



Introduction to modelization of thick and heterogeneous plates

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Introduction to modelization of thick and heterogeneous plates

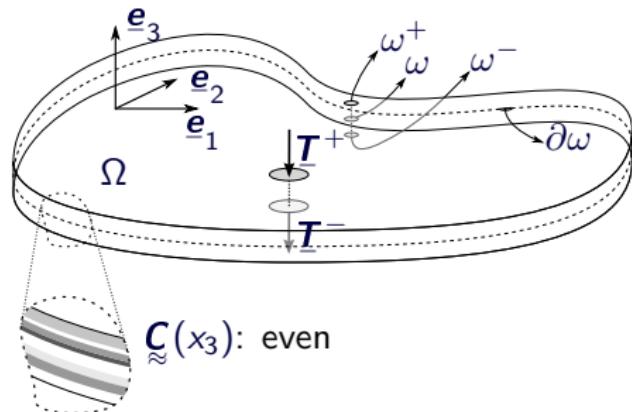
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The 3D Problem

$$\begin{cases} \sigma_{ij,j}^t = 0 & \text{on } \Omega, \\ \sigma_{ij}^t = C_{ijkl}(x_3)\varepsilon_{kl}^t & \text{on } \Omega, \\ \sigma_{i3}^t = \pm T_i^\pm & \text{on } \omega^\pm, \\ \varepsilon_{ij}^t = u_{(i,j)}^t & \text{on } \Omega, \\ u_i^t = 0 & \text{on } \partial\omega \times]-t/2, t/2[\end{cases}$$



- ▶ monoclinic $\tilde{\mathbf{C}}$:
- $C_{\alpha\beta\gamma 3} = C_{333\alpha} = 0$
- ▶ symmetrically laminated plate
- ▶ symmetric transverse load
- $T^\pm = \frac{p_3}{2} \mathbf{e}_3$

⇒ pure bending:

- ▶ u_3 and $\sigma_{\alpha 3}$ even / x_3
- ▶ u_α , $\sigma_{\alpha\beta}$ and σ_{33} odd / x_3

Building a plate model?

For typical width L let $t \rightarrow 0$

- ▶ Solve a 2D problem
- ▶ “fair” 3D displacement localization
- ▶ “fair” 3D stress localization

Some energy principles...

- ▶ Statically admissible fields:

$$SA^{3D} = \left\{ \sigma_{ij} \quad / \quad \sigma_{ij,j} = 0 \quad \text{and} \quad \underline{\sigma} \left(\pm \frac{t}{2} \right) \cdot \pm \underline{e}_3 = \frac{P_3}{2} \underline{e}_3 \right\}$$

$$SA^{3D,0} \Leftrightarrow P_3 = 0$$

- ▶ Kinematically admissible fields:

$$KA^{3D} = \left\{ u_i \quad / \quad u_i = 0 \text{ on } \partial\omega \times \left[-\frac{t}{2}, \frac{t}{2} \right] \right\}$$

$$KA^{3D,0} = KA^{3D}$$

- ▶ Orthogonality between $KA^{3D,0}$ and $SA^{3D,0}$:

$$\forall \underline{u} \in KA^{3D,0}, \underline{\sigma} \in SA^{3D,0}, \quad \int_{\Omega} \underline{\sigma} : \underline{\varepsilon}(\underline{u}) d\Omega = 0$$

Some energy principles...

- ▶ Potential energy:

$$\underline{\boldsymbol{u}}^{3D} = \operatorname{argmin}_{\underline{\boldsymbol{u}} \in KA^{3D}} \left\{ W^{3D} (\underline{\boldsymbol{\varepsilon}}(\underline{\boldsymbol{u}})) - \int_{\omega} P_3 \frac{\underline{u}_3^+ + \underline{u}_3^-}{2} d\omega \right\}$$

where $W^{3D} = \frac{1}{2} \int_{\Omega} \underline{\boldsymbol{\varepsilon}}(\underline{\boldsymbol{u}}) : \underline{\boldsymbol{C}} : \underline{\boldsymbol{\varepsilon}}(\underline{\boldsymbol{u}}) d\Omega$

- ▶ Complementary energy:

$$\underline{\boldsymbol{\sigma}}^{3D} = \operatorname{argmin}_{\underline{\boldsymbol{\sigma}} \in SA^{3D}} \left\{ W^{*3D} (\underline{\boldsymbol{\sigma}}) \right\} \quad \text{where} \quad W^{*3D} = \frac{1}{2} \int_{\Omega} \underline{\boldsymbol{\sigma}} : \underline{\boldsymbol{S}} : \underline{\boldsymbol{\sigma}} d\Omega$$

$$\underline{\boldsymbol{S}} = \underline{\boldsymbol{C}}^{-1}$$

The 2-energy principle

$$\forall \underline{\hat{u}} \in KA^{3D}, \quad \forall \underline{\hat{\sigma}} \in SA^{3D}:$$

$$W^{3D} \left(\underline{\varepsilon}(\underline{\hat{u}}) - \underline{\underline{S}} : \underline{\hat{\sigma}} \right) = W^{*3D} \left(\underline{\hat{\sigma}} - \underline{\underline{C}} : \underline{\varepsilon}(\underline{\hat{u}}) \right)$$

$$= W^{*3D} \left(\underline{\hat{\sigma}} - \underline{\sigma}^{3D} \right) + W^{3D} \left(\underline{\varepsilon} \left(\underline{\hat{u}} - \underline{u}^{3D} \right) \right)$$

$\Rightarrow W^{*3D} \left(\underline{\hat{\sigma}} - \underline{\underline{C}} : \underline{\varepsilon}(\underline{\hat{u}}) \right)$ provides an error estimate in terms of constitutive equation.

Prager and Synge (1947); Morgenstern (1959); Braess et al. (2010)

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Why all periodic plates are not “Reissner” like...

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Applications

Natural scaling of the stress

$$SA^{3D} \left\{ \begin{array}{l} \sigma_{\alpha\beta,\beta} + \sigma_{\alpha 3,3} = 0 \\ \sigma_{\alpha 3,\alpha} + \sigma_{33,3} = 0 \\ \sigma_{33}(\pm t/2) = \pm P_3/2 \\ \sigma_{\alpha 3}(\pm t/2) = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \sigma_{\alpha 3} = - \int_{-t/2}^{x_3} \sigma_{\alpha\beta,\beta} dz \\ \sigma_{33} = - \int_{t/2}^{x_3} \sigma_{\alpha 3,\alpha} dz - P_3/2 \end{array} \right.$$

$$\sigma_{\alpha\beta} \sim t^0 \quad \Rightarrow \quad \sigma_{\alpha 3} \sim t^1, \quad \sigma_{33} \sim t^2 \quad \text{and} \quad P_3 \sim t^2$$

From 3D equilibrium to 2D

Plate generalized stresses:

$$\begin{cases} M_{\alpha\beta}(x_1, x_2) = \langle x_3 \sigma_{\alpha\beta} \rangle \sim t^2 \\ Q_\alpha(x_1, x_2) = \langle \sigma_{\alpha 3} \rangle \sim t^2 \end{cases}$$

$$\langle \bullet \rangle = \int_{-\frac{t}{2}}^{\frac{t}{2}} \bullet dx_3$$

2D equilibrium equations:

$$\begin{cases} \langle \sigma_{\alpha 3, \alpha} + \sigma_{33, 3} \rangle = 0 \\ \langle x_3 (\sigma_{\alpha\beta, \beta} + \sigma_{\alpha 3, 3}) \rangle = 0 \end{cases} \Rightarrow \begin{cases} Q_{\alpha, \alpha} + P_3 = 0 \\ M_{\alpha\beta, \beta} - Q_\alpha = 0 \end{cases}$$

Boussinesq (1871); Mindlin (1951)...

“Kirchhoff's assumption”

At leading order in t :

$$\begin{cases} \varepsilon_{\alpha\beta} = x_3 K_{\alpha\beta} & \text{where } K_{\alpha\beta} = -U_{3,\alpha\beta} \\ \sigma_{i3} \simeq 0 + O(t^1) & \text{plane stress} \end{cases}$$

$$\varepsilon_{33} \neq 0!!$$

In-plane stress:

$$\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}^\sigma \varepsilon_{\delta\gamma} = x_3 C_{\alpha\beta\gamma\delta}^\sigma K_{\delta\gamma}$$

where $C_{\alpha\beta\gamma\delta}^\sigma = C_{\alpha\beta\gamma\delta} - \frac{C_{\alpha\beta 33} C_{33\gamma\delta}}{C_{3333}}$:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix}$$

The Kirchhoff-Love plate problem

Bending constitutive equation:

$$M_{\alpha\beta} = \langle x_3 \sigma_{\alpha\beta} \rangle = \langle x_3^2 C_{\alpha\beta\gamma\delta}^\sigma \rangle K_{\delta\gamma} = -D_{\alpha\beta\gamma\delta} U_{3,\delta\gamma}$$

$$\mathbf{D} \approx \frac{t^3}{12} \mathbf{C}^\sigma$$

Statically admissible fields:

$$SA^{KL} : \{ M_{\alpha\beta}/M_{\alpha\beta,\alpha\beta} + P_3 = 0 \}$$

Kinematically compatible fields:

$$KA^{KL} : \{ U_3/U_3 = 0 \text{ and } U_{3,n} = 0 \text{ on } \partial\omega, \mathbf{n} \text{ outer normal to } \omega \}$$

always "hard support": $U_3 = 0 \text{ on } \partial\omega \Rightarrow U_{3,t} = 0 \text{ on } \partial\omega$

Building SA^{3D} fields

$$\hat{\sigma}_{\alpha\beta} = x_3 C_{\alpha\beta\gamma\delta}^\sigma d_{\delta\gamma\epsilon\zeta} M_{\zeta\epsilon} = \frac{12x_3}{t^3} M_{\alpha\beta} \sim t^0$$

$$\hat{\sigma}_{\alpha 3} = - \int_{-t/2}^{x_3} \sigma_{\alpha\beta,\beta} dz = \frac{3}{2t} \left(1 - \frac{4x_3^2}{t^2} \right) M_{\alpha\beta,\beta} \sim t^1$$

$$\hat{\sigma}_{33} = - \int_{-t/2}^{x_3} \sigma_{\alpha 3, \alpha} dz - P_3/2 = \frac{3x_3}{2t} \left(1 - \frac{4x_3^2}{3t^2} \right) P_3 \sim t^2$$

$$\Leftrightarrow \hat{\sigma} = \hat{s}^M(x_3) : \hat{M} + \hat{s}^Q(x_3) \cdot (\hat{M} \cdot \nabla) + \hat{s}^{P_3}(x_3) P_3$$

$$\hat{\sigma} \in SA^{3D}$$

Building KA^{3D} fields

Strains

$$\hat{\varepsilon} = \begin{cases} \hat{\varepsilon}_{\alpha\beta} &= x_3 d_{\alpha\beta\gamma\delta} M_{\gamma\delta} \\ \hat{\varepsilon}_{\alpha 3} &= \frac{3}{4Gt} \left(1 - \frac{4x_3^2}{t^2}\right) M_{\alpha\beta,\beta} \\ \hat{\varepsilon}_{33} &= -\frac{12\nu x_3}{Et^3} M_{\alpha\alpha} \end{cases} = S_{\alpha\beta\gamma\delta} \hat{\sigma}_{\delta\gamma} + \cancel{S_{\alpha\beta 33} \hat{\sigma}_{33}}$$

$$= 2S_{\alpha 3\beta 3} \hat{\sigma}_{\beta 3}$$

$$= S_{33\alpha\beta} \hat{\sigma}_{\alpha\beta} + \cancel{S_{3333} \hat{\sigma}_{33}}$$

Building KA^{3D} fields

Integration

$$\hat{u}_3 = \int^{x_3} \hat{\varepsilon}_{33}(z) dz + U_3 = \underbrace{u_{3\alpha\beta}^M(x_3) M_{\beta\alpha}}_{\sim t^1} + \underbrace{U_3}_{\sim t^{-1}}$$

where: $u_{3\alpha\beta}^M(x_3) = \frac{\nu}{2Et} \left(\frac{12x_3^2}{t^2} - 1 \right) \delta_{\alpha\beta}$ and $\langle u_{3\alpha\beta}^M(x_3) \rangle = 0$

Building KA^{3D} fields

Integration

$$\hat{u}_3 = \int^{x_3} \hat{\varepsilon}_{33}(z) dz + U_3 = \underbrace{u_{3\alpha\beta}^M(x_3) M_{\beta\alpha}}_{\sim t^1} + \underbrace{U_3}_{\sim t^{-1}}$$

$$\hat{u}_\alpha = \int^{x_3} 2\hat{\varepsilon}_{\alpha 3}(z) - \hat{u}_{3,\alpha} dz = \underbrace{u_{\alpha\beta\gamma\delta}^{M \otimes \nabla}(x_3) M_{\delta\gamma,\beta}}_{\sim t^2} - \underbrace{x_3 U_{3,\alpha}}_{\sim t^0}$$

where:

$$u_{\alpha\beta\gamma\delta}^{M \otimes \nabla}(x_3) M_{\delta\gamma,\beta} = \frac{x_3}{2Et} \left(6(1+\nu) \left(1 - \frac{4x_3^2}{3t^2} \right) M_{\alpha\beta,\beta} + \nu \left(1 - \frac{4x_3^2}{t^2} \right) M_{\beta\beta,\alpha} \right)$$

and $\langle u_{\alpha\beta\gamma\delta}^{M \otimes \nabla}(x_3) \rangle = 0$

Building KA^{3D} fields

Integration

$$\hat{u}_3 = \int^{x_3} \hat{\varepsilon}_{33}(z) dz + U_3 = \underbrace{u_{3\alpha\beta}^M(x_3) M_{\beta\alpha}}_{\sim t^1} + \underbrace{U_3}_{\sim t^{-1}}$$

$$\hat{u}_\alpha = \int^{x_3} 2\hat{\varepsilon}_{\alpha 3}(z) - \hat{u}_{3,\alpha} dz = \underbrace{u_{\alpha\beta\gamma\delta}^{M \otimes \nabla}(x_3) M_{\delta\gamma,\beta}}_{\sim t^2} - \underbrace{x_3 U_{3,\alpha}}_{\sim t^0}$$

$$\Rightarrow \hat{\underline{u}} = U_3 \underline{e}_3 - x_3 U_{3,\alpha} \underline{e}_\alpha + u_{3\alpha\beta}^M(x_3) M_{\beta\alpha} \underline{e}_3 + u_{\alpha\beta\gamma\delta}^{M \otimes \nabla}(x_3) M_{\delta\gamma,\beta} \underline{e}_\alpha$$

with:

$$\hat{\underline{\varepsilon}} = \underline{\varepsilon}(\hat{\underline{u}})$$

Application of the Two-Energy principle

Consider U_3^{KL} and $\tilde{\mathbf{M}}^{KL}$ the solution of the Kirchhoff-Love plate problem and define:

$$\tilde{\boldsymbol{\sigma}}^{KL} = \tilde{\mathbf{s}}^M(x_3) : \tilde{\mathbf{M}}^{KL} + \tilde{\mathbf{s}}^Q(x_3) \cdot (\tilde{\mathbf{M}}^{KL} \cdot \nabla) + \tilde{\mathbf{s}}^{P_3}(x_3) P_3$$

$$\underline{\mathbf{u}}^{KL} = U_3^{KL} \underline{\mathbf{e}}_3 - x_3 U_{3,\alpha}^{KL} \underline{\mathbf{e}}_\alpha + \mathbf{u}_{3\alpha\beta}^M(x_3) M_{\beta\alpha}^{KL} \underline{\mathbf{e}}_3 + \mathbf{u}_{\alpha\beta\gamma\delta}^{M \otimes \nabla}(x_3) M_{\delta\gamma,\beta}^{KL} \underline{\mathbf{e}}_\alpha$$

We have:

$$\tilde{\boldsymbol{\varepsilon}}(\underline{\mathbf{u}}^{KL}) - \tilde{\mathbf{s}} : \tilde{\boldsymbol{\sigma}}^{KL} = \tilde{\mathbf{s}}_{33}^{P_3} P_3 \begin{pmatrix} S_{\alpha\beta 33} & 0 \\ 0 & S_{3333} \end{pmatrix} \sim t^2$$

Application of the Two-Energy principle

$$\tilde{\varepsilon}(\underline{u}^{KL}) - \tilde{\mathbf{S}} : \underline{\sigma}^{KL} \sim t^2 \quad \Rightarrow \quad W^{3D} \left(\tilde{\varepsilon}(\underline{u}^{KL}) - \tilde{\mathbf{S}} : \underline{\sigma}^{KL} \right) \sim t^5$$

Would lead to a relative error in t^2 ...

$$\underline{\sigma}^{KL} \in SA^{3D} \quad \text{but} \quad \underline{u}^{KL} \notin KA^{3D}$$

At best: relative error in $t^{1/2}$...

Braess et al. (2010) among many others!

Reissner's original plate model (1945)

min of W^{*3D}

Let us consider:

$$\hat{\sigma} = \underline{\mathbb{S}}^M(x_3) : \underline{\mathbf{M}} + \underline{\mathbb{S}}^Q(x_3) \cdot \underline{\mathbf{Q}} + \underline{\mathbb{S}}^{P_3}(x_3) P_3$$

with

$$SA^{RM} = \left\{ (\underline{\mathbf{M}}, \underline{\mathbf{Q}}) / Q_{\alpha,\alpha} + P_3 = 0 \quad \text{and} \quad M_{\alpha\beta,\beta} - Q_\alpha = 0 \quad \text{on } \omega \right\}$$

$$W^{*3D}(\underline{\boldsymbol{\sigma}}^{3D}) \leq \min_{(\underline{\mathbf{M}}, \underline{\mathbf{Q}}) \in SA^{RM}} W^{*3D}(\hat{\sigma}) \leq W^{*3D}(\underline{\boldsymbol{\sigma}}^{KL})$$

is a better approximation of $W^{*3D}(\underline{\boldsymbol{\sigma}}^{3D})$

Reissner's original plate model (1945)

Dualization

$$\begin{cases} Q_{\alpha,\alpha} + P_3 = 0 & \times U_3 \\ M_{\alpha\beta,\beta} - Q_\alpha = 0 & \times \varphi_\alpha \end{cases} \Rightarrow \begin{cases} Q_\alpha & \leftrightarrow \quad \gamma_\alpha \quad = \varphi_\alpha + U_{3,\alpha} \\ M_{\alpha\beta} & \leftrightarrow \quad \chi_{\alpha\beta} \quad = \varphi_{(\alpha,\beta)} \end{cases}$$

$$KA^{RM} = \{(U_3, \varphi_\alpha) / U_3 = 0 \text{ and } \varphi_\alpha = 0 \text{ on } \partial\omega\}$$

Reissner's original plate model (1945)

Constitutive equation

$$W^{*RM} (\underline{\mathbf{M}}, \underline{\mathbf{Q}}) = W^{*3D} (\hat{\boldsymbol{\varphi}})$$

$$= \frac{1}{2} \int_{\omega}^T \begin{pmatrix} \underline{\mathbf{M}} \\ \underline{\mathbf{Q}} \\ P_3 \end{pmatrix} \begin{pmatrix} \underline{\mathbf{d}} & 0 & \frac{6\nu}{5Et}\underline{\boldsymbol{\delta}} \\ \underline{0} & \frac{6}{5Gt}\underline{\boldsymbol{\delta}} & 0 \\ \frac{6\nu}{5Et}\underline{\boldsymbol{\delta}} & 0 & \frac{17t}{140E} \end{pmatrix} \begin{pmatrix} \underline{\mathbf{M}} \\ \underline{\mathbf{Q}} \\ P_3 \end{pmatrix} d\omega$$

$$\Rightarrow \begin{cases} \underbrace{\chi_{\alpha\beta}}_{\sim t^{-1}} = \underbrace{d_{\alpha\beta\gamma\delta} M_{\delta\gamma}}_{\sim t^{-1}} + \underbrace{\frac{6\nu}{5Et} \delta_{\alpha\beta} P_3}_{\sim t^1} \\ \gamma_{\alpha} = \frac{6}{5Gt} Q_{\alpha} \sim t^1 \end{cases}$$

The contribution of P_3 is almost always dropped in the literature



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Field Localization

Following the same procedure leads to:

$$\hat{\sigma} = \hat{s}^M(x_3) : \hat{M} + \underbrace{\hat{s}^R(x_3) : \hat{M} \otimes \nabla}_{\sim t^1} + \underbrace{\hat{s}^T(x_3) : \hat{M} \otimes \nabla^2}_{\sim t^2} + \hat{s}^{P_3}(x_3) P_3$$

$$\hat{u} = U_3 \hat{e}_3 - x_3 U_{3,\alpha} \hat{e}_\alpha + u_{3\alpha\beta}^M(x_3) M_{\beta\alpha} \hat{e}_3 + u_{\alpha\beta\gamma\delta}^R(x_3) M_{\delta\gamma,\beta} \hat{e}_\alpha$$

where $\hat{R} = \hat{M} \otimes \nabla$ and $\hat{T} = \hat{R} \otimes \nabla$

... Kirchhoff-Love error estimates still hold.

