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Stress-gradient materials: an analytical exploration

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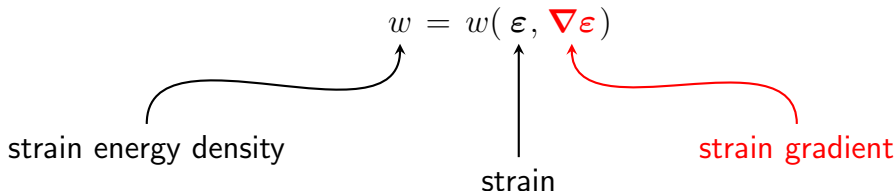
Strain gradient elasticity theory

$$w = w(\boldsymbol{\varepsilon}, \nabla \boldsymbol{\varepsilon})$$

strain energy density

strain

strain gradient



Strain gradient elasticity theory

$$w = w(\boldsymbol{\varepsilon}, \nabla \boldsymbol{\varepsilon})$$

Stress gradient elasticity theory

$$w^* = w^*(\boldsymbol{\sigma}, \nabla \boldsymbol{\sigma})$$

stress energy density

stress

stress gradient



Motivation (3)

At first glance...

Strain gradient elasticity theory

$$w = w(\boldsymbol{\varepsilon}, \nabla \boldsymbol{\varepsilon})$$

Stress gradient elasticity theory

$$w^* = w^*(\boldsymbol{\sigma}, \nabla \boldsymbol{\sigma})$$

Is stress gradient elasticity **equivalent** or **complementary** to strain gradient elasticity?



Objective

Outline

- ? How does stress gradient elasticity theory (Forest and Sab, 2012) differ from strain gradient one?
- Stress gradient elasticity theory (Forest and Sab, 2012).
- Closed form solution to Eshelby spherical inhomogeneity problem.
- Homogenization of heterogeneous stress-gradient materials (Mori Tanaka estimate)

Decomposition of stress gradient tensor

The spherical and deviatoric part of a third order tensor

Strain gradient tensor

- "free"

Stress gradient tensor

- constrained

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0 \Leftrightarrow (\nabla \boldsymbol{\sigma}) : \boldsymbol{\delta} + \mathbf{b} = 0$$

Decomposition of stress gradient tensor

$$\nabla \boldsymbol{\sigma} = \mathbf{J} \therefore \nabla \boldsymbol{\sigma} + \mathbf{K} \therefore \nabla \boldsymbol{\sigma}$$

spherical projector

deviatoric projector

$$J_{ijklmn} = \frac{1}{8} (\delta_{ik}\delta_{jl}\delta_{mn} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jk}\delta_{mn} + \delta_{im}\delta_{jk}\delta_{ln})$$

$$K_{ijklmn} = I_{ijklmn} - J_{ijklmn}$$



Stress gradient elasticity theory

Linear, isotropic stress gradient materials

Stress energy density function

$$w^* = w^*(\boldsymbol{\sigma}, \underbrace{\mathbf{K} \cdot \cdot \nabla \boldsymbol{\sigma}}_{\mathbf{R}})$$

For isotropic stress gradient materials:

$$w^*(\boldsymbol{\sigma}, \mathbf{R}) = \frac{1}{2} \boldsymbol{\sigma} : \mathbf{S} : \boldsymbol{\sigma} + \frac{1}{2} \mathbf{R} \cdot \cdot \mathbf{M} \cdot \cdot \mathbf{R}$$

Constitutive laws

$$\mathbf{e} = \frac{\partial w^*}{\partial \boldsymbol{\sigma}} = \mathbf{S} : \boldsymbol{\sigma}, \quad \boldsymbol{\Phi} = \frac{\partial w^*}{\partial \mathbf{R}} = \mathbf{M} \cdot \cdot \mathbf{R}$$

NOTA: $\boldsymbol{\Phi}, \mathbf{R}$ have to be deviatoric tensors!

$$\mathbf{M} = \mathbf{K} \cdot \cdot \mathbf{M} \cdot \cdot \mathbf{K}$$



Simplified stress / strain gradient elasticity

A comparison of field equations (no body force)

