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Dynamic optimal execution in a mixed-market-impact Hawkes price model

Aurélien Alfonsi, Pierre Blanc*

April 2, 2014

Abstract

We study a linear price impact model including other liquidity takers, whose flow of orders either follows a Poisson or a Hawkes process. The optimal execution problem is solved explicitly in this context, and the closed-formula optimal strategy describes in particular how one should react to the orders of other traders. This result enables us to discuss the viability of the market. It is shown that Poissonian arrivals of orders lead to quite robust Price Manipulation Strategies in the sense of Huberman and Stanzl [24]. Instead, a particular set of conditions on the Hawkes model balances the self-excitation of the order flow with the resilience of the price, excludes Price Manipulation Strategies and gives some market stability.

Keywords: Market Impact Model, Optimal Execution, Hawkes Processes, Market Microstructure, High-frequency Trading, Price Manipulations.

AMS (2010): 91G99, 91B24, 91B26, 60G55, 49J15.

1 Introduction

When modeling the price of an asset, we typically distinguish at least three different time scales. At the low-frequency level, the price can often be well approximated by a diffusive process. At the other end, when dealing with very high frequencies, some key features of the Limit Order Book (LOB) dynamics have to be modeled. In between, price impact models consider an intra-day mesoscopic time scale, somewhere between seconds and hours. They usually ignore most of the LOB events (limit orders, cancellations, market orders, etc.) and focus on describing the price impact of the transactions. Their goal is to be more tractable than high-frequency models and to bring quantitative results on practical issues such as optimal execution strategies. The pioneering price impact models of Bertsimas and Lo [11] and Almgren and Chriss [6] consider a linear immediate and permanent impact on the price. These models ignore the transient part of the impact which is due to the resilience of the market and cannot be neglected when trading frequently. For that purpose, Obizhaeva and Wang [28] have considered a model that includes in addition a linear transient impact that decays exponentially. However, empirical evidence on market data shows that the price impact is not linear but rather concave, see e.g. Potters and Bouchaud [29], Eisler et al. [16], Mastromatteo, Tóth

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and Bouchaud [27] and Donier [15]. Extensions or alternatives to the Obizhaeva and Wang model that include non-linear price impact have been proposed by Alfonsi, Fruth and Schied [2], Predoiu, Shaikhet and Shreve [30], Gatheral [19] and Guéant [21] to mention a few. Similarly, the exponential decay of the transient impact is not truly observed on market data, and one should consider more general decay kernels. Alfonsi, Schied and Slynko [5] and Gatheral, Schied and Slynko [20] consider the extension of the Obizhaeva and Wang model when the transient impact has a general decay kernel. Another simplification made by these models is that they generally assume that there is only one large trader, and basically ignore the market orders issued by other participants. However, if one wants to use these models at a higher frequency, they would naturally wonder how these orders (at least the largest ones) can be taken into account. This is one of the contributions of the present paper.

On the other hand, high-frequency price models aim at reproducing some statistical observations made on market data such as the autocorrelation in the signs of trades, the volatility clustering effect, the high-frequency resilience of the price, etc., and to obtain low-frequency asymptotics that are consistent with continuous diffusions. At very high frequencies, one then has to describe LOB dynamics, or a part of it. Such models have been proposed by Abergel and Jedidi [1], Huang, Lehalle and Rosenbaum [23], Cont and de Larrard [13], Garèche et al. [18], among others. However, as stressed in [13], LOB events are much more frequent than price moves. Thus, it may be relevant to model the price at the slightly lower frequency of midpoint price changes. For example, Robert and Rosenbaum [31] have proposed a model based on a diffusion with uncertainty zones that trigger the price changes. Recently, Bacry et al. [7] presented a tick-by-tick price model based on Hawkes processes, that reproduces well some empirical facts of market data. This model has then been enriched by Bacry and Muzy [9] to describe jointly the order flow and the price moves. In fact, there is a very recent and active literature that focuses on the use of mutually exciting Hawkes processes in high-frequency price models. Without being exhaustive, we mention here the works of Da Fonseca and Zaatour [14], Zheng, Roueff and Abergel [34], Filimonov and Sornette [17] and Hardiman, Bercot and Bouchaud [22]. Asymptotic and low-frequency behaviour of such models has been investigated recently by Bacry et al. [8] and Jaisson and Rosenbaum [25].