Building SA^{3D} fields

$$\begin{aligned}
 \hat{\sigma}_{\alpha\beta} &= x_3 C_{\alpha\beta\gamma\delta}^\sigma d_{\delta\gamma\epsilon\zeta} M_{\zeta\epsilon} & = \$_{\alpha\beta\gamma\delta}^M(x_3) M_{\delta\gamma} \\
 \hat{\sigma}_{\alpha 3} &= - \int_{-t/2}^{x_3} \sigma_{\alpha\beta,\beta} dz & = \$_{\alpha 3\beta\gamma\delta}^R(x_3) M_{\delta\gamma,\beta} \\
 \hat{\sigma}_{33} &= - \int_{-t/2}^{x_3} \sigma_{\alpha 3,\alpha} dz - P_3/2 & = \$_{33\alpha\beta\gamma\delta}^T(x_3) M_{\delta\gamma,\beta\alpha} + \$_{33}^{P_3}(x_3) P_3
 \end{aligned}$$

The Bending-Gradient constitutive equation

Extending Reissner's approach?:

$$\hat{\sigma} = \underbrace{\mathbf{S}^M(x_3) : \mathbf{M}}_{\approx} + \underbrace{\mathbf{S}^R(x_3) : \mathbf{R}}_{\approx} + \cancel{\mathbf{S}^T(x_3) : \mathbf{T}} + \cancel{\mathbf{S}^{P_3}(x_3) P_3}$$

Let us define:

$$\hat{\sigma}^* = \underbrace{\mathbf{S}^M(x_3) : \mathbf{M}}_{\approx} + \underbrace{\mathbf{S}^R(x_3) : \mathbf{R}}_{\approx}$$

with: $\hat{\sigma}^* \cdot \nabla = 0 + O(t^1)$ only.

$$W^{*BG}(\mathbf{M}, \mathbf{R}) = W^{*3D}(\hat{\sigma}^*) = \frac{1}{2} \int_{\omega} \left(\mathbf{M} : \mathbf{d} : \mathbf{M} + \mathbf{R} : \mathbf{f} : \mathbf{R} \right) d\omega$$

The Bending-Gradient theory for thick plates

- Equilibrium equations:

$$\text{BG: } \begin{cases} R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma} \\ R_{\alpha\beta\beta,\alpha} + P_3 = 0 \end{cases}$$

$$\text{RM: } \begin{cases} Q_\alpha = M_{\alpha\beta,\beta} \\ Q_{\alpha,\alpha} + P_3 = 0 \end{cases}$$

- Mechanical meaning of \underline{R}

$$Q_\alpha = R_{\alpha\beta\beta} \Leftrightarrow \begin{cases} Q_1 = R_{111} + R_{122} = M_{11,1} + M_{12,2} \\ Q_2 = R_{121} + R_{222} = M_{21,1} + M_{22,2} \end{cases}$$

The Bending-Gradient theory for thick plates

- Equilibrium equations:

$$\text{BG: } \begin{cases} R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma} & \times \Phi_{\alpha\beta\gamma} \\ R_{\alpha\beta\beta,\alpha} + P_3 = 0 & \times U_3 \end{cases}$$

$$\text{RM: } \begin{cases} Q_\alpha = M_{\alpha\beta,\beta} & \times \varphi_\alpha \\ Q_{\alpha,\alpha} + P_3 = 0 & \times U_3 \end{cases}$$

The Bending-Gradient theory for thick plates

- Equilibrium equations:

$$\text{BG: } \begin{cases} R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma} \\ R_{\alpha\beta\beta,\alpha} + P_3 = 0 \end{cases} \quad \times \Phi_{\alpha\beta\gamma} \quad \times U_3$$

$$\text{RM: } \begin{cases} Q_\alpha = M_{\alpha\beta,\beta} \\ Q_{\alpha,\alpha} + P_3 = 0 \end{cases} \quad \times \varphi_\alpha \quad \times U_3$$

| Equilibrium | Work of internal forces | Work on Boundary |
|---|--|--|
| BG: $\begin{cases} R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma} \\ R_{\alpha\beta\beta,\alpha} + P_3 = 0 \end{cases}$ | $M_{\alpha\beta}\Phi_{\alpha\beta\epsilon,\epsilon} + R_{\alpha\beta\gamma}(\Phi_{\alpha\beta\gamma} + I_{\alpha\beta\gamma\epsilon}U_{3,\epsilon})$ | $M_{\alpha\beta}\Phi_{\alpha\beta\gamma}n_\gamma + R_{\alpha\beta\beta}n_\alpha U_3$ |
| RM: $\begin{cases} Q_\alpha = M_{\alpha\beta,\beta} \\ Q_{\alpha,\alpha} + P_3 = 0 \end{cases}$ | $M_{\alpha\beta}\varphi_{(\alpha,\beta)} + Q_\alpha(\varphi_\alpha + U_{3,\alpha})$ | $M_{\alpha\beta}n_\beta\varphi_\alpha + Q_\alpha n_\alpha U_3$ |

$$I_{\alpha\beta\gamma\delta} = \frac{1}{2} (\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \text{ identity for in-plane elasticity}$$

The Bending-Gradient theory for thick plates

- Equilibrium equations:

$$\text{BG: } \begin{cases} R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma} \\ R_{\alpha\beta\beta,\alpha} + P_3 = 0 \end{cases} \quad \times \Phi_{\alpha\beta\gamma} \quad \times U_3$$

$$\text{RM: } \begin{cases} Q_\alpha = M_{\alpha\beta,\beta} \\ Q_{\alpha,\alpha} + P_3 = 0 \end{cases} \quad \times \varphi_\alpha \quad \times U_3$$

| Equilibrium | Work of internal forces | Work on Boundary |
|---|--|--|
| BG: $\begin{cases} R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma} \\ R_{\alpha\beta\beta,\alpha} + P_3 = 0 \end{cases}$ | $M_{\alpha\beta}\Phi_{\alpha\beta\epsilon,\epsilon} + R_{\alpha\beta\gamma}(\Phi_{\alpha\beta\gamma} + I_{\alpha\beta\gamma\epsilon}U_{3,\epsilon})$ | $M_{\alpha\beta}\Phi_{\alpha\beta\gamma}n_\gamma + R_{\alpha\beta\beta}n_\alpha U_3$ |
| RM: $\begin{cases} Q_\alpha = M_{\alpha\beta,\beta} \\ Q_{\alpha,\alpha} + P_3 = 0 \end{cases}$ | $M_{\alpha\beta}\varphi_{(\alpha,\beta)} + Q_\alpha(\varphi_\alpha + U_{3,\alpha})$ | $M_{\alpha\beta}n_\beta\varphi_\alpha + Q_\alpha n_\alpha U_3$ |

When the plate is homogeneous, BG is exactly turned into RM
 (loss of invertibility of $\underline{\underline{f}}$)

The Bending-Gradient theory for thick plates

- Equilibrium equations:

$$\text{BG: } \begin{cases} R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma} \\ R_{\alpha\beta\beta,\alpha} + P_3 = 0 \end{cases} \quad \times \Phi_{\alpha\beta\gamma} \quad \times U_3$$

$$\text{RM: } \begin{cases} Q_\alpha = M_{\alpha\beta,\beta} \\ Q_{\alpha,\alpha} + P_3 = 0 \end{cases} \quad \times \varphi_\alpha \quad \times U_3$$

| Equilibrium | Work of internal forces | Work on Boundary |
|---|--|--|
| BG: $\begin{cases} R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma} \\ R_{\alpha\beta\beta,\alpha} + P_3 = 0 \end{cases}$ | $M_{\alpha\beta}\Phi_{\alpha\beta\epsilon,\epsilon} + R_{\alpha\beta\gamma}(\Phi_{\alpha\beta\gamma} + I_{\alpha\beta\gamma\epsilon}U_{3,\epsilon})$ | $M_{\alpha\beta}\Phi_{\alpha\beta\gamma}n_\gamma + R_{\alpha\beta\beta}n_\alpha U_3$ |
| RM: $\begin{cases} Q_\alpha = M_{\alpha\beta,\beta} \\ Q_{\alpha,\alpha} + P_3 = 0 \end{cases}$ | $M_{\alpha\beta}\varphi_{(\alpha,\beta)} + Q_\alpha(\varphi_\alpha + U_{3,\alpha})$ | $M_{\alpha\beta}n_\beta\varphi_\alpha + Q_\alpha n_\alpha U_3$ |

Low order interpolation for U_3 and Φ !

Local Fields reconstruction

Once the plate problem is solved (U_3^{BG} , $\underline{\Phi}^{BG}$, \underline{M}^{BG} , \underline{R}^{BG} known), we suggest the following field reconstruction:

- ▶ $\sigma^{BG} = \underline{\underline{s}}^M(x_3) : \underline{\underline{M}} + \underline{\underline{s}}^R(x_3) : \underline{\underline{R}} + \underline{\underline{s}}^T(x_3) : \underline{\underline{R}} \otimes \nabla + \underline{\underline{s}}^{P_3}(x_3) P_3$
- ▶ $\underline{u}^{BG} = U_3 \underline{e}_3 - x_3 U_{3,\alpha} \underline{e}_\alpha + \underline{u}_{3\alpha\beta}^M(x_3) M_{\beta\alpha} \underline{e}_3 + \underline{u}_{\alpha\beta\gamma\delta}^R(x_3) R_{\delta\gamma\beta} \underline{e}_\alpha$

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Why all periodic plates are not “Reissner” like...

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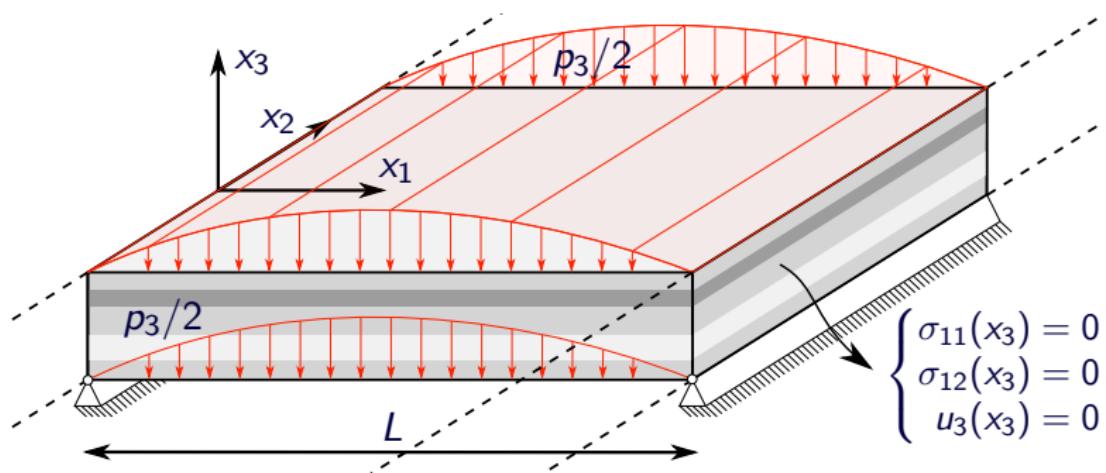
Extension to periodic plates

The case of cellular sandwich panels

Why all periodic plates are not “Reissner” like...

Pagano's boundary value problem (Pagano, 1969)

CFRP layers with different orientations:



Invariant in x_2 -Direction, “periodic” in x_1 -Direction
 \Rightarrow No boundary layer!

Practical Localization...

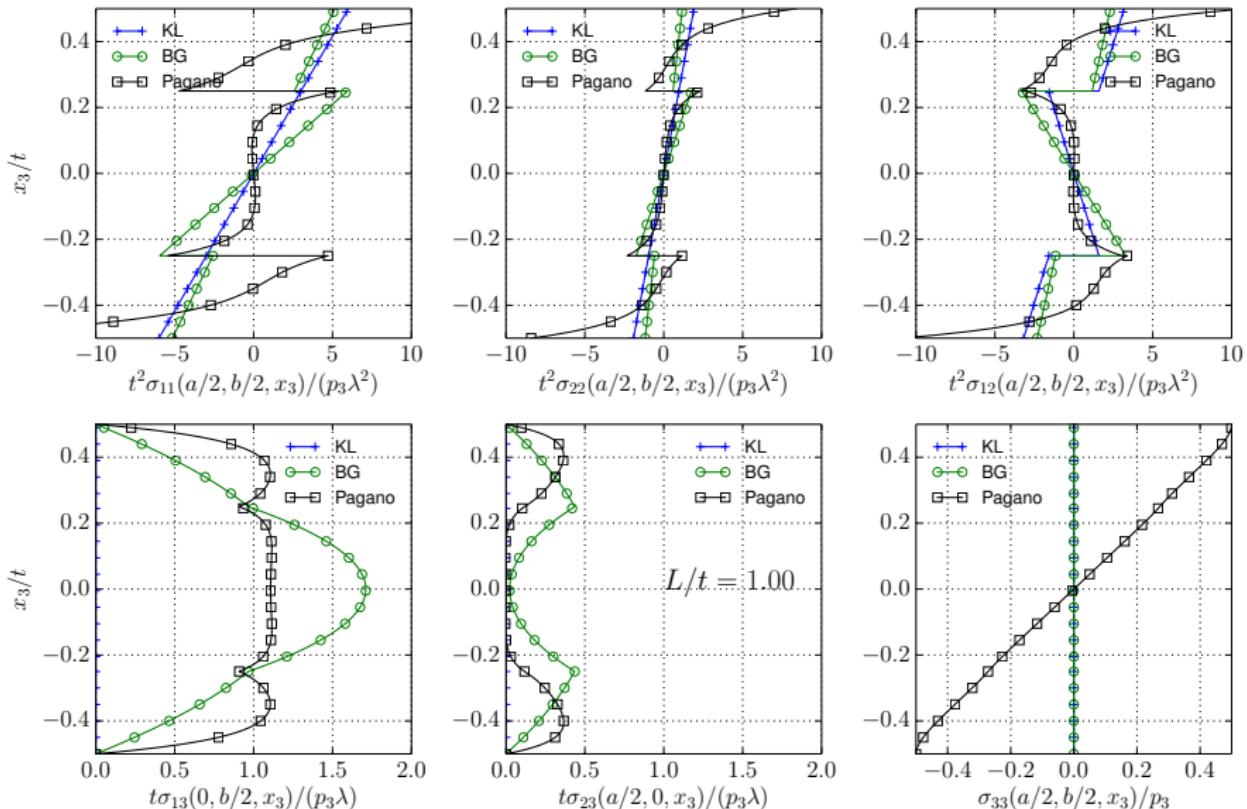
Kirchhoff-Love

- ▶ $\underline{\sigma}^{KL} = \underline{\underline{s}}^M(x_3) : \underline{\underline{M}}^{KL} + \cancel{\underline{\underline{s}}^Q(x_3) \cdot (\underline{\underline{M}}^{KL} \cdot \nabla)} + \cancel{\underline{\underline{s}}^{P_3}(x_3) P_3}$
- ▶ $\underline{u}^{KL} = U_3^{KL} \underline{\mathbf{e}}_3 - x_3 U_{3,\alpha}^{KL} \underline{\mathbf{e}}_\alpha + \underline{u}_{3\alpha\beta}^M(x_3) M_{\beta\alpha}^{KL} \underline{\mathbf{e}}_3 + \cancel{\underline{u}_{\alpha\beta\gamma\delta}^{M \otimes \nabla}(x_3) M_{\delta\gamma,\beta}^{KL} \underline{\mathbf{e}}_\alpha}$

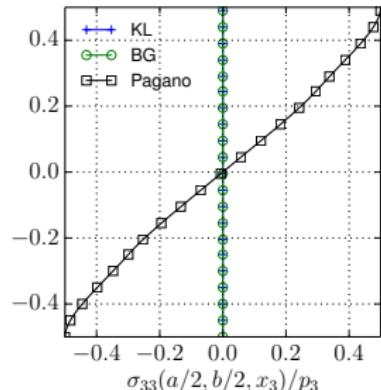
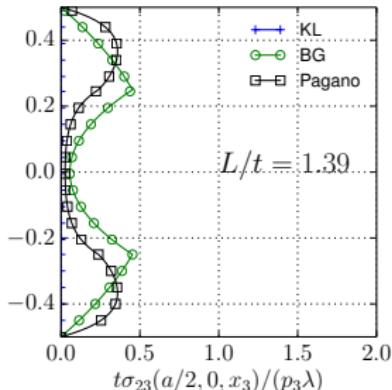
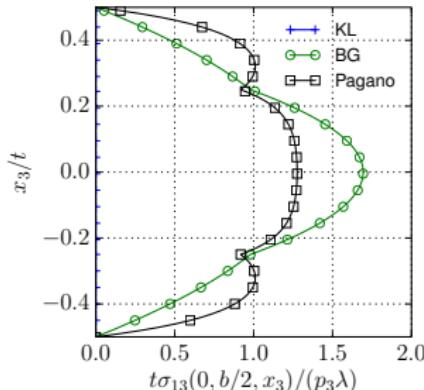
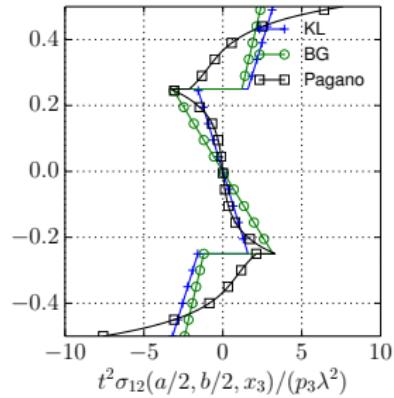
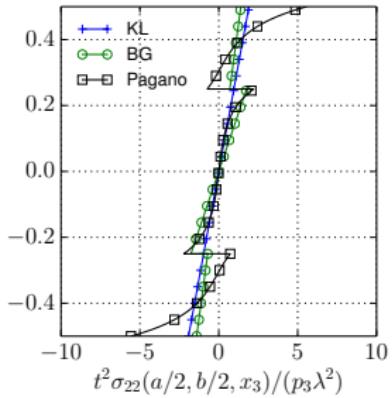
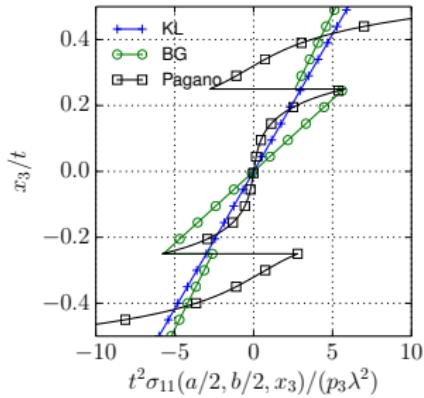
Bending-Gradient

- ▶ $\underline{\sigma}^{BG} = \underline{\underline{s}}^M(x_3) : \underline{\underline{M}} + \cancel{\underline{\underline{s}}^R(x_3) : \underline{\underline{R}}} + \cancel{\underline{\underline{s}}^T(x_3) : \underline{\underline{R}} \otimes \nabla} + \cancel{\underline{\underline{s}}^{P_3}(x_3) P_3}$
- ▶ $\underline{u}^{BG} = U_3 \underline{\mathbf{e}}_3 - x_3 U_{3,\alpha} \underline{\mathbf{e}}_\alpha + \underline{u}_{3\alpha\beta}^M(x_3) M_{\beta\alpha} \underline{\mathbf{e}}_3 + \underline{u}_{\alpha\beta\gamma\delta}^R(x_3) R_{\delta\gamma\beta} \underline{\mathbf{e}}_\alpha$

