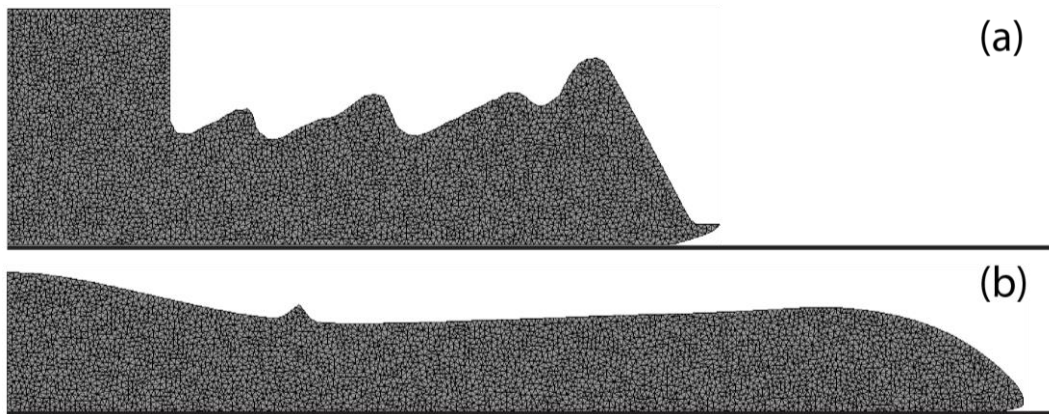
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819 Figure 12. Curves of the front location and the centre height of the column against  
820 normalized time

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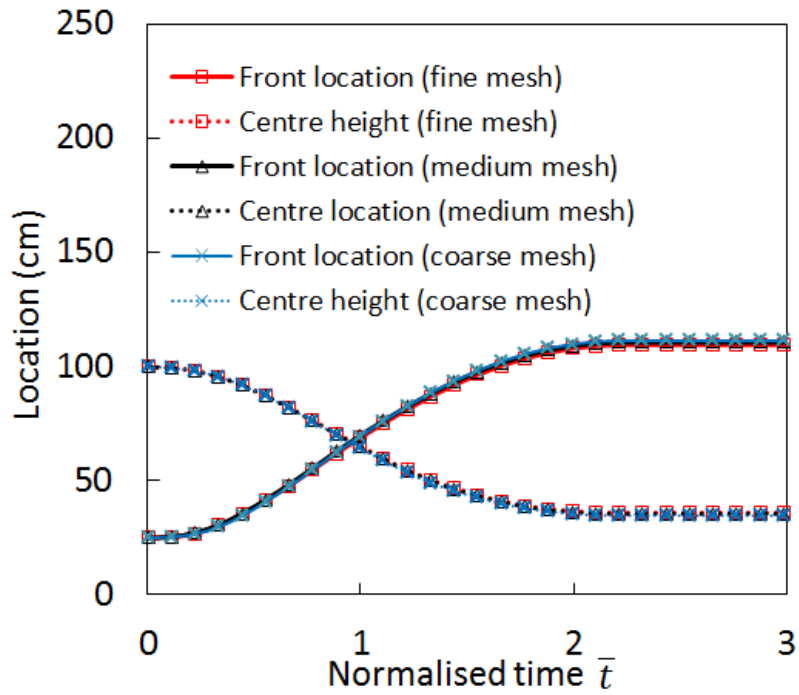
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Figure 13. Final configuration with mesh topology illustrated



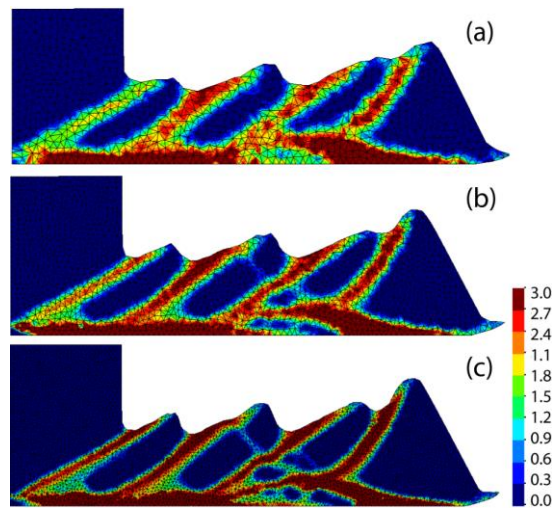
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827 Figure 14. Effects of the utilized mesh size on the curves for front location and centre height  
 828 of the column against normalized time