	Simplified stress gradient	Simplified strain gradient
Equilibrium	$\sigma_{ij,j} = 0$	$\tau_{ij,j} = 0$
Gradient	$R_{ijk} = \sigma_{ij,k}$	$\kappa_{ijk} = \varepsilon_{ij,k}$
Constitutive laws	$e_{ij} = S_{ijkl}\sigma_{kl}$ $e_{ij} = \varepsilon_{ij} + \Phi_{ijk,k}$ $\Phi_{ijk} = \ell^2 K_{ijkpqr} S_{pqmn} R_{mnr}$	$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ $\tau_{ij} = \sigma_{ij} - \mu_{ijk,k}$ $\mu_{ijk} = \ell^2 C_{ijmn}\kappa_{mnk}$
SUBC	$\sigma_{ij} = \Sigma_{ij}$	$\sigma_{ij}n_j - (\mu_{ijk}n_k)_{,j} + (\mu_{ijk}n_k n_l)_{,l}n_j = t_i$ <p>on $\partial\Omega$ (smooth part)</p> $\mu_{ijk}n_j n_k = q_i$ on $\partial\Omega$ (smooth part) $[[\mu_{ijk}m_j n_k]] = 0$ on the edge of $\partial\Omega$

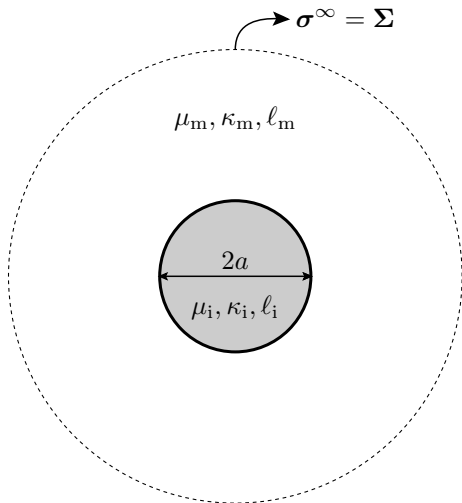
S.Forest, K. Sab Mechanics Research Communications, 40, pp. 16–25, 2012

Altan, B.S., Aifantis, E.C., Journal of the Mechanical Behavior of Materials, 8(3), pp. 231–282, 1997

Gao, X.-L., Park, S.K. International Journal of Solids and Structures, 44, pp. 7486–7499, 2007

Eshelby's spherical inhomogeneity problem (1)

Geometry + mechanical properties



Stress gradient (this work):

- Uniform isotropic stress:

$$\Sigma = \mathbf{e}_x \otimes \mathbf{e}_x + \mathbf{e}_y \otimes \mathbf{e}_y + \mathbf{e}_z \otimes \mathbf{e}_z$$

- Uniform axial stress:

$$\Sigma = \mathbf{e}_z \otimes \mathbf{e}_z$$

Strain gradient:

- Gao, X.-L., Ma, H.M Acta Mechanica, 207 (3), pp. 163-181, 2009
- Gao, X.-L., Ma, H.M Journal of the Mechanics and Physics of Solids, 58 (5), pp. 779-797, 2010

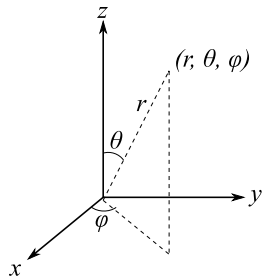
Postulated stress form in spherical coordinates (similar to Love's approach)

- Uniform isotropic stress prescribed at boundary:

$$\sigma = \sigma(r)$$

- Uniform axial stress prescribed at boundary:

$$\sigma = \sigma(r, \theta)$$



Continuity conditions at the surface matrix-inclusion

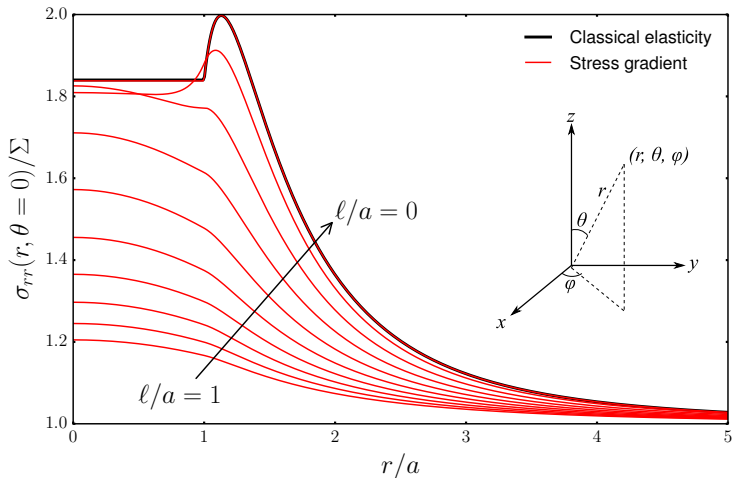
$$\sigma_m(r = a) = \sigma_i(r = a)$$

$$[\mathbf{u} \otimes^s \mathbf{n} + \Phi \cdot \mathbf{n}]_m (r = a) = [\mathbf{u} \otimes^s \mathbf{n} + \Phi \cdot \mathbf{n}]_i (r = a)$$