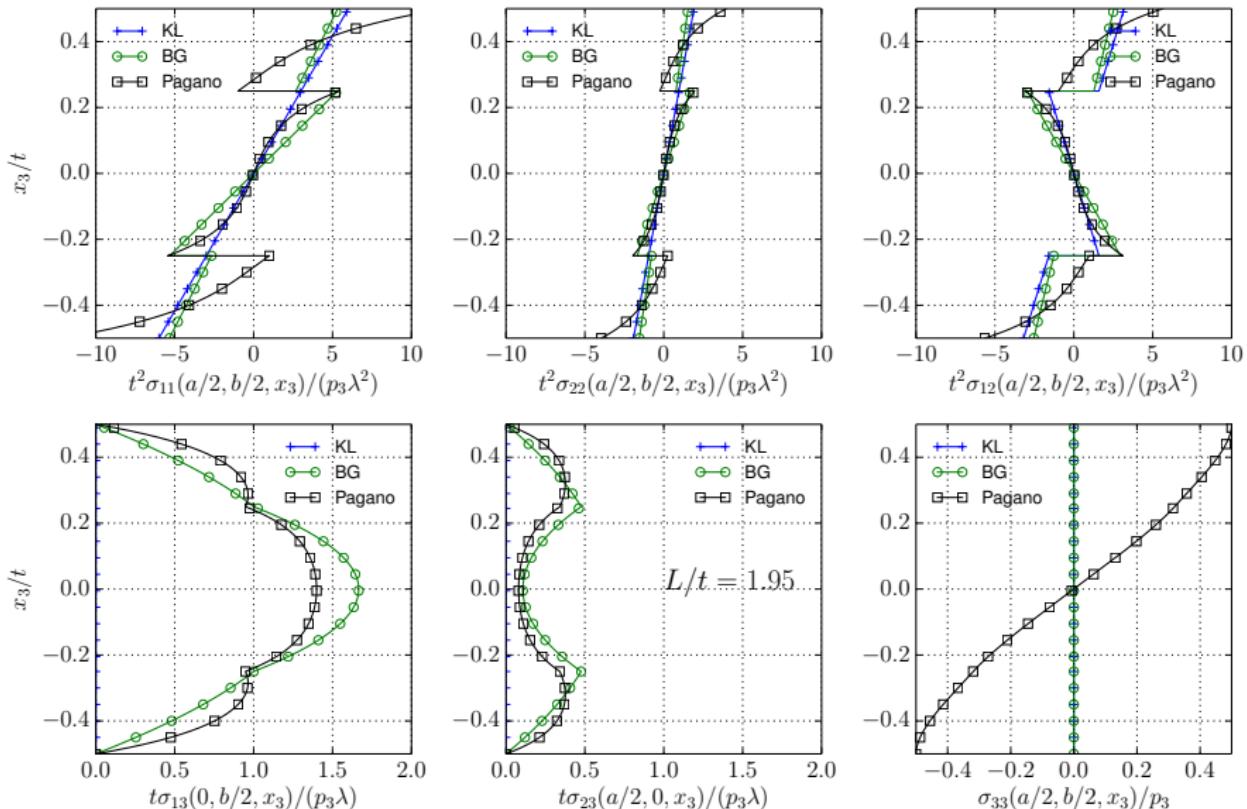
Stress distributions for a $[30^\circ, -30^\circ, 30^\circ]$ stack



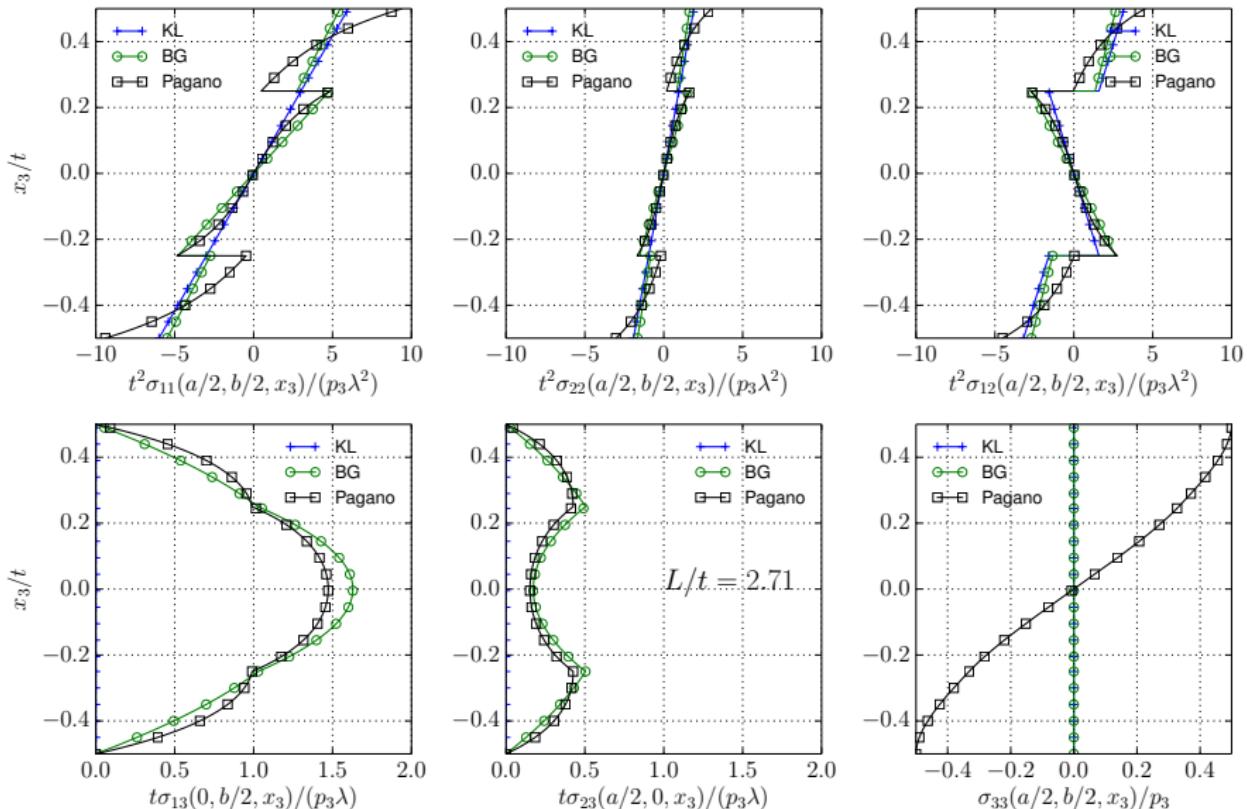
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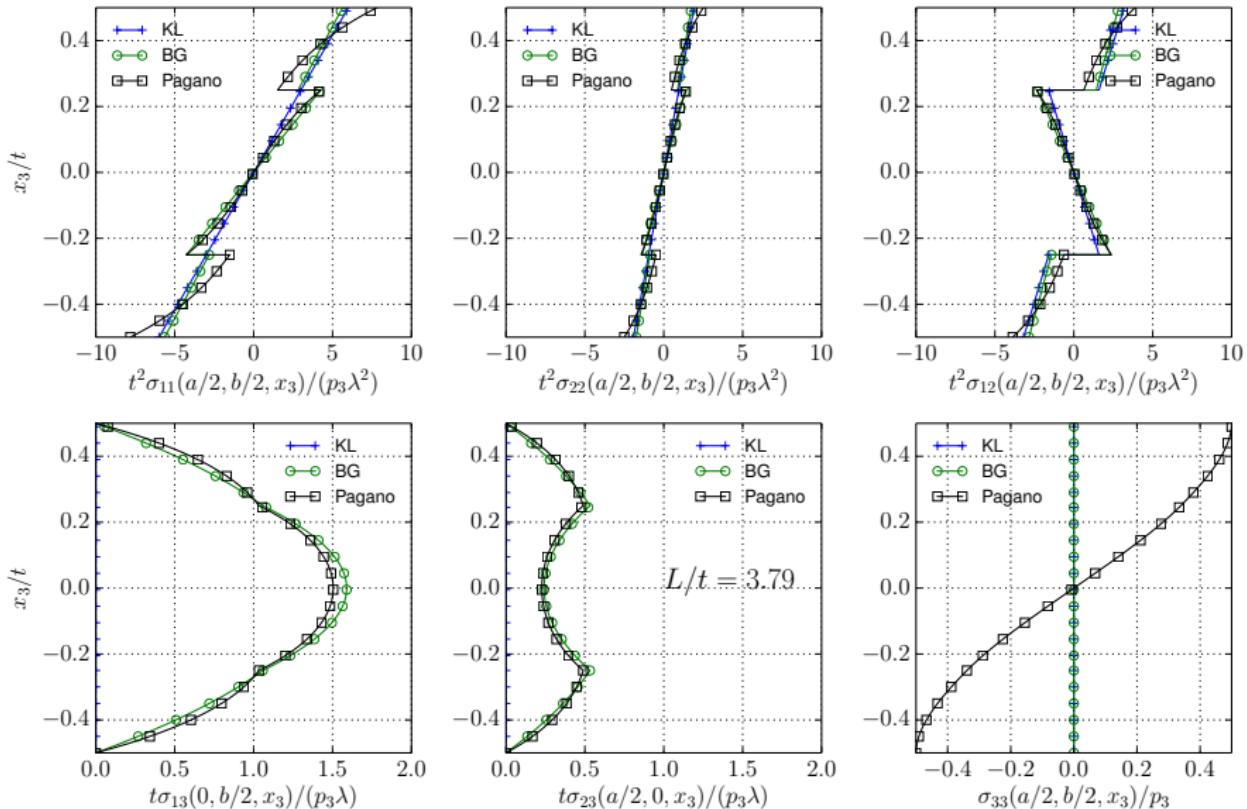
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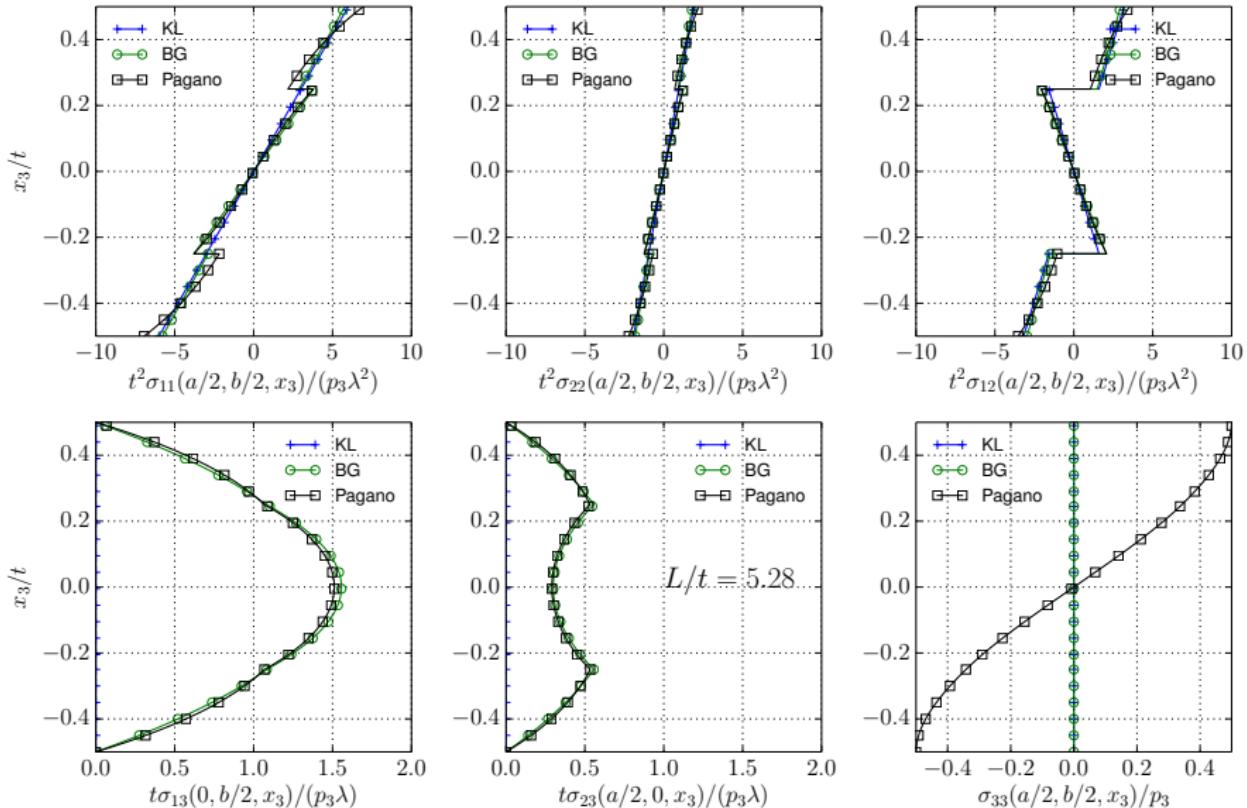
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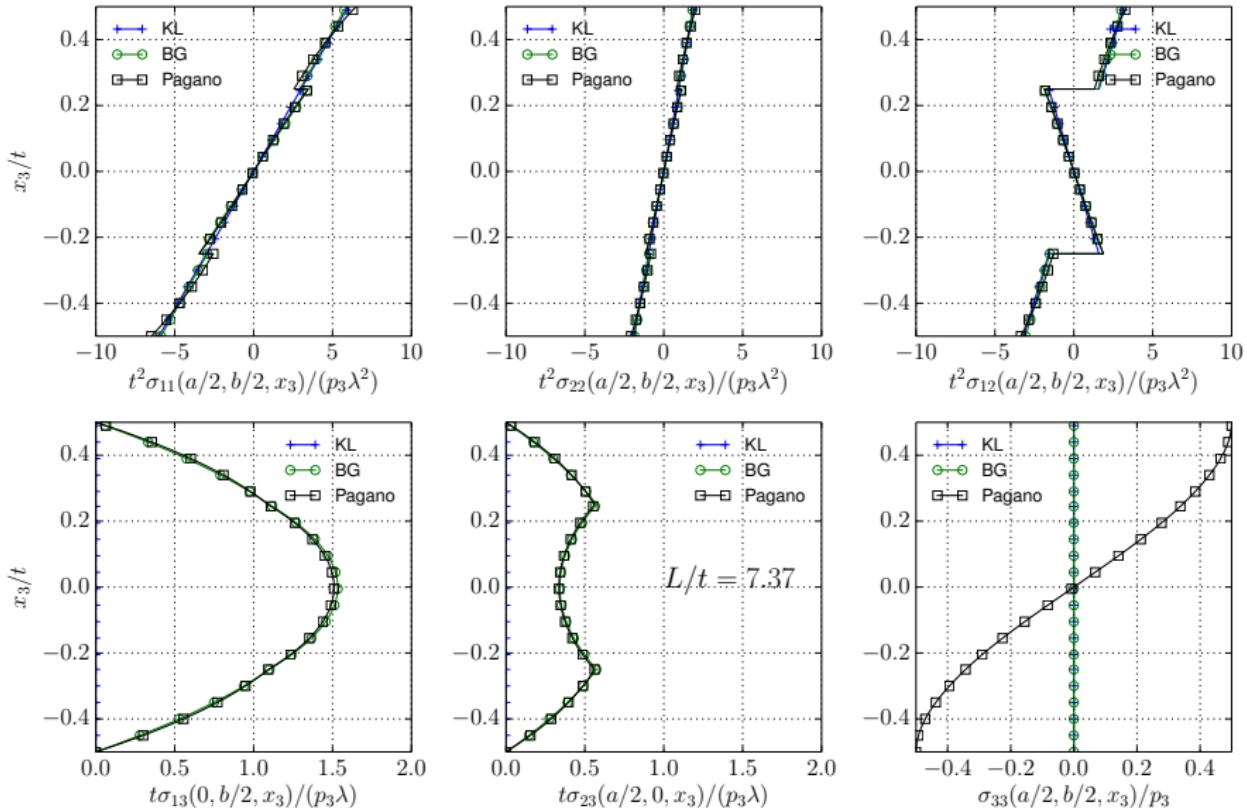
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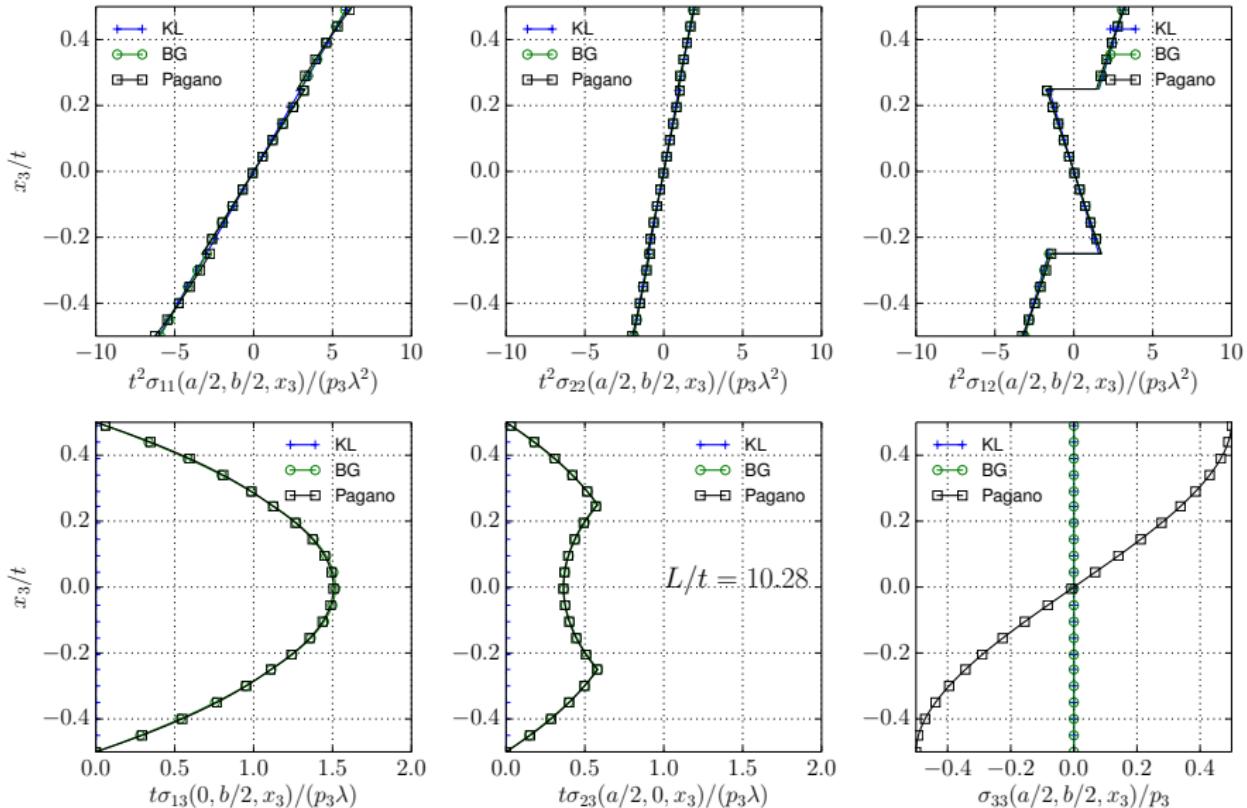
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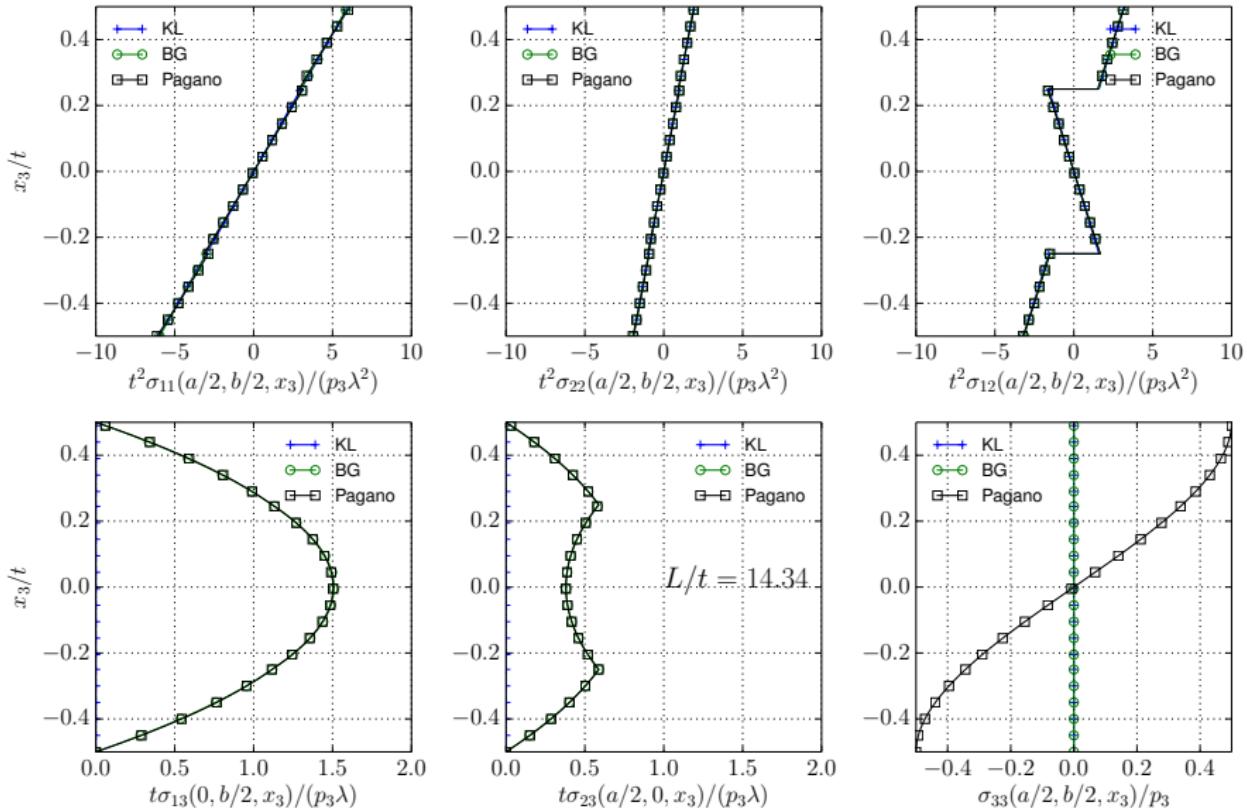
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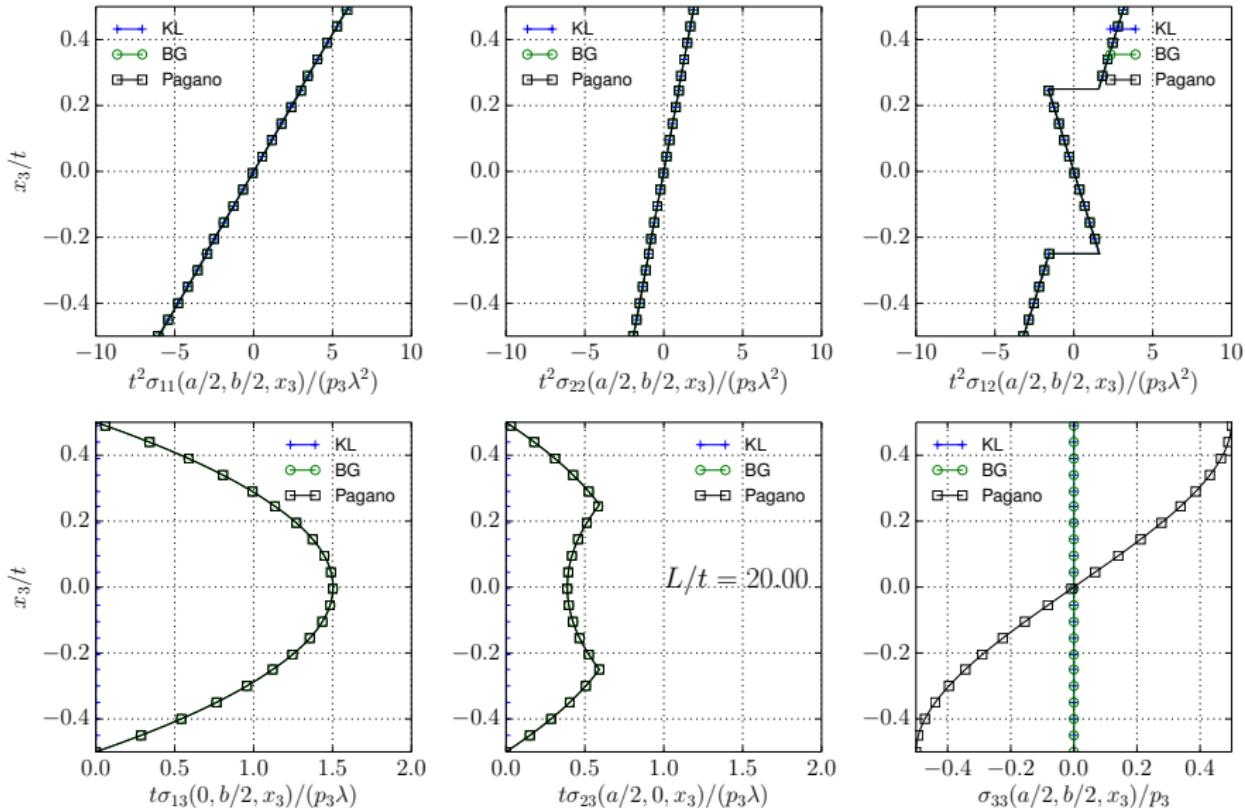
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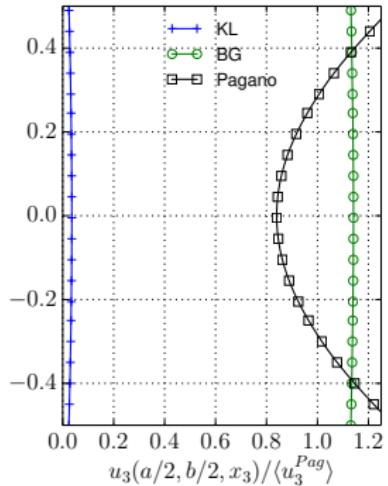
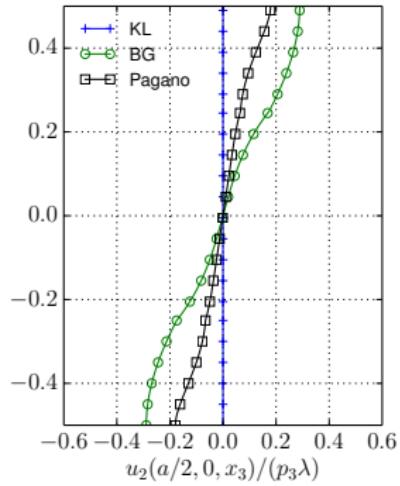
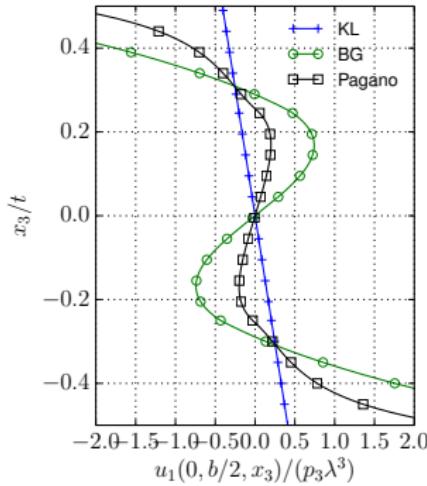
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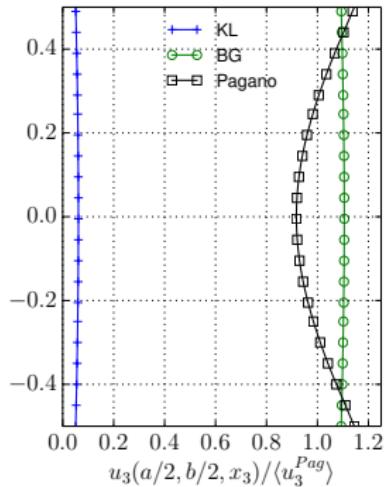
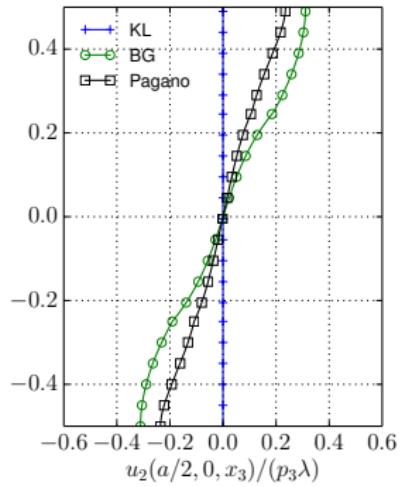
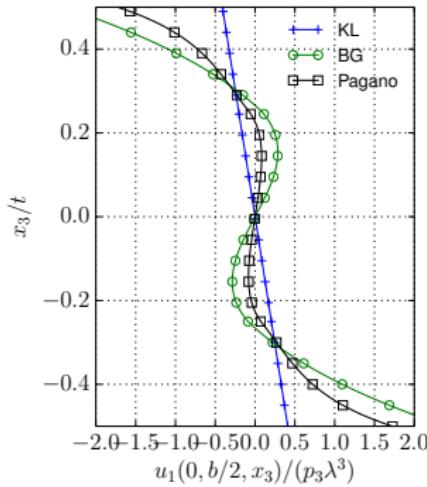
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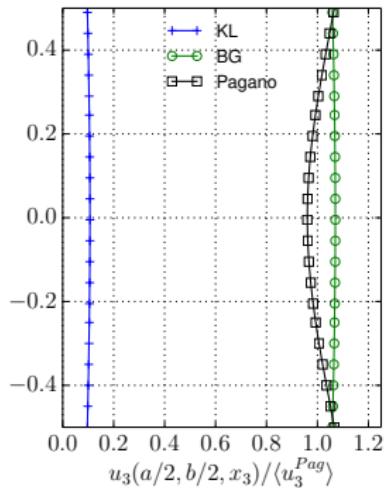
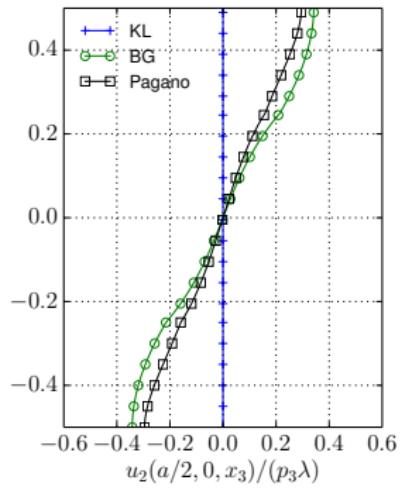
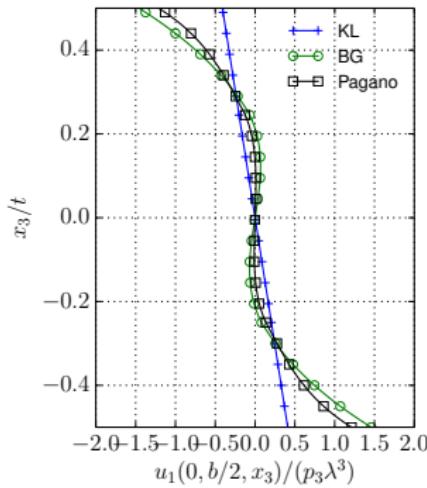
Displacement distributions for a $[30^\circ, -30^\circ, 30^\circ]$ stack