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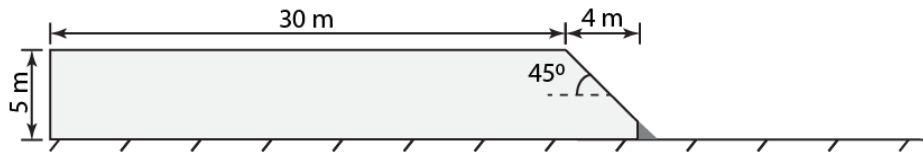
833 Figure 15. Final configurations and shear band distributions of the column collapse using (a)  
 834 coarse meshes, (b) medium meshes, and (c) fine meshes. Colours are proportional to the  
 835 accumulated equivalent plastic strain

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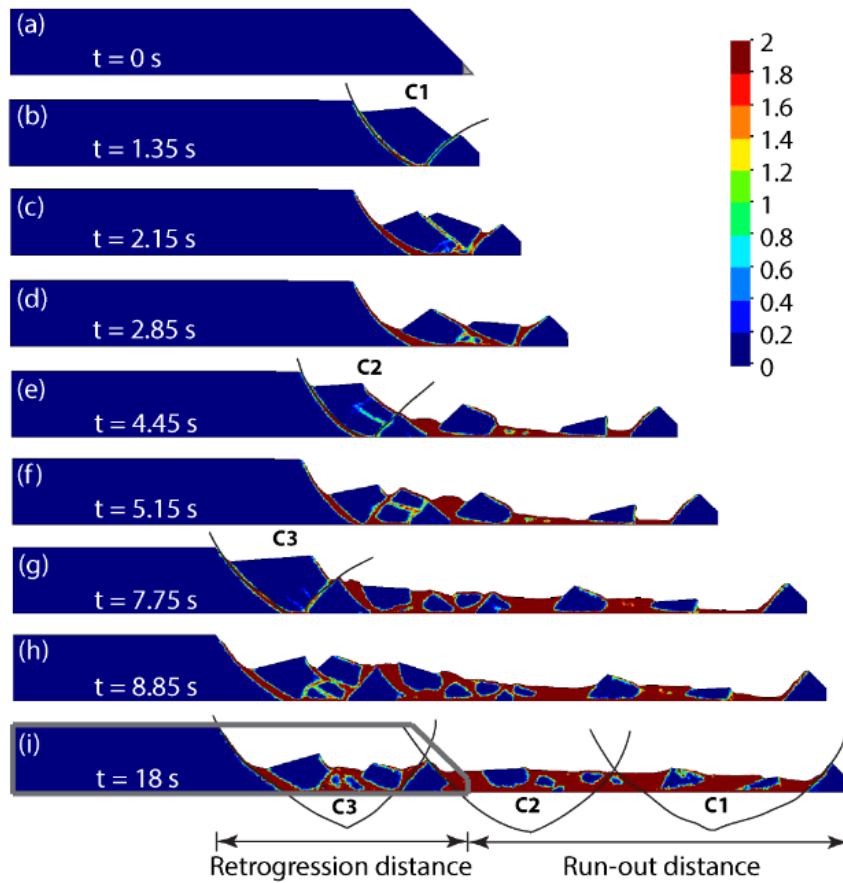


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841 Figure 16. Schematic diagram for the retrogressive failure of a slope in sensitive clays.

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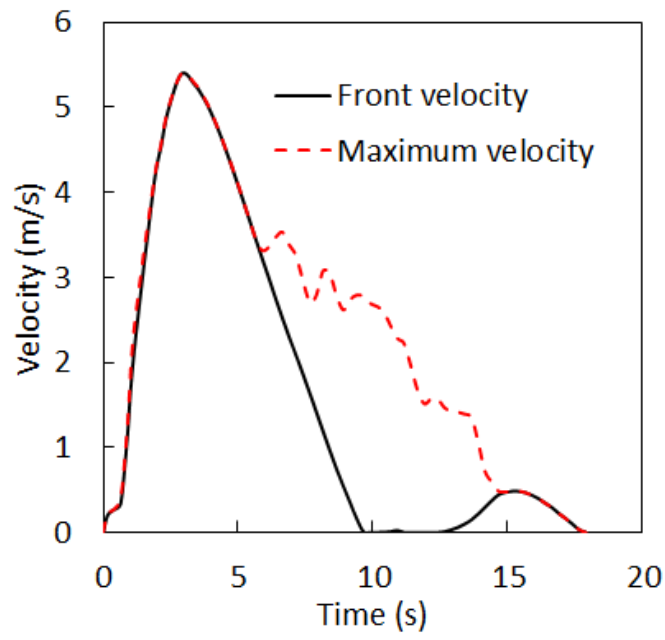