The present paper is a contribution to this also mutually exciting literature. Its main goal is to make a bridge between high-frequency price models and optimal execution frameworks. On the one hand, Hawkes processes seem to be rich enough to describe satisfactorily the flow of market orders. On the other hand, price impact models are tractable and well-designed to calculate trading costs. The aim of our model is to grasp these two features. Thus, we consider an Obizhaeva and Wang framework where market buy and sell orders issued by other traders are modeled through Hawkes processes. This enables us to make quantitative calculations and to solve the optimal execution problem explicitly. Also, we obtain a necessary and sufficient condition on the parameters of the Hawkes model to rule out Price Manipulation Strategies that can be seen as high-frequency arbitrages.

The paper is organized as follows. In Section 2, we set up the model and present a general criterion to exclude Price Manipulation Strategies. Section 3 summarizes our main results. In Section 4, we present the optimal execution strategy when market orders are Poissonian and discuss on the robustness of Price Manipulation Strategies in this case. Section 5 gives the main result on the optimal execution problem along with several comments and insights on the optimal strategy. Technical proofs are gathered in the Appendix.

2 Model setup and the optimal execution problem

We start by describing the price model itself, without considering the execution problem. We consider a single asset and denote by P_t its price at time t. We assume that we can write it as the sum of a "fundamental price" component S_t and a "mesoscopic price deviation" D_t:

$$P_t = \underbrace{S_t}_{\text{fundamental price}} + \underbrace{D_t}_{\text{mesoscopic price deviation}}. \tag{1}$$

Typically, these quantities are respectively related to the permanent and the transient impact of the market orders. We now precise this and consider the framework of Obizhaeva and Wang [28] where these impacts are linear. Let N_t be the sum of the signed volumes of past market orders on the book between time 0 and time t. By convention, a buy order is counted positively in N while a sell order makes N decrease, and we assume besides that N is a càdlàg (right continuous with left limits) process. We assume that an order modifies the price proportionally to its size, which would correspond to a block-shaped limit order book. A proportion $\nu \in [0,1]$ of the price impact is permanent, while the remaining proportion $1-\nu$ is transient with an exponential decay of speed $\rho > 0$. This mean-reversion effect can be seen as the feedback of market markers, who affect the price using limit orders and cancellations. Namely, we consider the following dynamics for S and D:

$$\begin{aligned} \mathrm{d}S_t &= \frac{\nu}{q} \underbrace{\mathrm{d}N_t}_{\text{market orders}} \\ \mathrm{d}D_t &= \underbrace{-\rho\, D_t\, \mathrm{d}t}_{\text{market resilience}} + \frac{1-\nu}{q} \underbrace{\mathrm{d}N_t}_{\text{market orders}}, \end{aligned}$$

with $q > 0$. As usual, we consider $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$ a filtered probability space where $\mathbb{P}$ weights the probability of the market events and the filtration $(\mathcal{F}_t)_{t\geq 0}$ describes the market information at time $t \geq 0$. We assume that the process $(N_t)_{t\geq 0}$ is $(\mathcal{F}_t)$-adapted with bounded variation and square integrable, i.e. $\sup_{s\in[0,t]} \mathbb{E}[N_s^2] < \infty$ for any $t \geq 0$. We will specify later on which dynamics we consider for N.

We now consider a particular trader who wants to buy or sell a given quantity of assets on the time interval $[0,T]$. Through the paper, we will call this trader the "strategic trader" to make the distinction between his market orders and all the other market orders, that are described by N. We will denote by X_t the quantity of shares owned by the strategic trader at time t. We assume that the process is $(\mathcal{F}_t)$-adapted, with bounded variation and càglàd (left continuous with right limits) which means that the strategic trader observes all the information available on the market, and that he can react instantly to the market orders issued by other traders. Thus, a strategy that liquidates x_0 assets on $[0,T]$ should satisfy $X_0 = x_0$ and $X_{T+} = 0$: $x_0 > 0$ (resp. $x_0 < 0$) corresponds to to a sell (resp. buy) program.