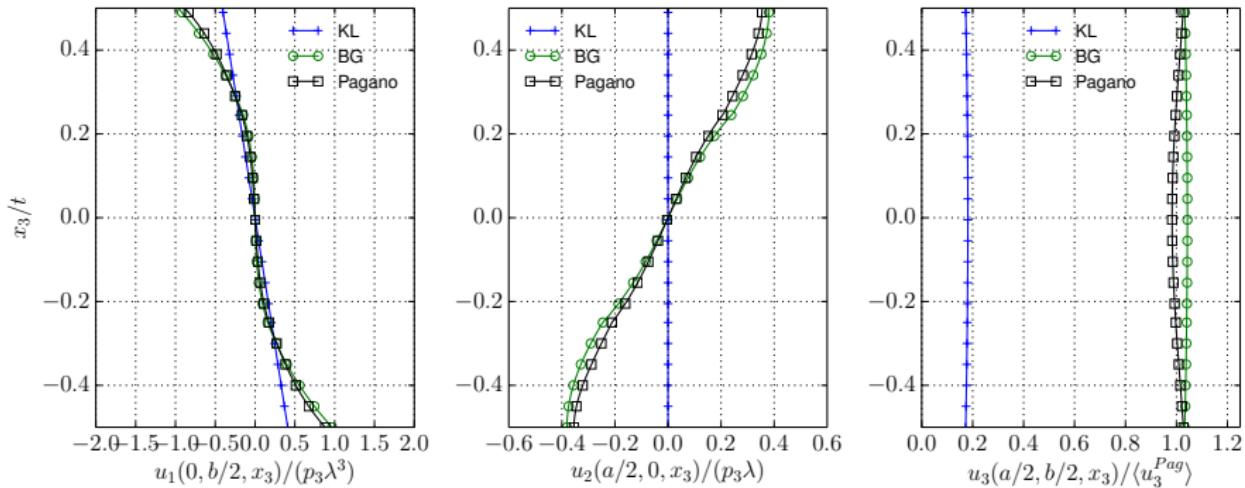
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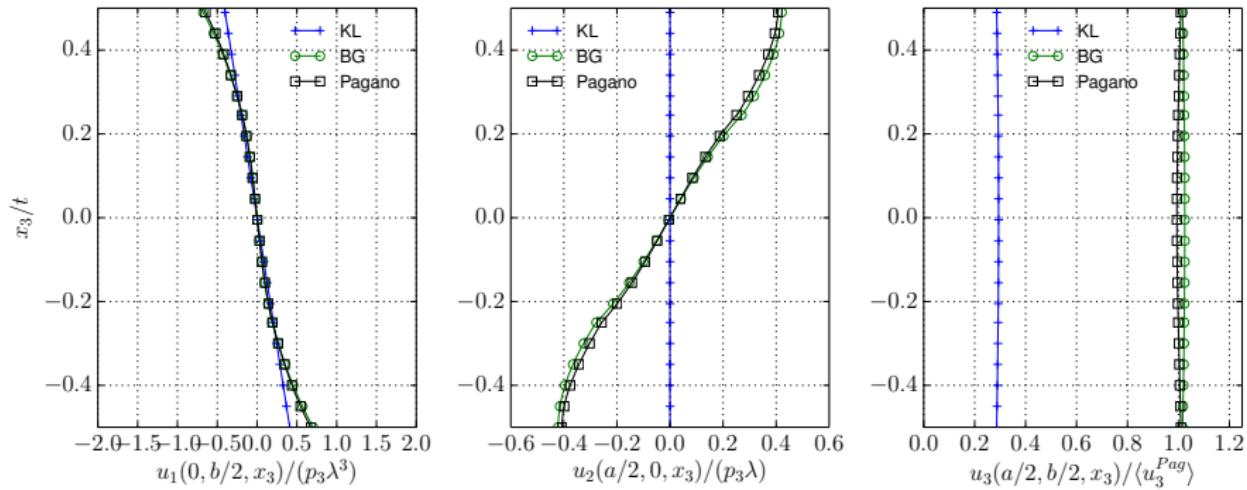
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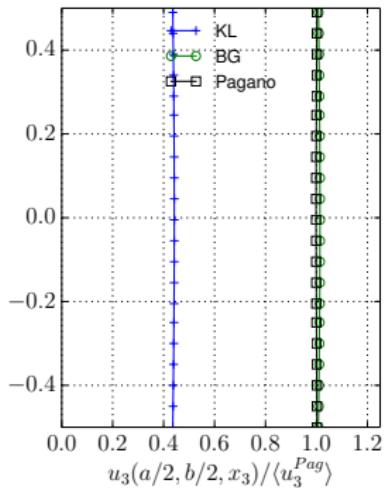
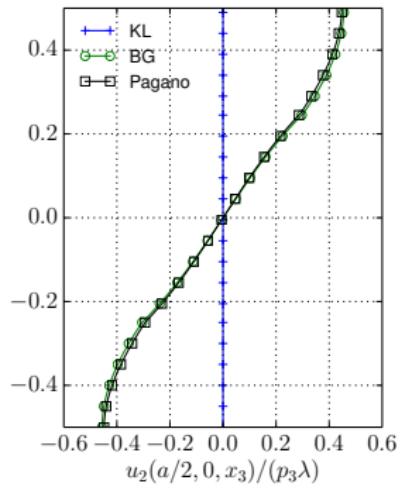
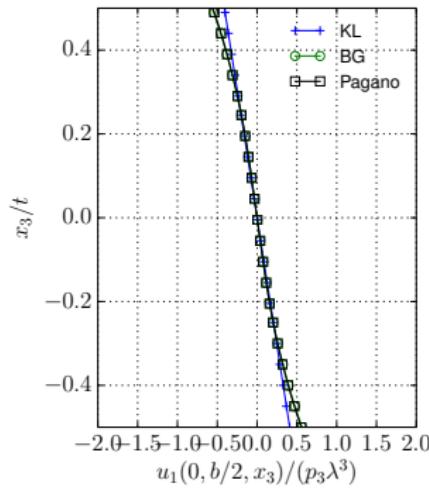
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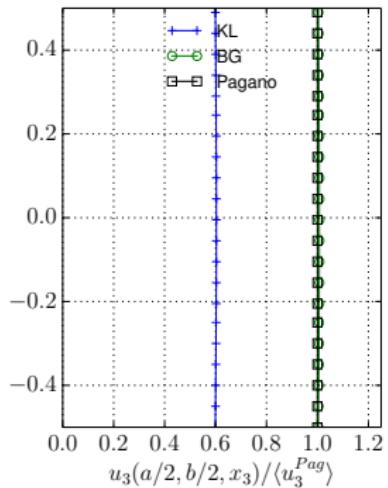
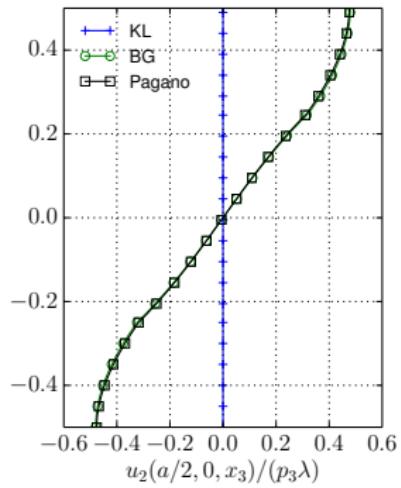
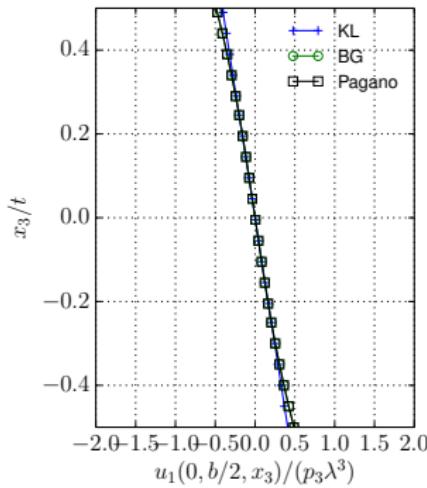
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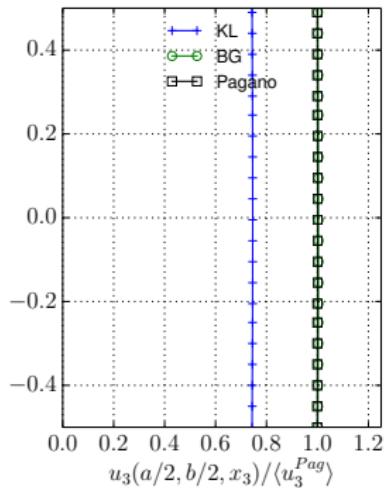
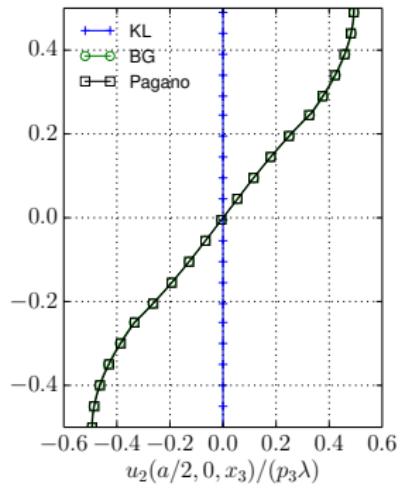
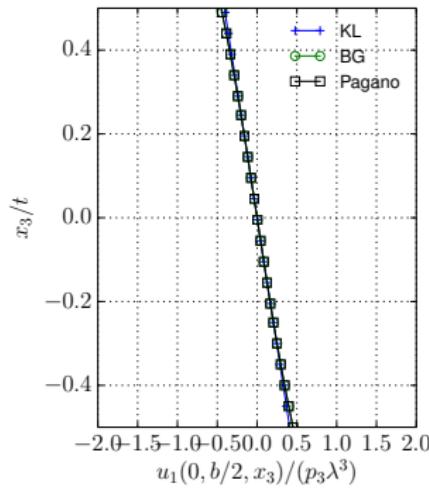
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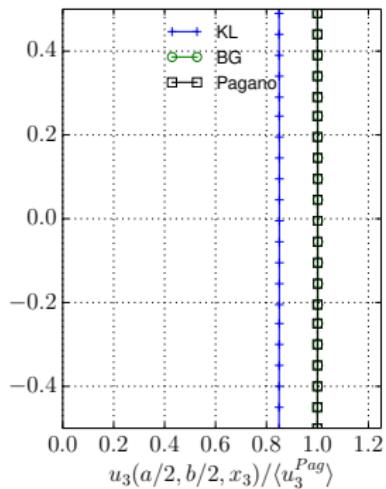
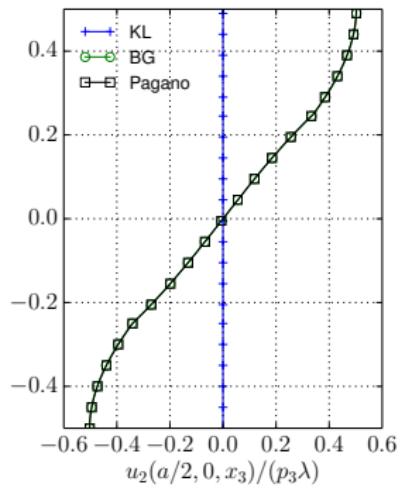
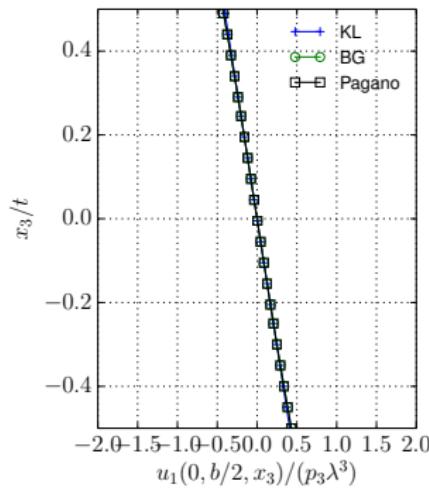
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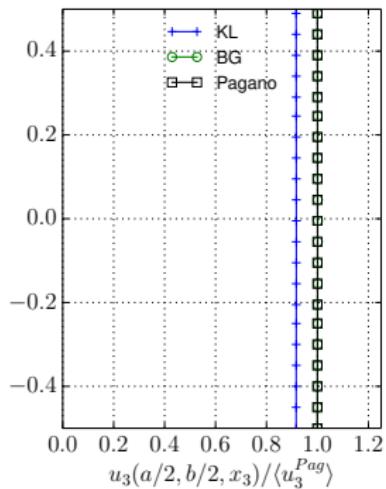
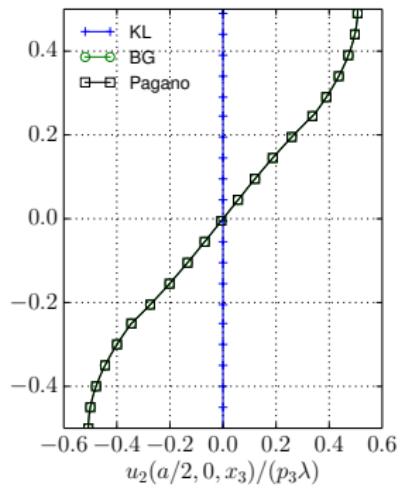
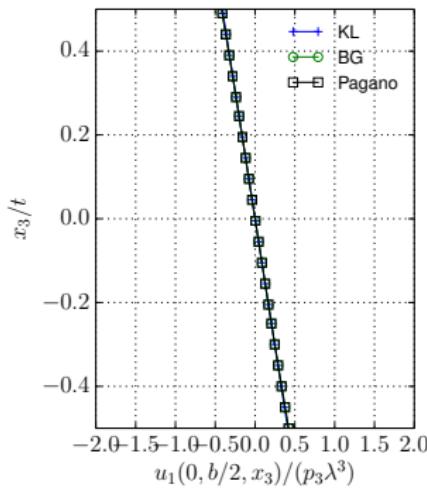
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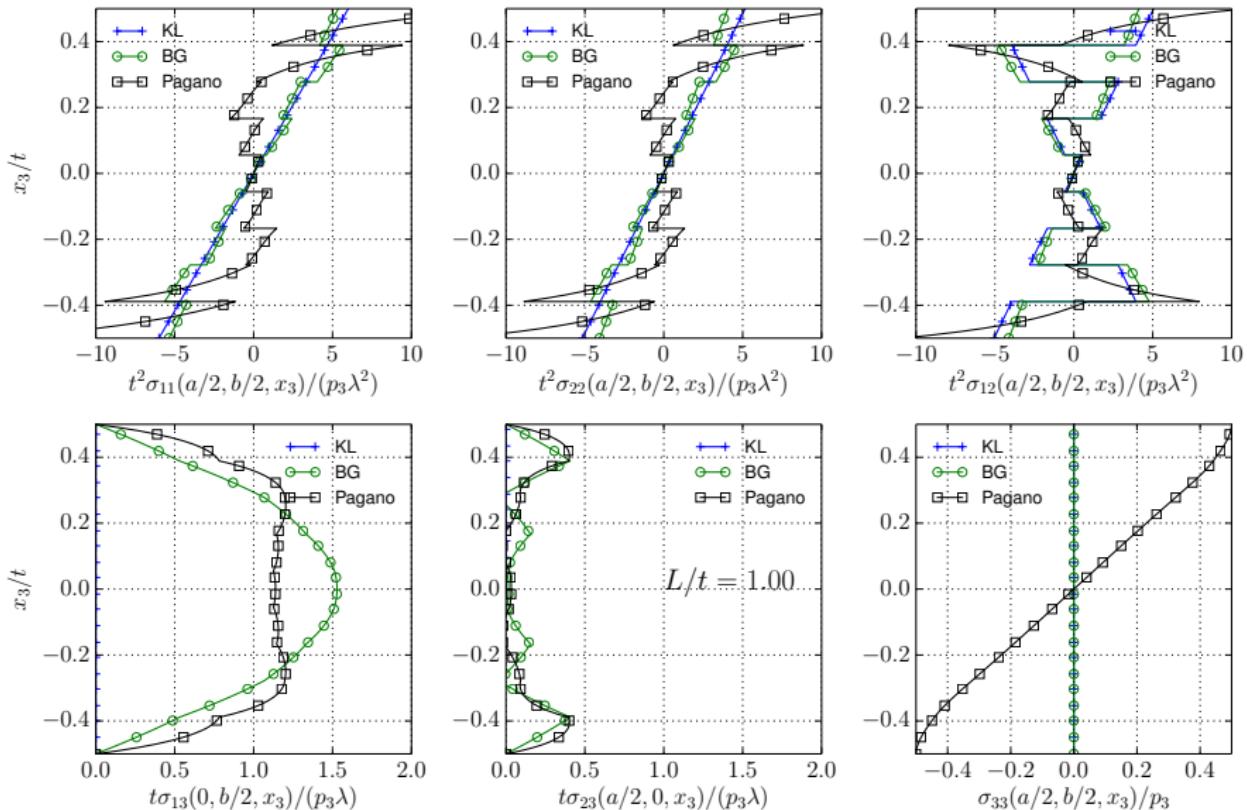
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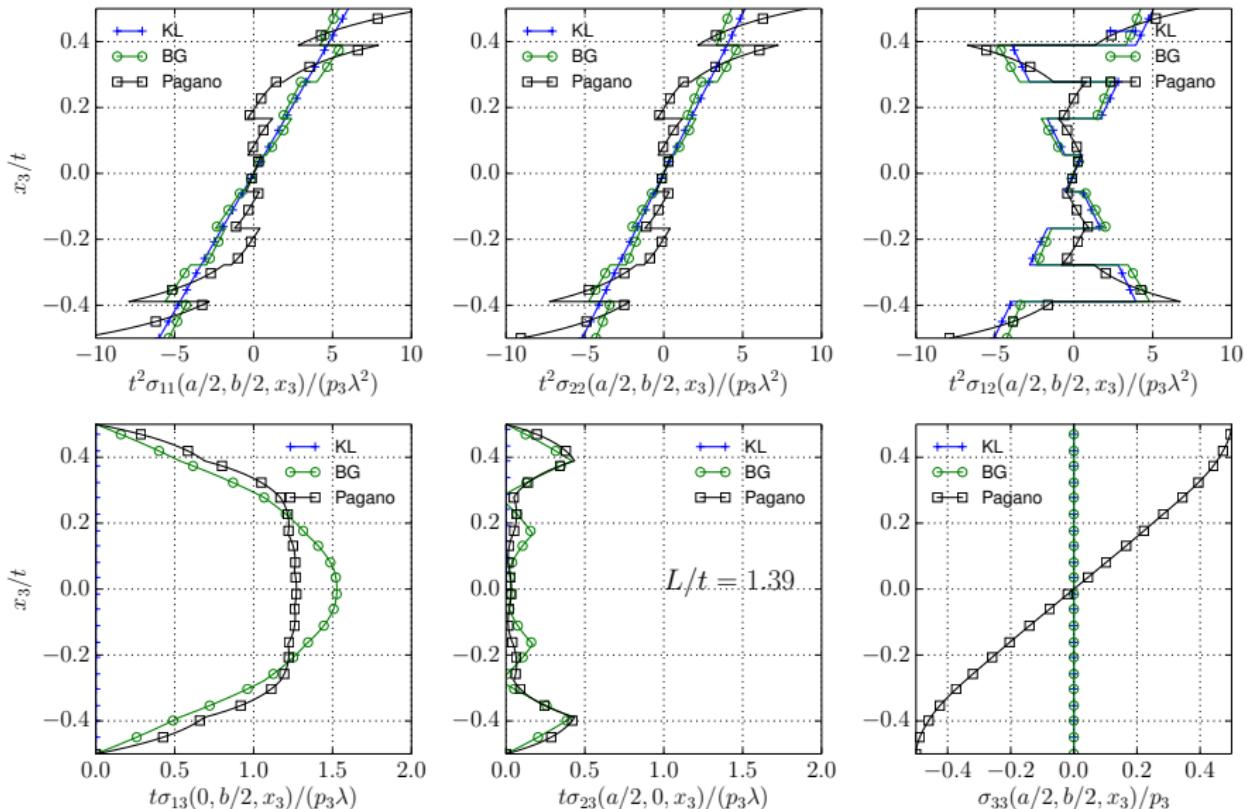
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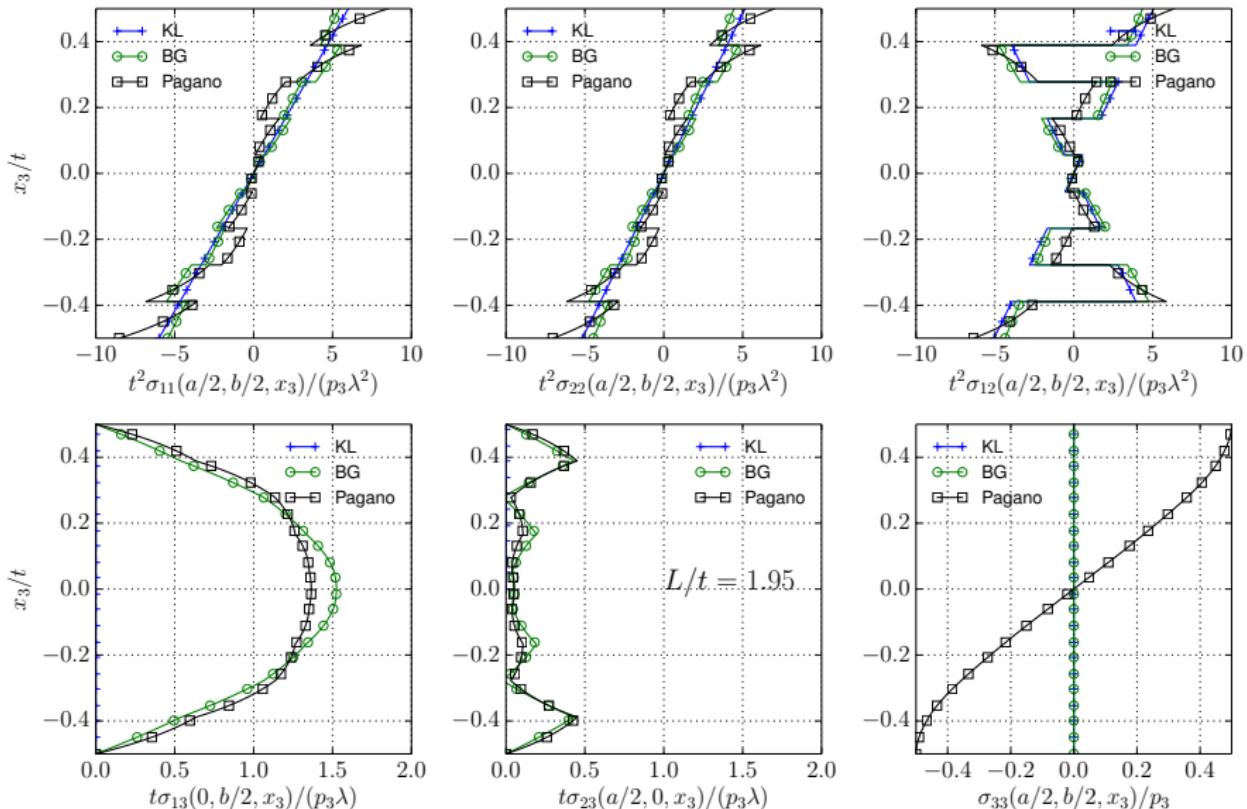
Stress distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