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845 Figure 17. Retrogressive failure procedures of the slope. Colours are proportional to  
846 accumulated equivalent plastic strain

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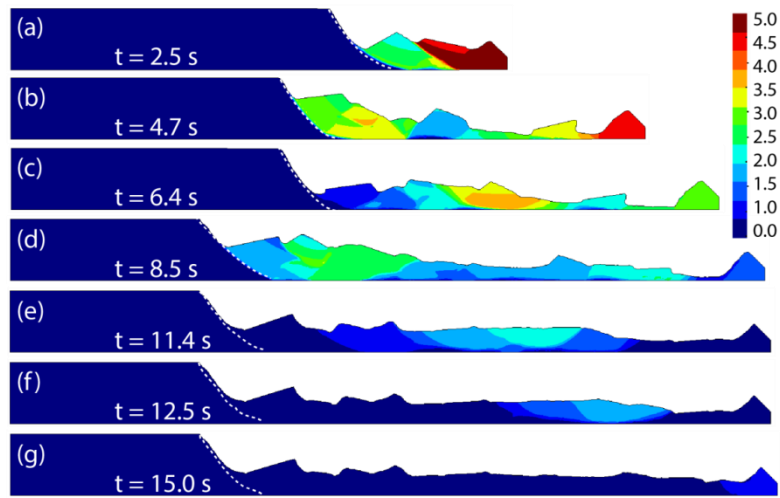


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849 Figure 18. Velocity of the sliding front and the maximum velocity against time

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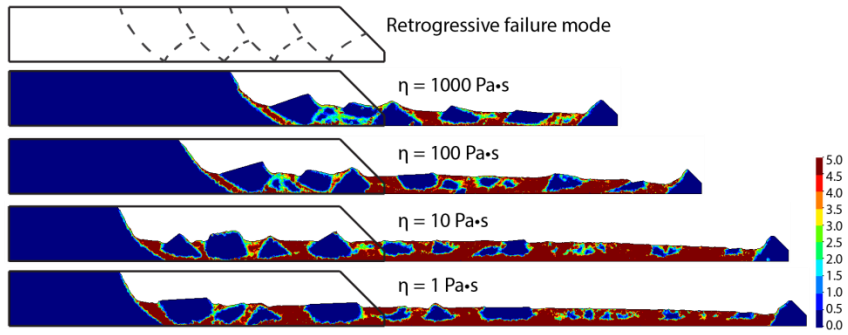
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853 Figure 19. Velocity contour during the retrogressive failure

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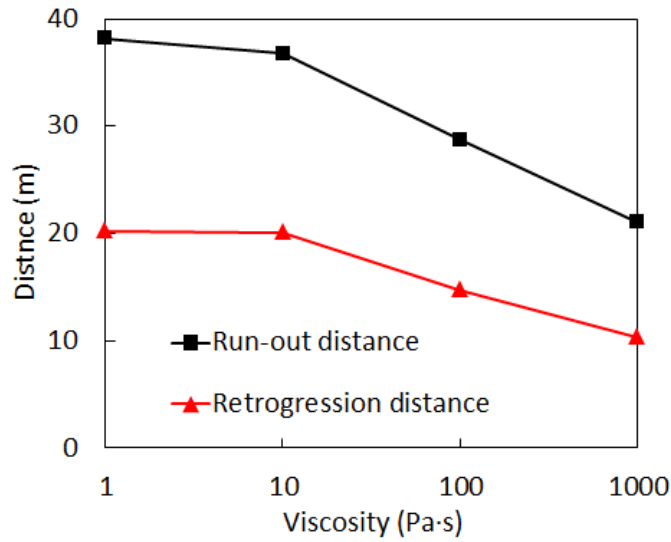
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858 Figure 20. Final deposits from the simulation using different viscosity coefficients for  
 859 sensitive clays

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864 Figure 21. Curves of run-out distance and retrogression distance against viscosity