Uniaxial problem $\Sigma = \mathbf{e}_z \otimes \mathbf{e}_z$

Radial stress field $\sigma_{rr}(r, \theta = 0)$

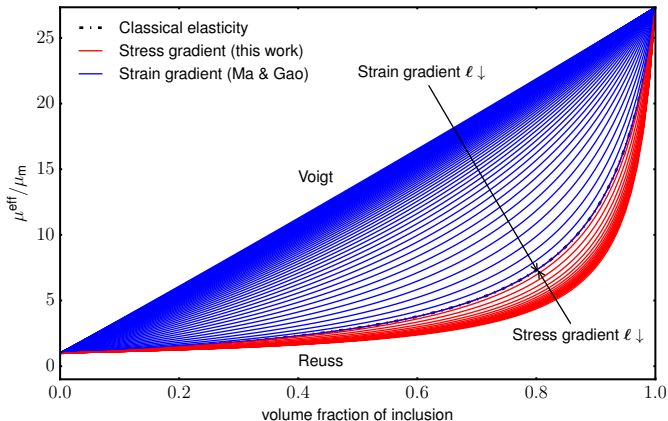
$$\mu_i = 27\mu_m, \nu_i = 0.4, \nu_m = 0.08, \ell_i = \ell_m = \ell$$



Effect of material internal length (1)

Effective shear moduli (Mori Tanaka estimate)

$$\mu_i = 27\mu_m, \nu_i = 0.4, \nu_m = 0.08, \ell_i = \ell_m = \ell$$



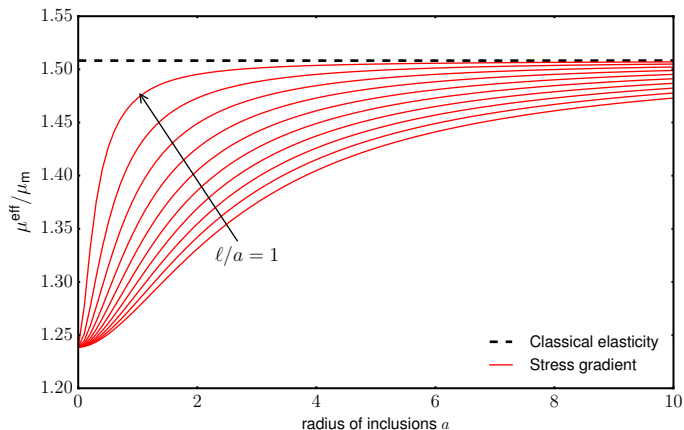
Stress gradient is complementary to strain gradient elasticity!

Ma, H.M, Gao, X.-L. Acta Mechanica, 225 (4-5), pp. 1075-1091, 2014

Effect of material internal length (2)

Effective shear moduli: $f = 20\%$

$$\mu_i = 27\mu_m, \nu_i = 0.4, \nu_m = 0.08, \ell_i = \ell_m = \ell$$



- **Stress gradient elasticity: capture size effects.**
- **For $a \gg \ell$, stress gradient effects vanish as expected.**



Softening size effect

... has been reported elsewhere (molecular dynamics simulations)!

To cite some:

- Polyimide matrix "*reinforced*" by silica spherical nanoparticles with various surface treatment.

Odegard, G.M., Clancy, T.C., Gates, T.S. *Polymer*, 46, pp.553–562, 2005

- Polystyrene matrix "*reinforced*" by silica spherical nanoparticles.

Davydov, D., et al *Soft Materials*, 12, S142–S151, 2014

- etc



Conclusion

- Solution to Eshelby's spherical inhomogeneity problem
- Mori Tanaka estimate for stress gradient materials
- Stress gradient elasticity differs from strain gradient elasticity
 - Strain gradient: stiffening size effect
 - Stress gradient: softening size effect

Perspective

- Numerical implementation: FEM, FFT
- Physical interpretation of material internal length?

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Thank you for your attention!

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