Definition 2.1. *A liquidating strategy X for the position $x_0 \in \mathbb{R}$ on $[0,T]$ is admissible if it is $(\mathcal{F}_t)$-adapted, càglàd, square integrable, with bounded variation and such that $X_0 = x_0$ and $X_{T^+} = 0$, a.s.*

One then has to specify how the strategic trader modifies the price, as well as the cost induced by his trading strategy. Again, we will consider the Obizhaeva and Wang model [28] with the same price impact as above. However, we let the possibility that the proportion $\epsilon \in [0,1]$ of permanent impact of the strategic trader could be different from the one of the other traders (which we note $\nu \in [0,1]$). For instance, one could justify that $0 \leq \epsilon < \nu$ should hold if the strategic trader does not hold any "real" economic information when he liquidates his position. We then assume the following dynamics

$$\mathrm{d}S_t = \frac{1}{q}\left(\nu \mathrm{d}N_t + \epsilon \mathrm{d}X_t\right), \tag{2}$$

$$\mathrm{d}D_t = -\rho\, D_t\, \mathrm{d}t + \frac{1}{q}\left((1-\nu)\mathrm{d}N_t + (1-\epsilon)\mathrm{d}X_t\right). \tag{3}$$

With the assumptions on N and X, the price processes P, S and D have left and right limits. More precisely, in case of discontinuity at time t, (2) and (3) have to be read here as follows

$$S_t - S_{t-} = \frac{\nu}{q}(N_t - N_{t-}),\ S_{t+} - S_t = \frac{\epsilon}{q}(X_{t+} - X_t),$$
$$D_t - D_{t-} = \frac{1-\nu}{q}(N_t - N_{t-}),\ D_{t+} - D_t = \frac{1-\epsilon}{q}(X_{t+} - X_t).$$

When the strategic trader places at time t an order of size $v \in \mathbb{R}$ ($v > 0$ for a buy order and $v < 0$ for a sell order), it has the following cost

$$\pi_t(v) \quad = \quad \underbrace{P_t\ v}_{\text{cost at the current price}} \quad + \quad \underbrace{\frac{v^2}{2q}}_{\text{impact cost}} \quad .$$

Since $P_{t+} = P_t + \frac{v}{q}$, this cost amounts to trade all the shares at the price $\frac{P_t + P_{t+}}{2}$ and corresponds to the cost given by a block-shaped Limit Order Book with depth $1/q$, since $\pi_t(v) = \int_0^v \left[P_t + \frac{1}{q}y\right] \mathrm{d}y$, see [28]. We stress here that if an order has just occurred, i.e. $N_t - N_{t-} \neq 0$, the value of P_t is different from P_{t-} and takes into account the price impact of this order. Therefore, the cost of an admissible strategy X is given by

$$\begin{aligned} C(X) &= \int_{[0,T)} P_u \ \mathrm{d}X_u \ + \ \frac{1}{2q} \sum_{0 \leq \tau < T} (\Delta X_\tau)^2 \ - \ P_T X_T \ + \ \frac{1}{2q} \ X_T^2 \qquad (4) \\ &= \int_{[0,T)} P_u \ \mathrm{d}X_u^c \ + \ \sum_{0 \leq \tau < T} P_\tau (\Delta X_\tau) + \ \frac{1}{2q} \sum_{0 \leq \tau < T} (\Delta X_\tau)^2 \ - \ P_T X_T \ + \ \frac{1}{2q} \ X_T^2, \end{aligned}$$

since at time T all the remaining shares have to be sold. Here, the sum brings on the countable times of discontinuity of X, $\Delta X_\tau = X_{\tau+} - X_\tau$, and $X_t^c = X_t - \sum_{0 \leq \tau < t} (\Delta X_\tau)$ is the continuous part of X. We note that all the terms involved in the cost function are integrable, thanks to the assumption on the square integrability of X and N.