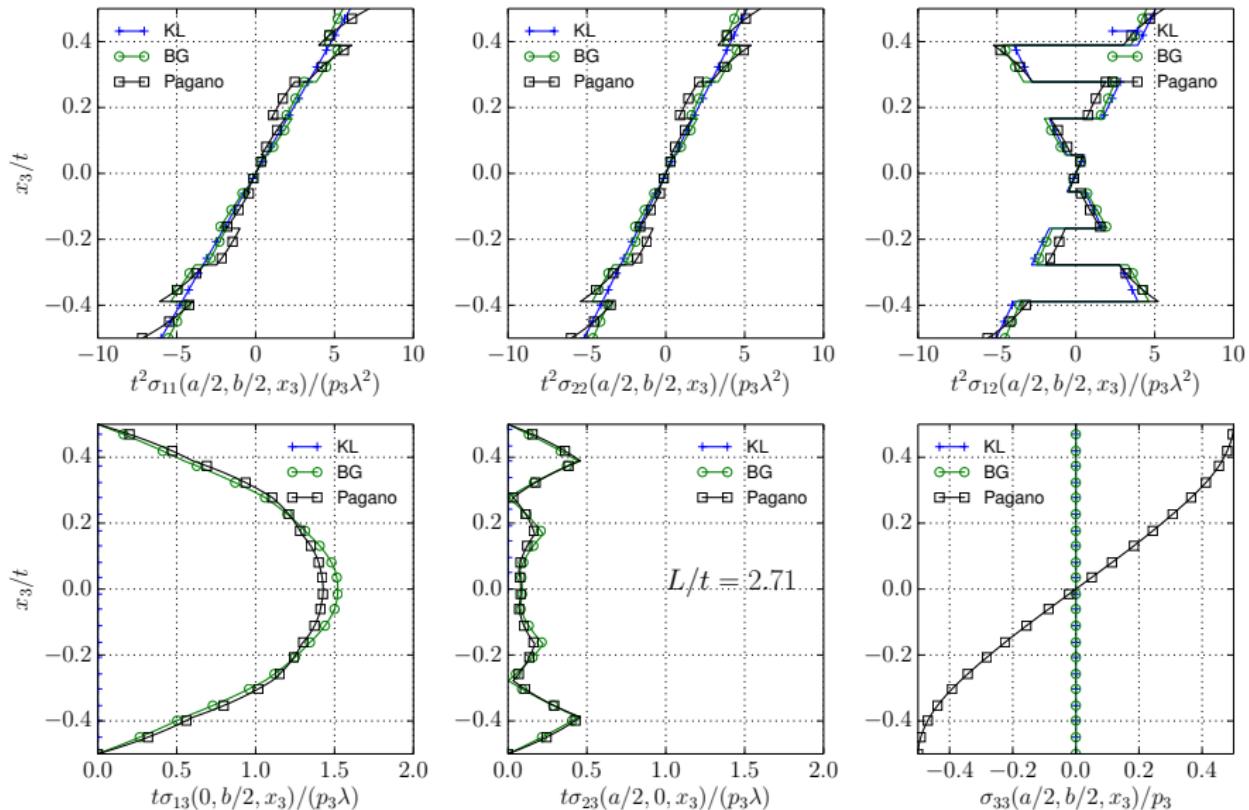
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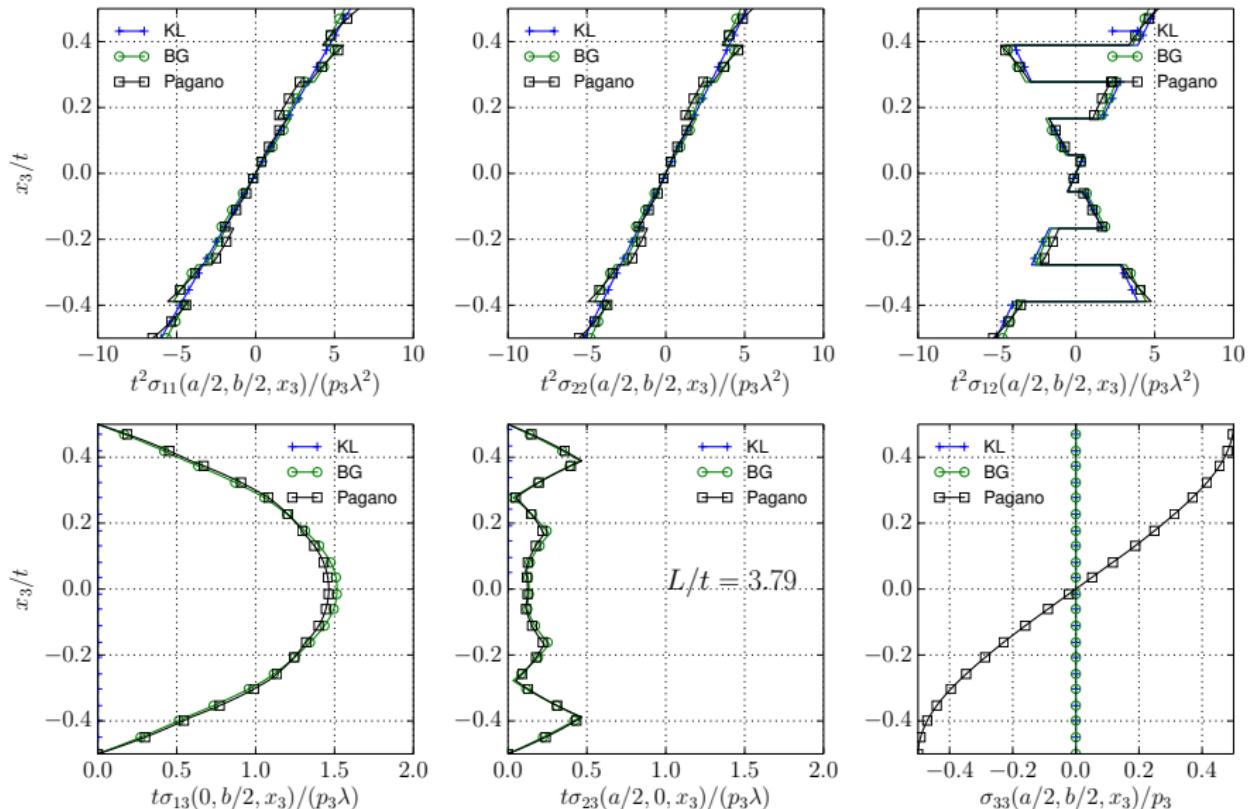
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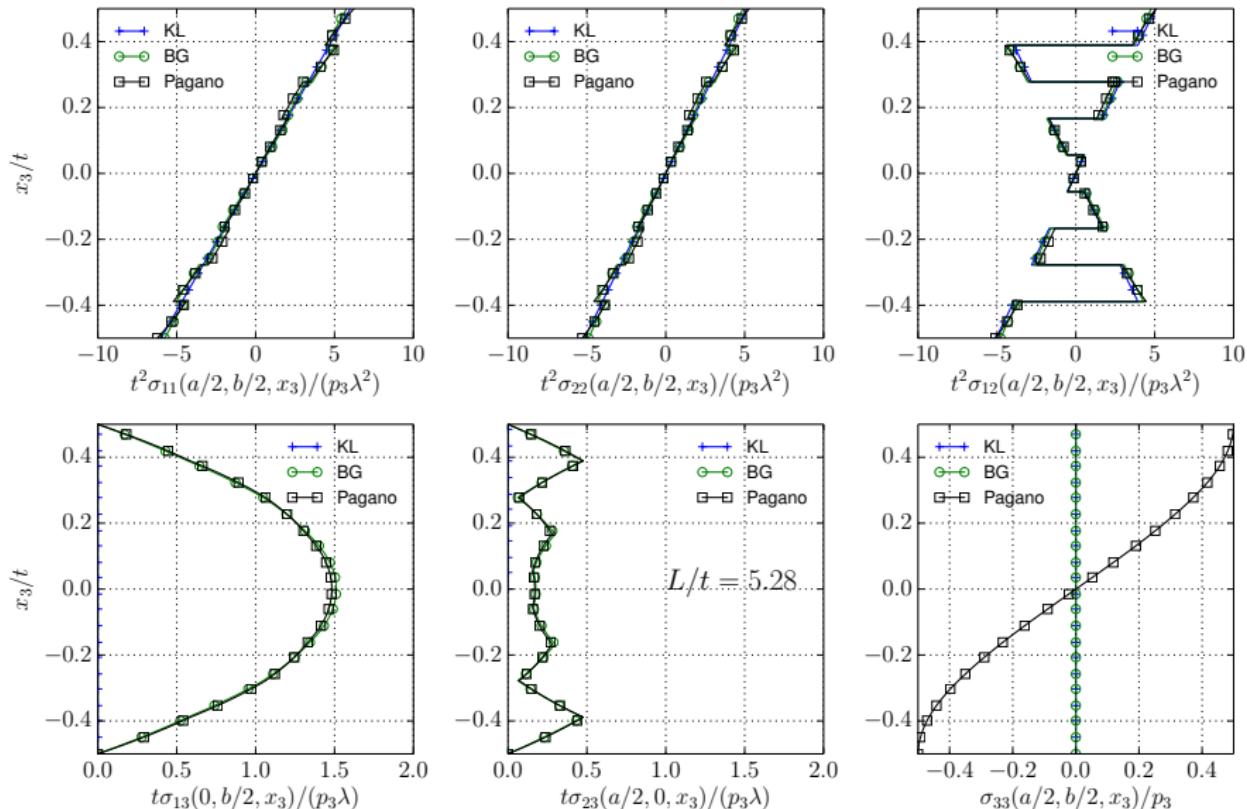
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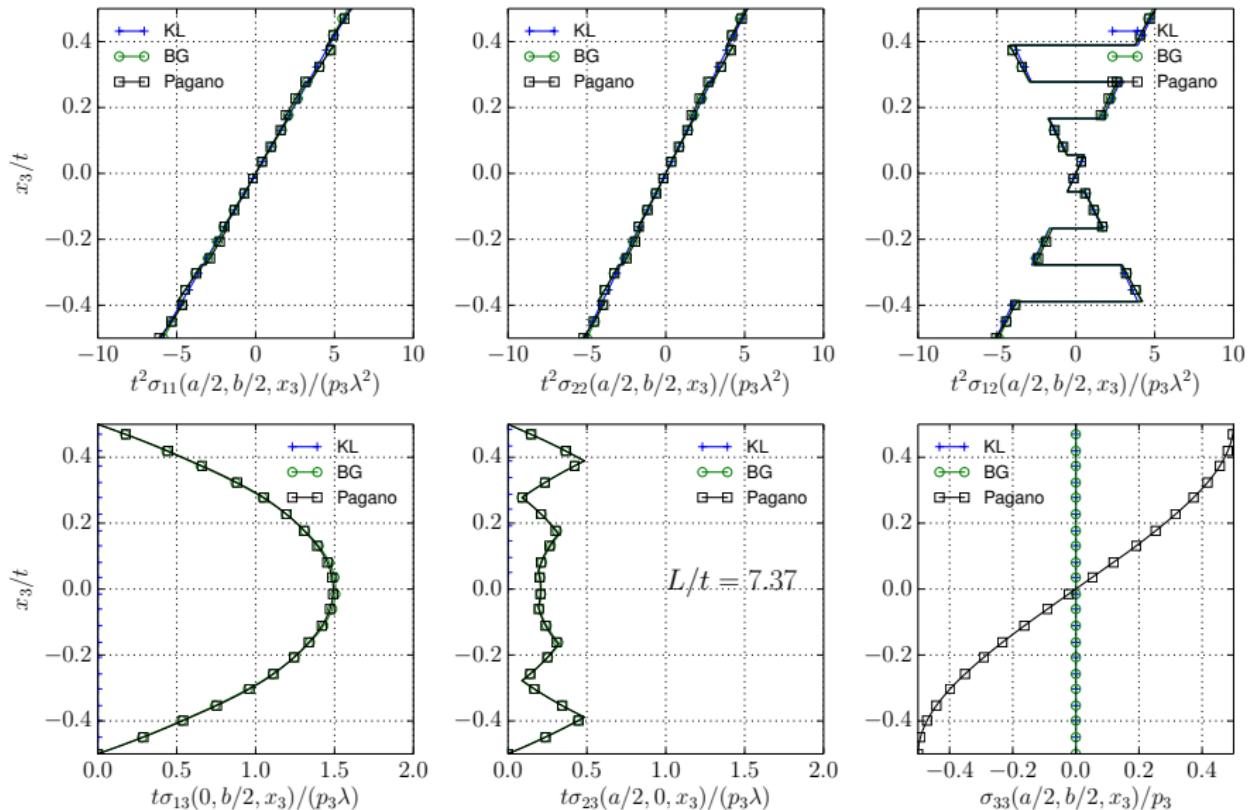
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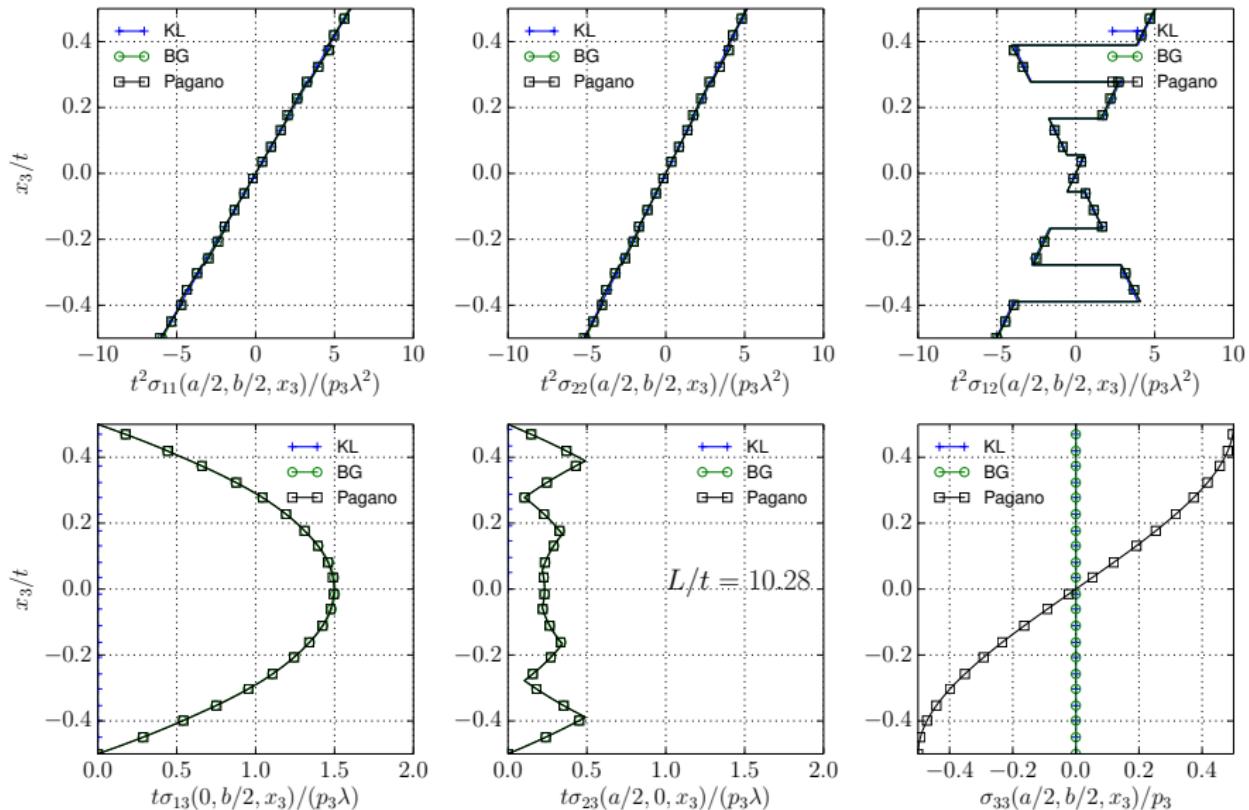
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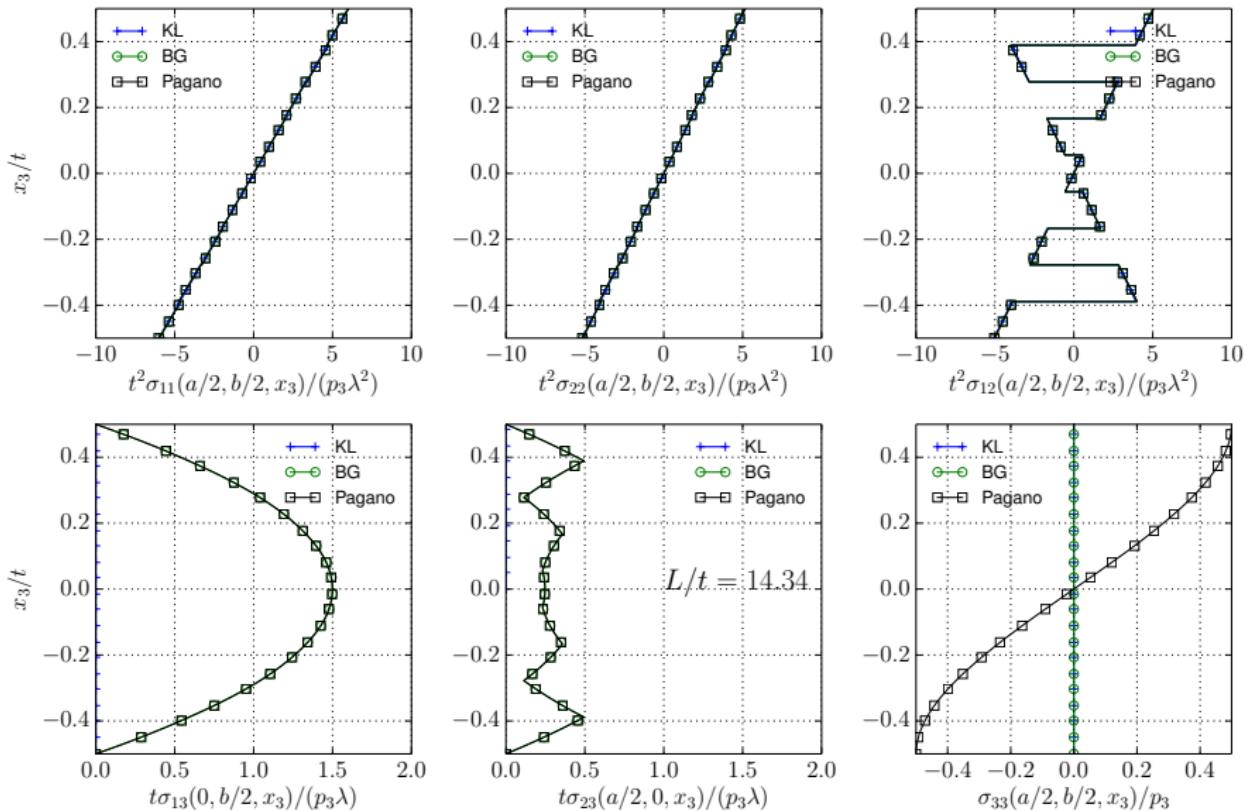
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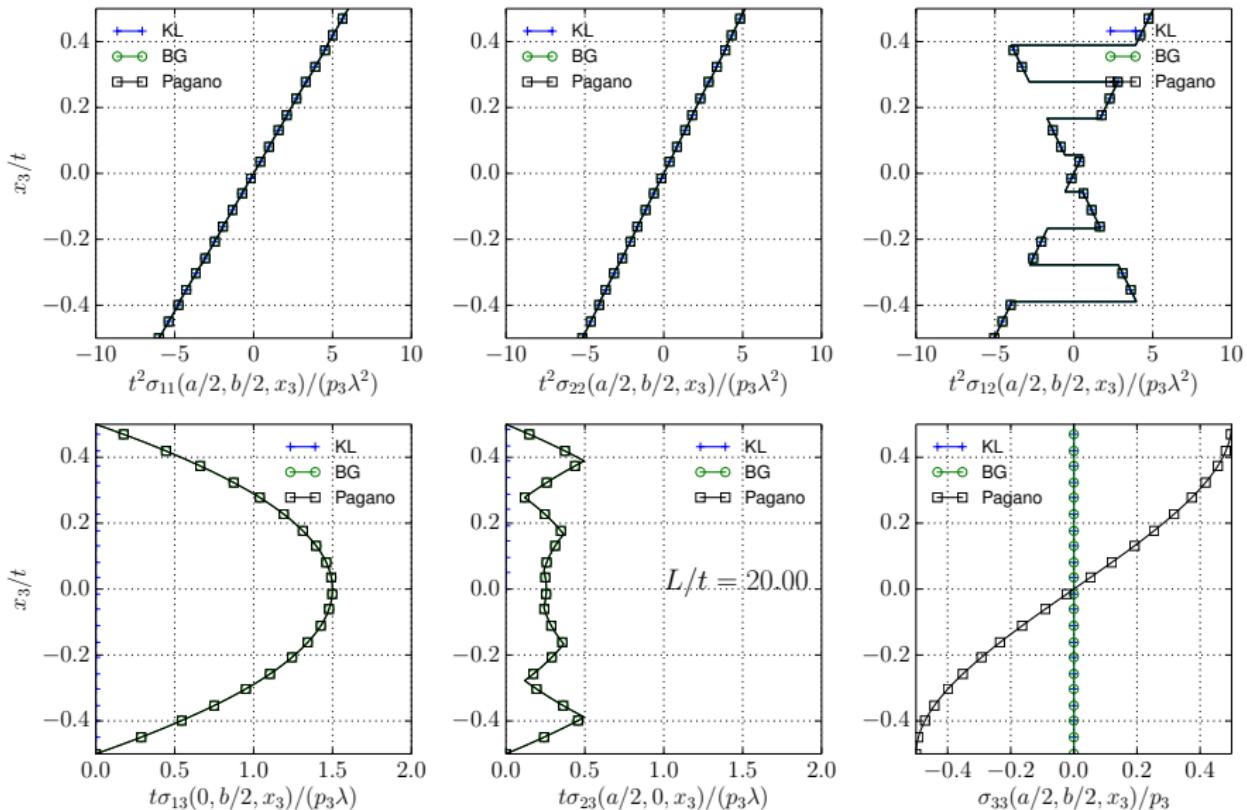
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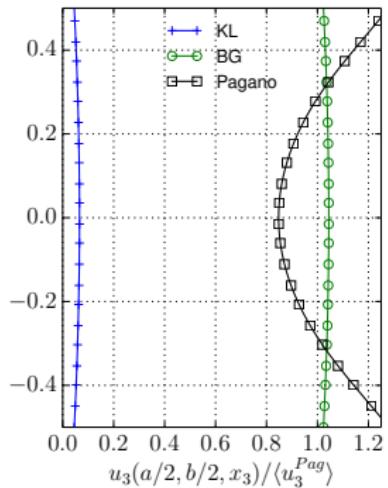
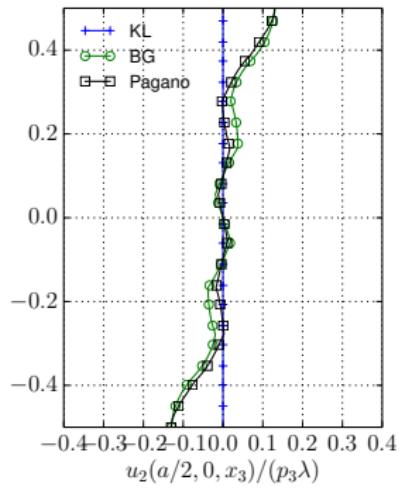
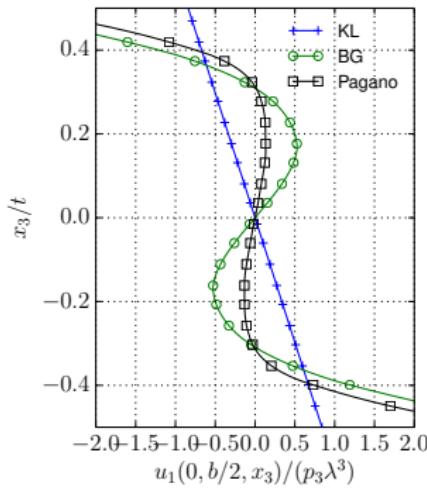
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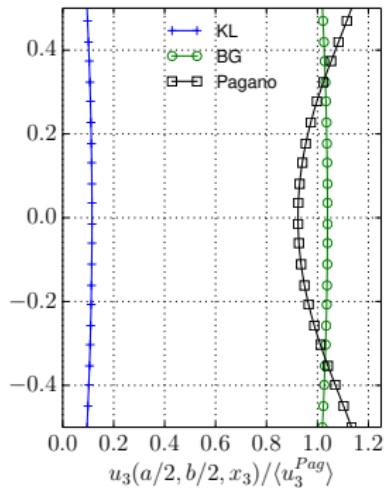
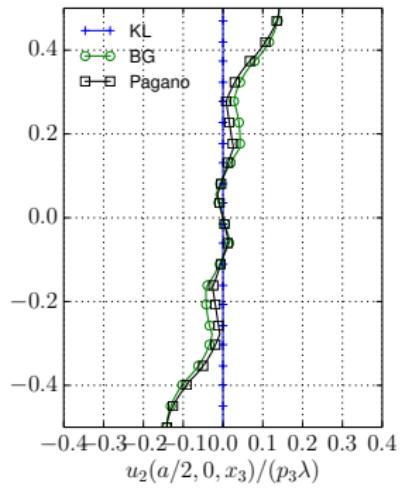
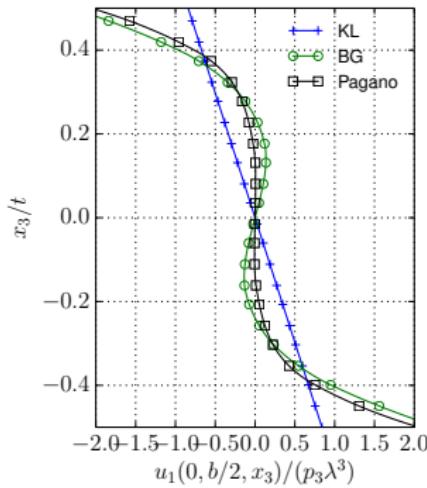
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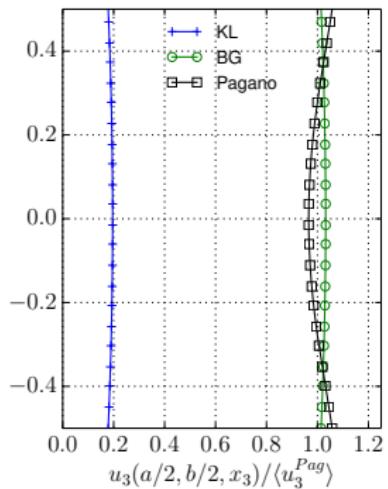
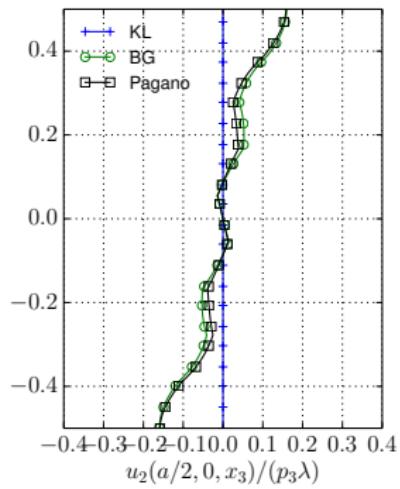
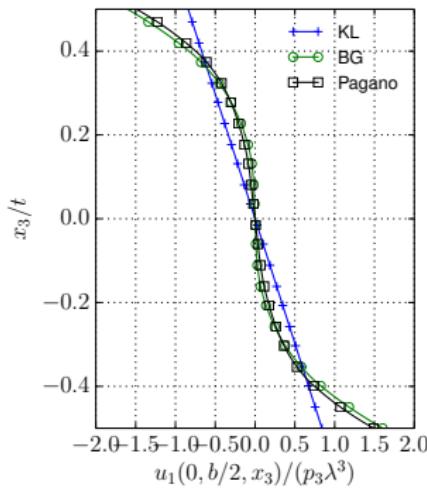
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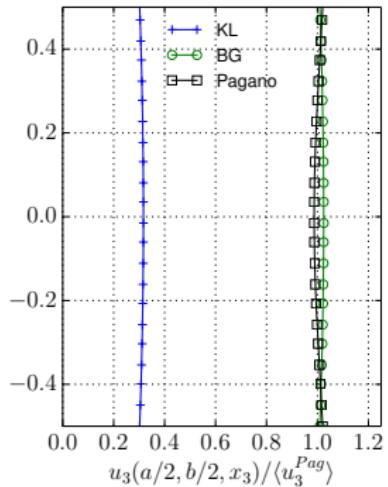
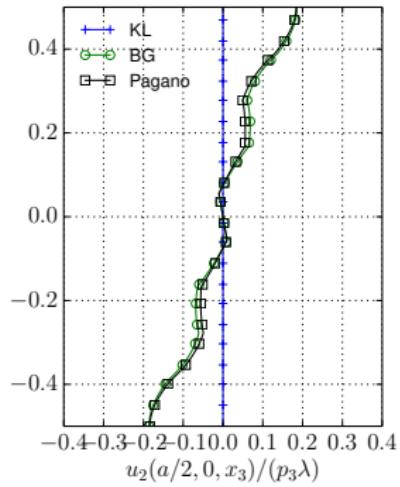
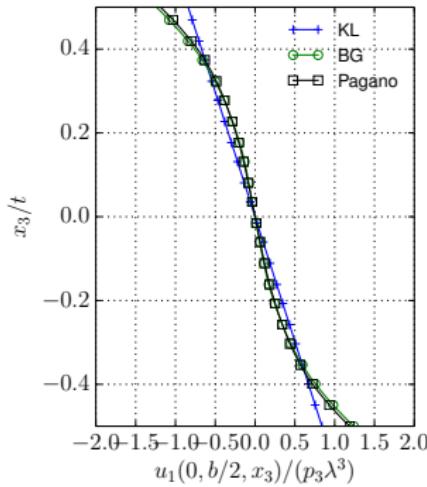
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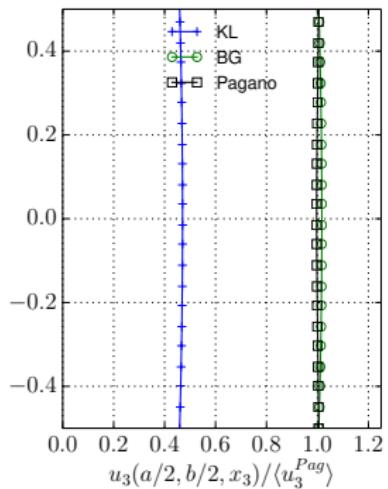
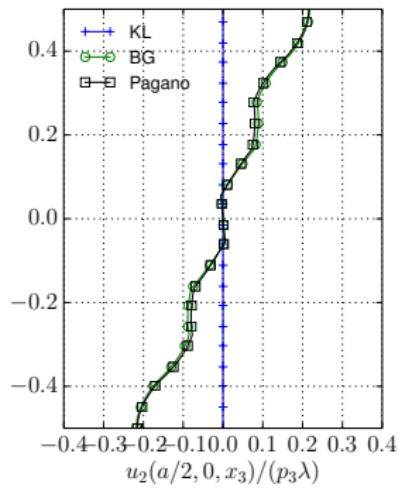
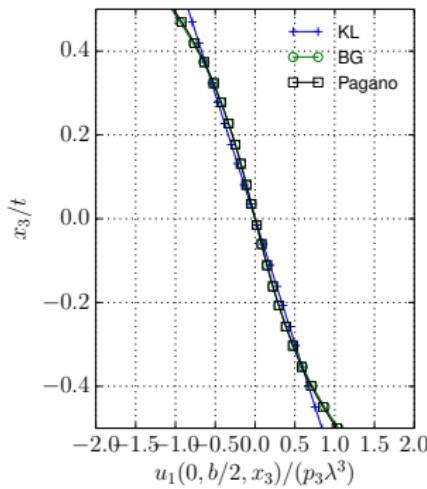
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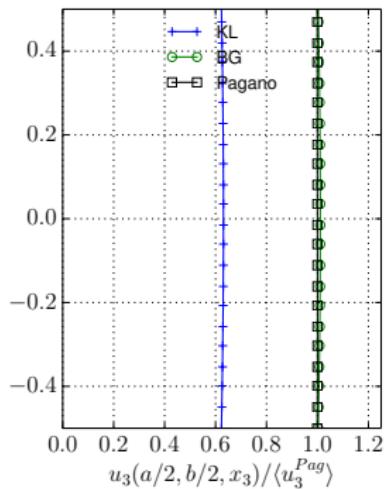
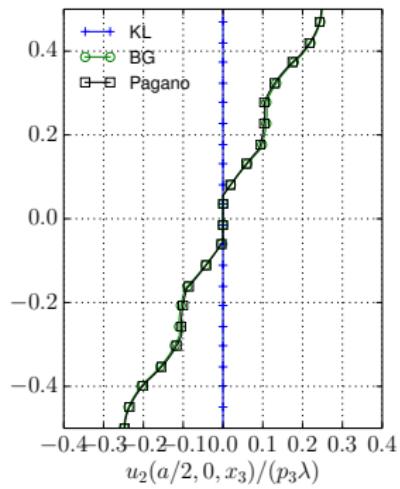
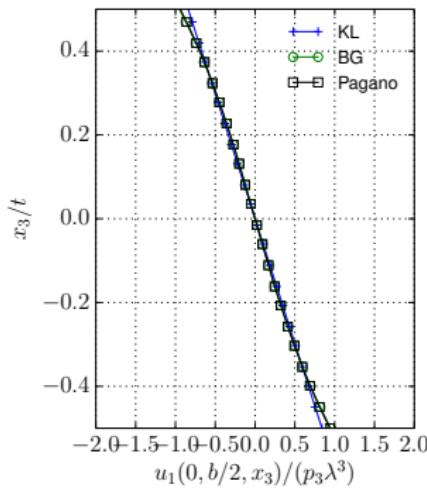
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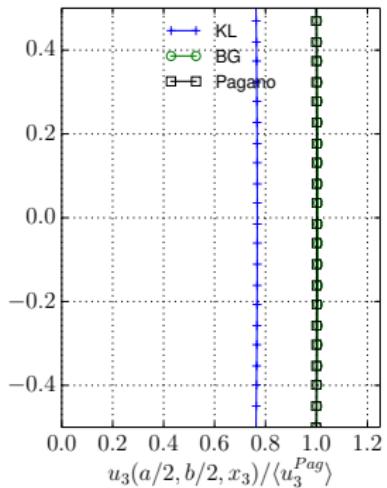
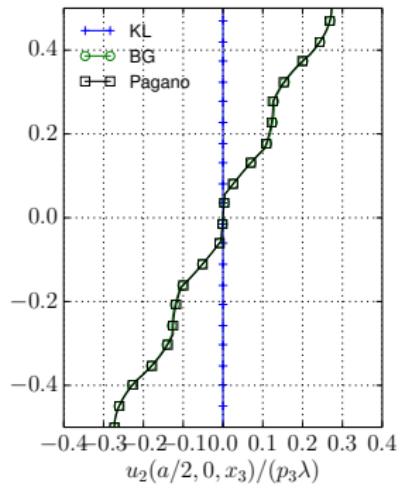
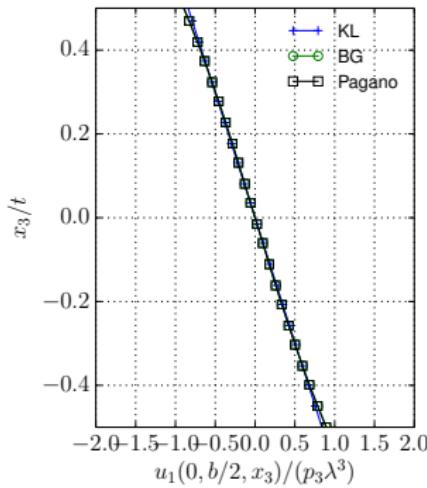
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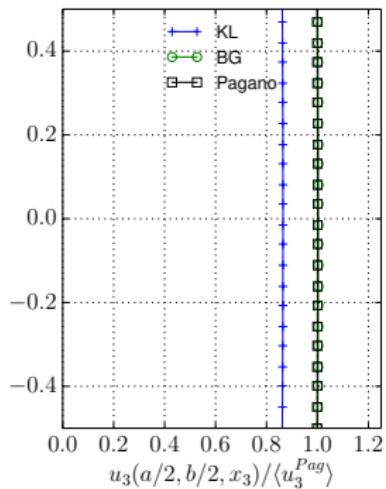
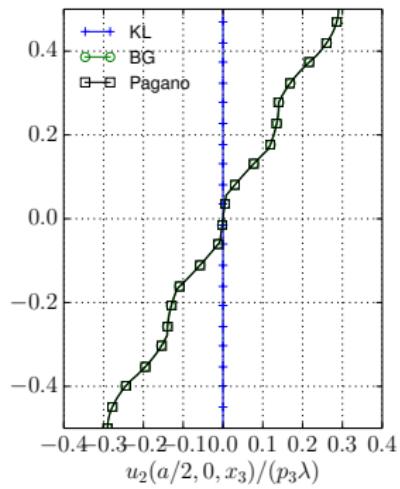
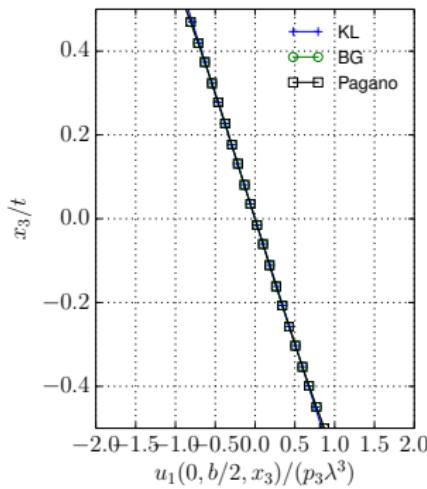
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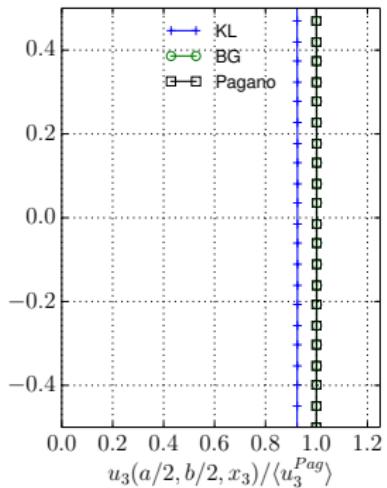
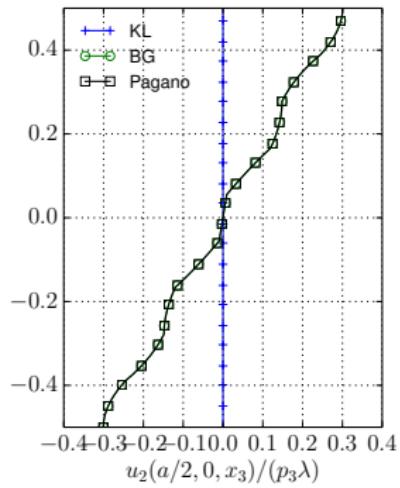
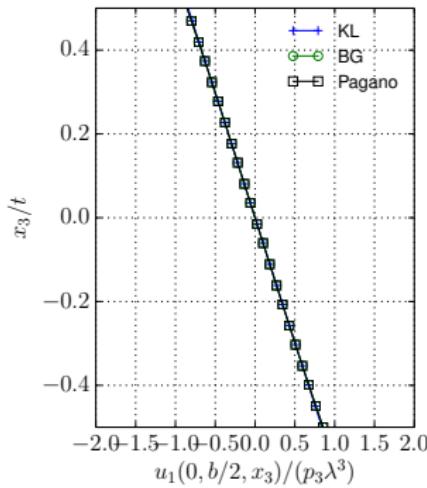
Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



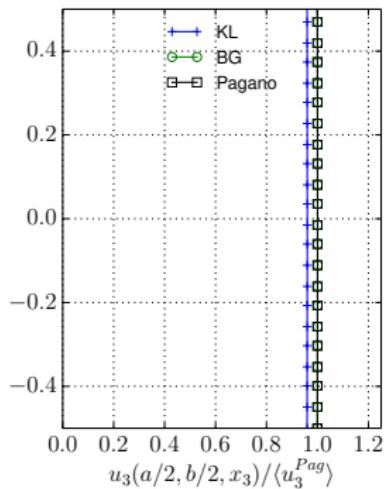
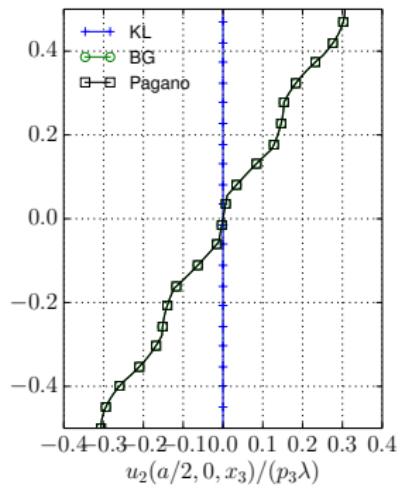
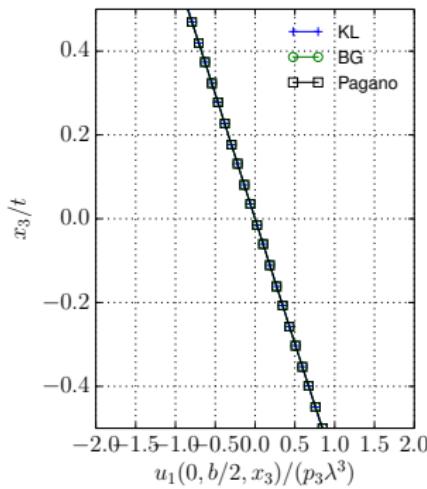
Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