Remark 2.1. *The cost defined by* (4) *in the price model* (1), (2) *and* (3) *is a deterministic function of* $(X_t)_{t \in [0,T]}$, $(N_t)_{t \in [0,T]}$, S_0, D_0 *and the parameters* q, ν, *and* ϵ. *In this remark, we denote by* $C(X, N, S_0, D_0, q)$ *this function when* ν *and* ϵ *are given. From* (2), (3) *and* (4), *we have the straightforward property*

$$C(X, N, S_0, D_0, q) = C(-X, -N, -S_0, -D_0, q). \qquad (5)$$

Observing that $qC(X) = \int_{[0,T)} qP_u \mathrm{d}X_u + \frac{1}{2} \sum_{0 \leq \tau < T} (\Delta X_\tau)^2 - (qP_T)X_T + \frac{1}{2}(X_T)^2$, *and remarking that* qS *and* qD *satisfy* (2) *and* (3) *with* $q = 1$, *we also get*

$$qC(X, N, S_0, D_0, q) = C(X, N, qS_0, qD_0, 1). \qquad (6)$$

Remark 2.2. *Since* X *is a càglàd process and* N *is a càdlàg process, we will have to work with làdlàg (with finite right-hand and left-hand limits) processes. When* Z *is a làdlàg process, we set* $\Delta^- Z_t = Z_t - Z_{t-}$ *and* $\Delta^+ Z_t = Z_{t^+} - Z_t$ *the left and right jumps of* Z, *and* $Z_t^c = Z_t - \sum_{0 \leq \tau < t} \Delta^+ Z_\tau - \sum_{0 < \tau \leq t} \Delta^- Z_\tau$ *the continuous part of* Z. *We also set* $\Delta Z_t = Z_{t+} - Z_{t-}$ *and use the shorthand notation* $\mathrm{d}Z_t = \mathrm{d}Z_t^c + \Delta Z_t$. *If* $\mathrm{d}Z_t = \mathrm{d}\tilde{Z}_t$ *for some other làdlàg process* $\tilde{Z}$, *this means that* $\mathrm{d}Z_t^c = \mathrm{d}\tilde{Z}_t^c$ *and* $\Delta Z_t = \Delta \tilde{Z}_t$. *In particular, when* Z *is càdlàg and* $\tilde{Z}$ *is càglàd, this means that* $Z_t - Z_{t-} = \tilde{Z}_{t^+} - \tilde{Z}_t$ *at the jump times.*

Then, the optimal execution problem consists in finding an admissible strategy X that minimizes the expected cost $\mathbb{E}[C(X)]$ for a given initial position $x_0 \in \mathbb{R}$. This problem for $x_0 = 0$ is directly related to the existence of Price Manipulation Strategies as defined below.

Definition 2.2. *A Price Manipulation Strategy (PMS) in the sense of Huberman and Stanzl [24] is an admissible strategy* X *such that* $X_0 = X_{T^+} = 0$ *a.s. for some* $T > 0$ *and* $\mathbb{E}[C(X)] < 0$.

We have the following result that gives a necessary and sufficient condition to exclude PMS.

Theorem 2.1. *The model does not admit PMS if, and only if the process P is a $(\mathcal{F}_t)$-martingale when $X \equiv 0$. In this case, the optimal strategy X^* is given by*

$$\Delta X_0^* = -\frac{x_0}{2+\rho T},\ \Delta X_T^* = -\frac{x_0}{2+\rho T},\ \mathrm{d}X_t^* = -\rho\frac{x_0}{2+\rho T}\mathrm{d}t \text{ for } t \in (0,T), \tag{7}$$

and has the expected cost $\mathbb{E}[C(X^*)] = -P_0 x_0 + \left[\frac{1-\epsilon}{2+\rho(T-t)} + \frac{\epsilon}{2}\right] x_0^2/q$.

Theorem 2.1 is proved in Appendix B. It indicates that suitable models for the order flow N should be such that P is a martingale when the strategic trader is absent. In this case, the optimal strategy is very robust in the sense that it does not depend on N, and is therefore the same as the one in the Obizhaeva and Wang model [28] that corresponds to $N \equiv 0$ and $D_0 = 0$. In fact, it does not depend either on ϵ and ν, and only depends on ρ. However, this result is obtained in an idealized framework where the market can be precisely described by the price model (1), (2) and (3) and where the strategic trader knows the model parameters. If this model can be well fitted to market data, one may expect that the process P to be, roughly speaking, nearly a martingale. This raises at least three questions. Which "simple" processes N can lead to a martingale price P? Can we characterize the optimal strategy when P is not a martingale? It would be interesting to understand how the optimal strategy (7) may be modified by other market orders. Last, when P is not a martingale, can we find strategies that still have a negative expected cost if the strategic trader has a wrong estimation of the parameters? In this paper, we study these questions when N follows either a Poisson or a Hawkes process.