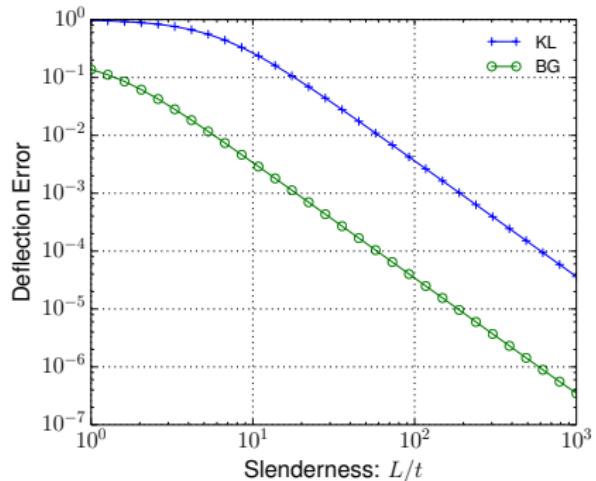
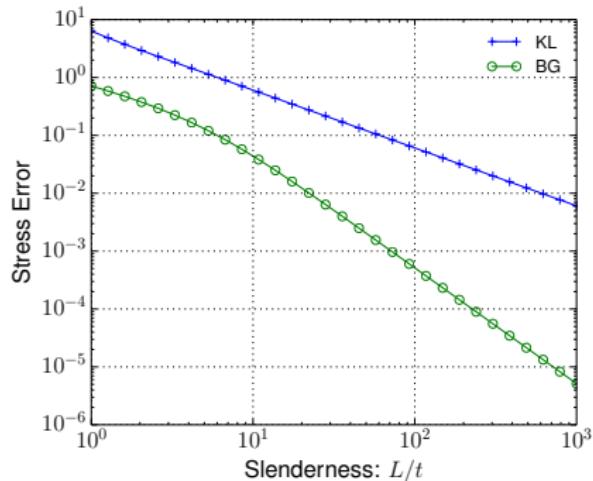
Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



Convergence for a $[30^\circ, -30^\circ, 30^\circ]$ stack



$\Delta(\sigma)$ rate: KL $\sim t$ and BG $\sim t^2$

$\Delta(U_3)$ rate: KL $\sim t^2$ and BG $\sim t^2$

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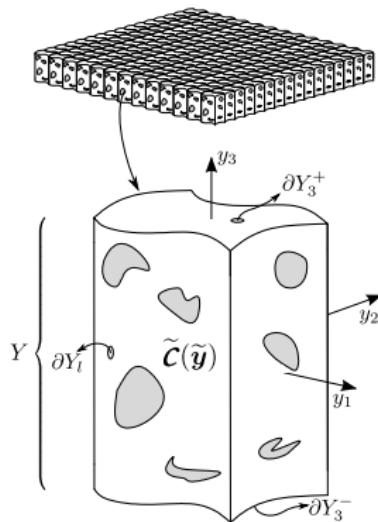
Extension to periodic plates

The case of cellular sandwich panels

Why all periodic plates are not “Reissner” like...

Extension to periodic plates

- ▶ Unit-cell and average estimates



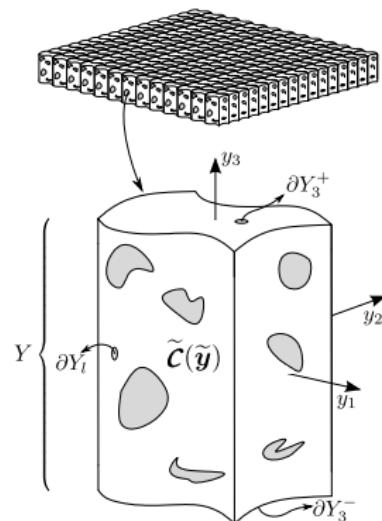
Extension to periodic plates

- ▶ Unit-cell and average estimates
- ▶ Bending auxiliary problem (Caillerie, 1984)

$$\mathcal{P}^K \left\{ \begin{array}{l} \tilde{\sigma}^K \cdot \nabla = 0 \\ \tilde{\sigma}^K = \mathbf{C}(\underline{y}) : \tilde{\varepsilon}^K \\ \tilde{\varepsilon}^K = y_3 \mathbf{K} + \nabla \otimes^s \underline{u}^{per} \\ \tilde{\sigma}^K \cdot \underline{e}_3 = 0 \text{ on free faces } \partial Y_3^\pm \\ \tilde{\sigma}^K \cdot \underline{n} \text{ skew-periodic on lateral edge } \partial Y_1 \\ \underline{u}^{per}(\underline{y}) \text{ } (y_1, y_2) \text{-periodic on lateral edge } \partial Y_1 \end{array} \right.$$

→ gives:

Localization \underline{u}^K $\tilde{\sigma}^K$ related to the curvature \mathbf{K}
 Bending stiffness: D_{\approx}



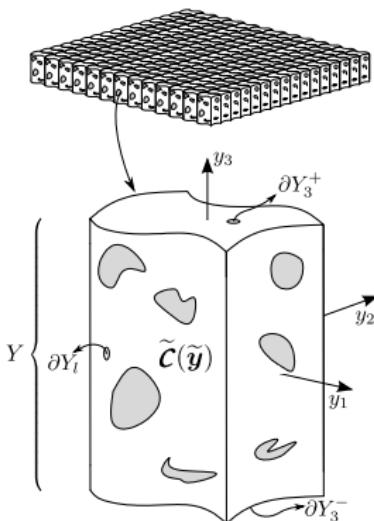
Extension to periodic plates

- ▶ Unit-cell and average estimates
- ▶ Bending auxiliary problem (Caillerie, 1984)
- ▶ Shear auxiliary problem

$$\mathcal{P}^R \left\{ \begin{array}{l} \underline{\sigma}^R \cdot \nabla + \underline{\sigma}^M(\underline{y}) = 0 \\ \underline{\sigma}^R = \underline{\mathbf{C}}(\underline{y}) : (\underline{\delta} \otimes^s \underline{u}^M + \nabla \otimes^s \underline{u}^R) \\ \underline{\sigma}^R \cdot \underline{e}_3 = 0 \text{ on free faces } \partial Y_3^\pm \\ \underline{\sigma}^R \cdot \underline{n} \text{ skew-periodic on lateral edge } \partial Y_l \\ \underline{u}^R(\underline{y}) \text{ } (y_1, y_2)\text{-periodic on lateral edge } \partial Y_l \end{array} \right.$$

→ gives:

Localization \underline{u}^R and $\underline{\sigma}^R$ related to \underline{R}
Shear compliance tensor: $\underline{\mathbf{f}}$



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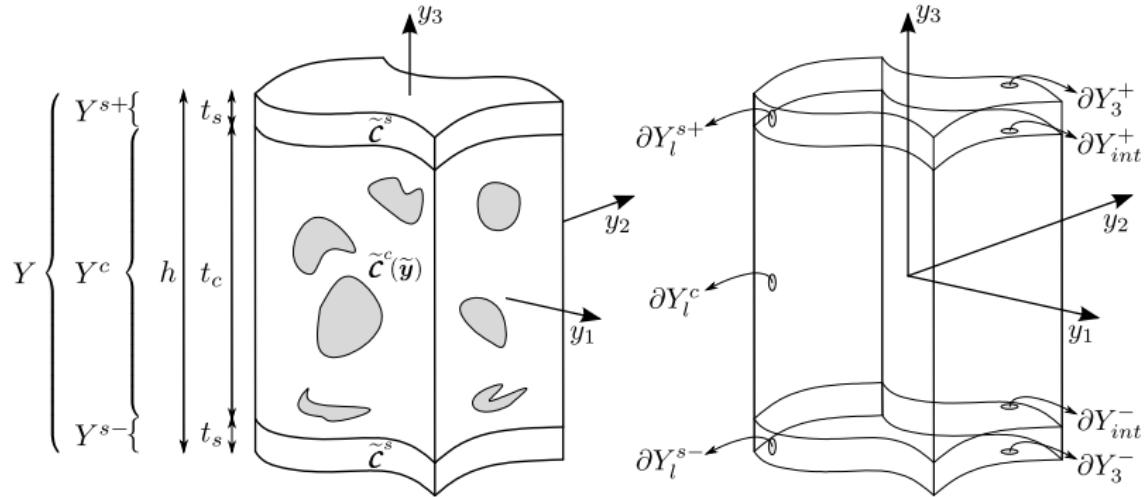
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Why all periodic plates are not “Reissner” like...

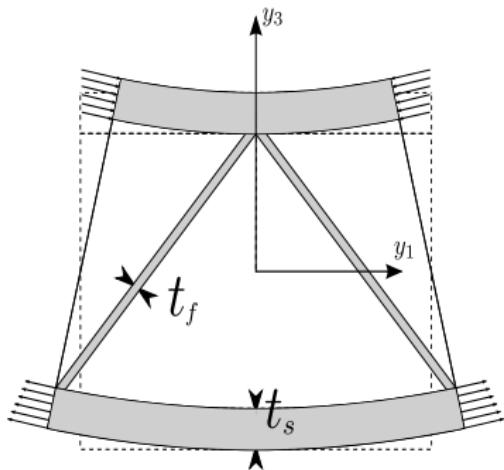
Justification of the Sandwich Theory

- Divide in 3 layers
(homogeneous skins and heterogeneous core)



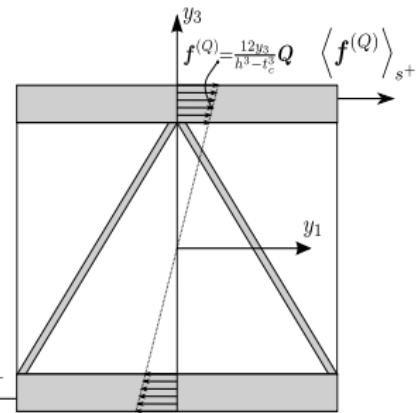
Justification of the Sandwich Theory

- ▶ Divide in 3 layers
(homogeneous skins and heterogeneous core)
- ▶ Bending auxiliary problem
 - ▶ Contrast assumption $\Leftrightarrow t_f \ll t_s$:
 $\rightarrow t_s/t_f$ Contrast ratio
 - \Rightarrow Skins under traction/compression
 - \Rightarrow Core not involved in Bending stiffness



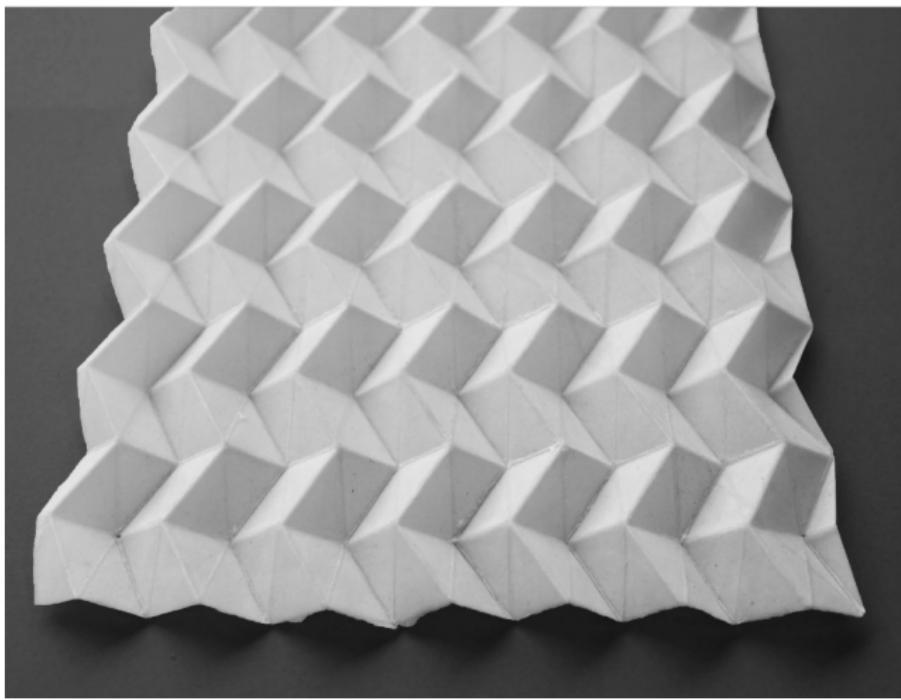
Justification of the Sandwich Theory

- ▶ Divide in 3 layers
(homogeneous skins and heterogeneous core)
- ▶ Bending auxiliary problem
- ▶ Shear auxiliary problem
 - ▶ \underline{f}^R becomes $\underline{f}^{(Q)}$ + Direct homogenization scheme
 - ▶ The BG is degenerated into RM model
 - ▶ $\underline{f}^{(Q)}$ confirms the classical intuition
 - ▶ Proof of the bounds from Kelsey et al. (1958)



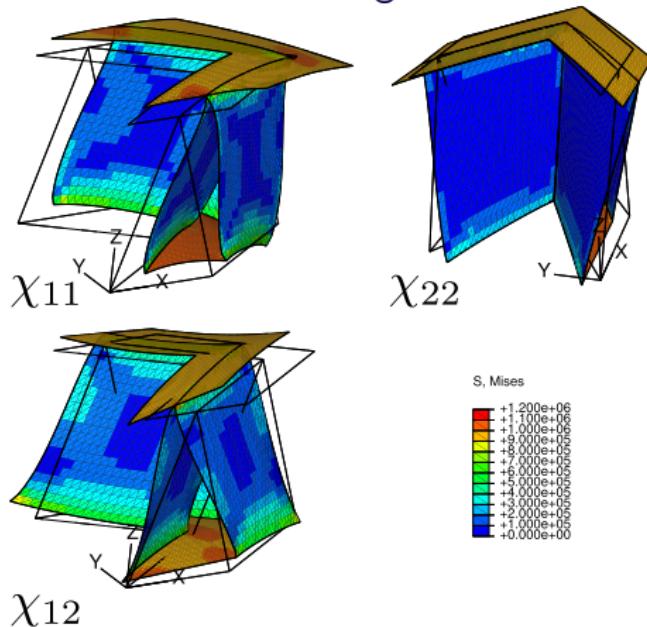
Lebée and Sab (2012a)

Application to the chevron pattern



Application to the chevron pattern

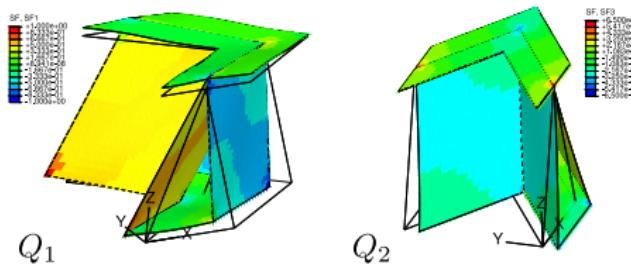
Bending:



Application to the chevron pattern

Shear forces
localization $\sigma^{(Q)}$

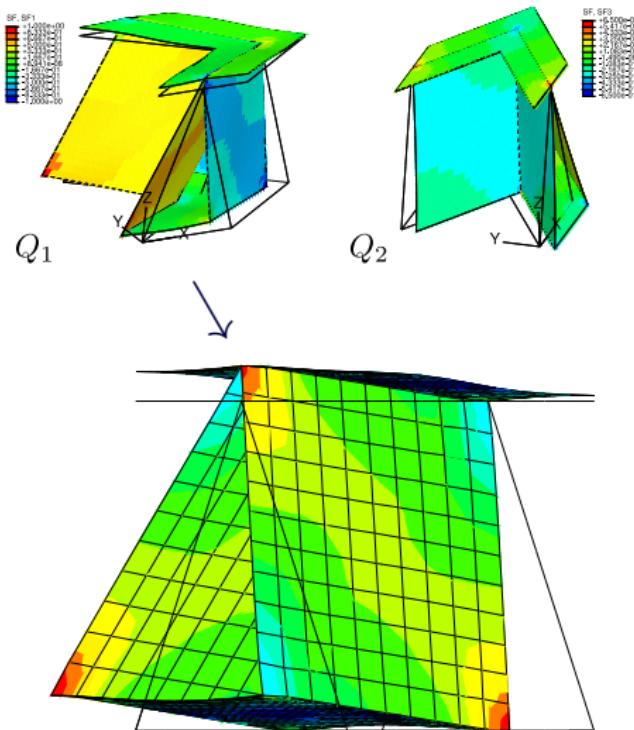
- Overall shearing
of the core



Application to the chevron pattern

Shear forces
localization $\sigma^{(Q)}$

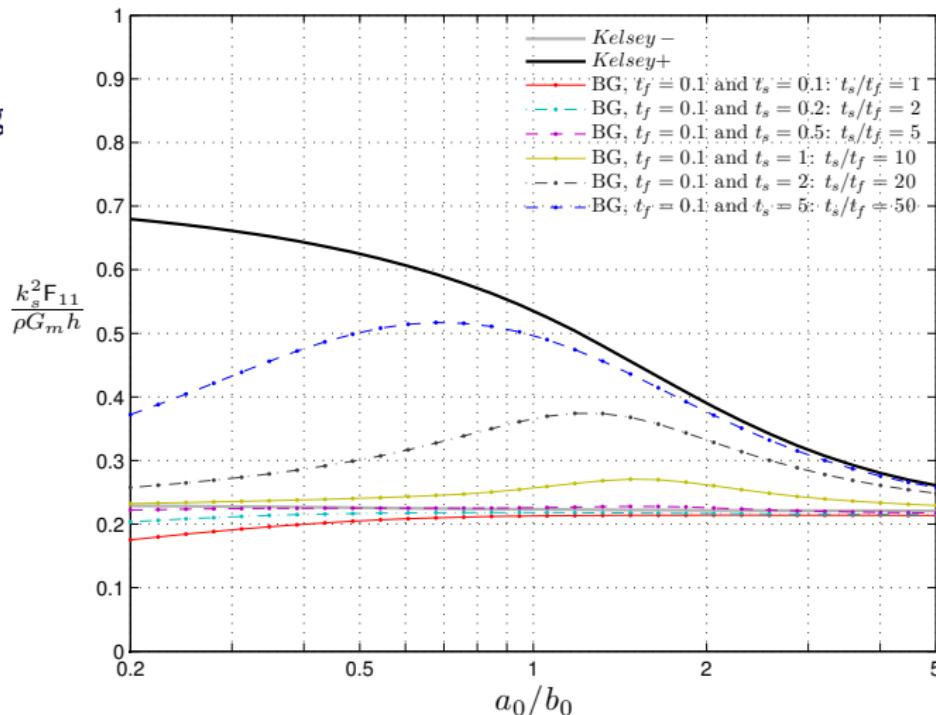
- ▶ Overall shearing of the core
- ▶ Out-of-plane skins distortion



Application to the chevron pattern

Shear forces
localization $\sigma^{(Q)}$

- ▶ Overall shearing of the core
- ▶ Out-of-plane skins distortion
- ▶ Critically influence shear force stiffness



Lebée and Sab (2012b)

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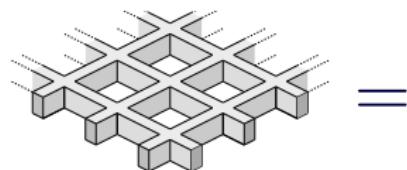
Cylindrical bending of laminates

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Why all periodic plates are not "Reissner" like...

Homogenizing an orthogonal beam lattice?

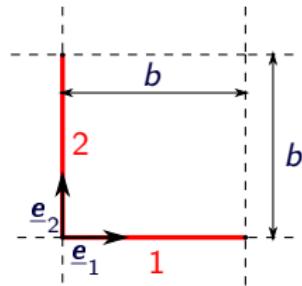
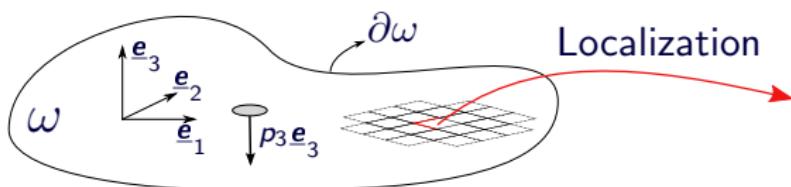


=

Thick-plate model (macro)

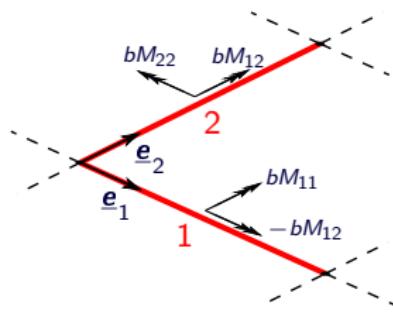
+

2 St-Venant Beams (micro)



Field localization

Bending moment $(\underline{r}^{(M)}, \underline{m}^{(M)})$:
 Apply $\underline{\underline{M}}$ "on average" on the unit-cell (Caillerie, 1984)



$${}^1\underline{r}^{(M)} = {}^2\underline{r}^{(M)} = \underline{0}$$

$${}^1\underline{m}^{(M)} = \begin{pmatrix} -bM_{12} \\ bM_{11} \\ 0 \end{pmatrix}_1 \quad \text{and} \quad {}^2\underline{m}^{(M)} = \begin{pmatrix} bM_{12} \\ bM_{22} \\ 0 \end{pmatrix}_2$$

Field localization

Bending moment $(\underline{r}^{(M)}, \underline{m}^{(M)})$:

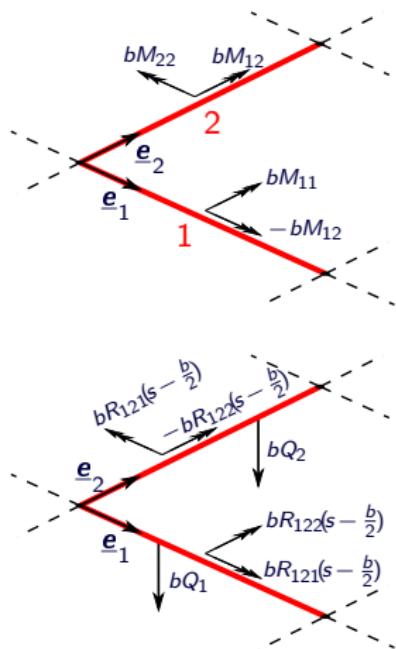
Apply \underline{M} "on average" on the unit-cell (Caillerie, 1984)

Bending gradient $(\underline{r}^{(R)}, \underline{m}^{(R)})$:

Assume $M_{\alpha\beta} = R_{\alpha\beta\gamma}x_\gamma$ (Lebée and Sab, 2013a)

$${}^1\underline{r}^{(R)} = \begin{pmatrix} 0 \\ 0 \\ b(R_{111} + R_{122}) \end{pmatrix}_1 \quad {}^1\underline{m}^{(R)} = \begin{pmatrix} bR_{121}\left(s - \frac{b}{2}\right) \\ bR_{122}\left(s - \frac{b}{2}\right) \\ 0 \end{pmatrix}_1$$

$${}^2\underline{r}^{(R)} = \begin{pmatrix} 0 \\ 0 \\ b(R_{121} + R_{222}) \end{pmatrix}_2 \quad {}^2\underline{m}^{(R)} = \begin{pmatrix} -bR_{122}\left(s - \frac{b}{2}\right) \\ bR_{121}\left(s - \frac{b}{2}\right) \\ 0 \end{pmatrix}_2$$



Field localization

Bending moment $(\underline{r}^{(M)}, \underline{m}^{(M)})$:

Apply \underline{M} "on average" on the unit-cell (Caillerie, 1984)

Bending gradient $(\underline{r}^{(R)}, \underline{m}^{(R)})$:

Assume $M_{\alpha\beta} = R_{\alpha\beta\gamma}x_\gamma$ (Lebée and Sab, 2013a)

Reissner-Mindlin $(\underline{r}^{(Q)}, \underline{m}^{(Q)})$:

Assume cylindrical bending (Whitney, 1969; Cecchi and Sab, 2007)

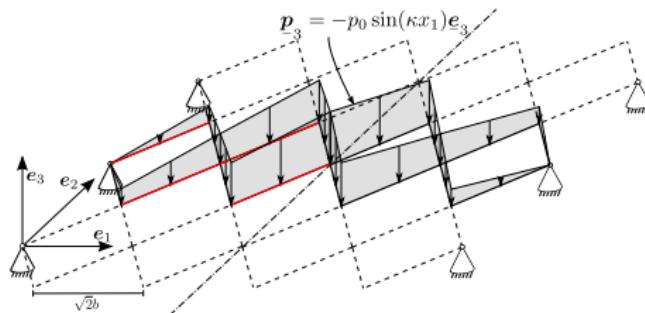
$$Q_1 = R_{111}, \quad Q_2 = R_{222}, \quad R_{121} = R_{122} = R_{221} = R_{112} = 0$$

$${}^1\underline{r}^{(Q)} = \begin{pmatrix} 0 \\ 0 \\ bQ_1 \end{pmatrix}_1 \quad \text{and} \quad {}^1\underline{m}^{(Q)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_1$$

$${}^2\underline{r}^{(Q)} = \begin{pmatrix} 0 \\ 0 \\ bQ_2 \end{pmatrix}_2 \quad \text{and} \quad {}^2\underline{m}^{(Q)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_2$$

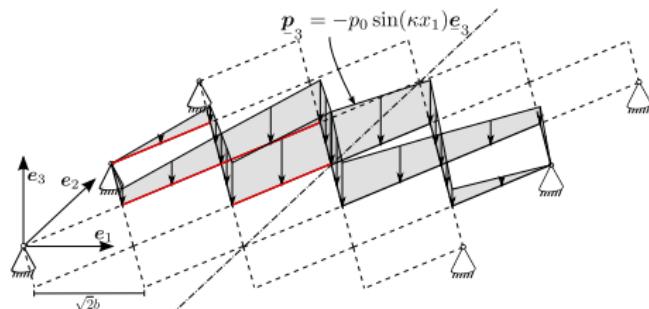
Navier

Application: lattice rotated 45° and cylindrical bending

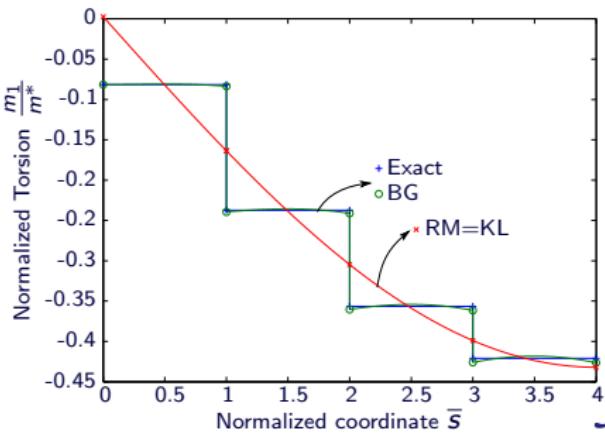
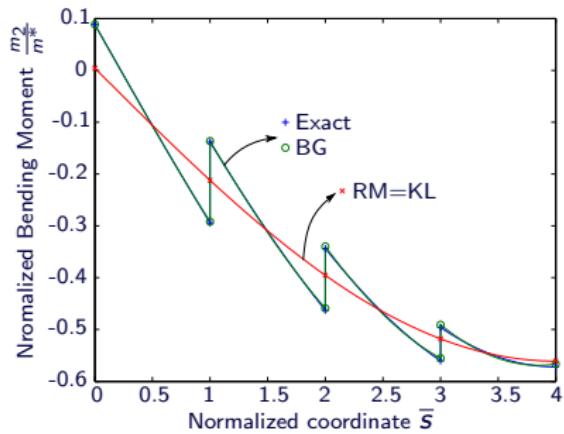


- ▶ Exact solution
- ▶ Plate solution + Localization (RM and BG)

Application: lattice rotated 45° and cylindrical bending



- ▶ Exact solution
- ▶ Plate solution + Localization (RM and BG)



Navier

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