Remark 2.3. *The model can be generalized by adding a càdlàg $(\mathcal{F}_t)$-martingale S^0 to the price process P, i.e. if we replace (1) by $P_t = S_t + D_t + S_t^0$, with $S_0^0 = 0$. This does not change the optimal execution problem since, using an integration by parts, S^0 adds the following term to the cost*

$$\begin{aligned}\int_{[0,T)} S_t^0\ \mathrm{d}X_t\ -\ S_T^0 X_T\ &=\ S_T^0 X_T\ -\ S_0^0 X_0\ -\ \int_{[0,T)} X_t\ \mathrm{d}S_t^0\ -\ S_T^0 X_T \\ &=\ -\int_{[0,T)} X_t\ \mathrm{d}S_t^0,\end{aligned}$$

which has a zero expected value from the martingale property. Let us note that there is no covariation between the processes X and S_0 since they do not jump simultaneously and X has bounded variations.

Remark 2.4. *Similarly, when N is a càdlàg $(\mathcal{F}_t)$-martingale and X is an admissible liquidating strategy for $X_0 = x_0$, we have*

$$\mathbb{E}[C(X)] = \mathbb{E}\left[\int_{[0,T)} D_u\ \mathrm{d}X_u\ +\ \frac{1-\epsilon}{2q}\sum_{0\le\tau<T}(\Delta X_\tau)^2\ -\ D_T X_T\ +\ \frac{1-\epsilon}{2q}\ X_T^2\right] + \frac{\epsilon}{2q}x_0^2,$$

since $x_0^2 = \int_{[0,T+]}\mathrm{d}[(X_t - X_0)^2] = 2\int_{[0,T)}(X_u - X_0)dX_u + \sum_{0\le\tau<T}(\Delta X_\tau)^2 - 2(X_T - X_0)X_T + X_T^2$. When $\epsilon \in [0,1)$, we set $X_t^\epsilon = (1-\epsilon)X_t$ and get

$$\mathbb{E}[C(X)] = \frac{1}{q(1-\epsilon)}\mathbb{E}\left[\int_{[0,T)} qD_u\mathrm{d}(X_u^\epsilon) + \frac{1}{2}\sum_{0\le\tau<T}(\Delta X_\tau^\epsilon)^2 - qD_T X_T^\epsilon + \frac{1}{2}(X_T^\epsilon)^2\right] + \frac{\epsilon}{2q}x_0^2. \tag{8}$$

Therefore, X is optimal if, and only if X^ϵ is optimal in the model with no permanent impact, $q = 1$ and an incoming flow of market orders equal to $(1-\nu)N$.

3 Main results

In this work, we consider either a Poisson or a Hawkes process for N, which leads to the two following models.

The Mixed-market-Impact Poisson (MIP) model. Poisson processes are often used to model the arrival of the customers in queuing theory. It is therefore natural to use them to model the flow of market orders, as it has been made for example by Bayraktar and Ludkovski [10] or Cont and de Larrard [13] in different frameworks. To be consistent with the parametrization of Hawkes process, we distinguish the incoming buy and sell orders. Thus, we set

$$N_t = N_t^+ - N_t^-, \tag{9}$$

where $(N_t^+)_{t\in[0,T]}$ and $(N_t^-)_{t\in[0,T]}$ are two independent compound Poisson processes of respective intensities κ_0^+ and κ_0^-. We assume that their jumps follow the same probability law μ on $\mathbb{R}_+$, and we define

$$m_k = \int_{\mathbb{R}^+} v^k \mu(\mathrm{d}v),\ k \in \mathbb{N},\ \delta_0 = \kappa_0^+ - \kappa_0^- \text{ and } \Sigma_0 = \kappa_0^+ + \kappa_0^-,$$

and assume that $m_2 < \infty$.

The Mixed-market-Impact Hawkes (MIH) model. The Hawkes model extends the previous one. It assumes that $N_t = N_t^+ - N_t^-$, where $(N_t^+)_{t\in[0,T]}$ and $(N_t^-)_{t\in[0,T]}$ are two self and mutually-exciting jump processes, with the same common law μ for the jump amplitudes. Their respective intensities κ_t^+ and κ_t^- are now assumed to be càdlàg processes which follow the Markovian Hawkes dynamics:

$$\mathrm{d}\kappa_t^+ \ = \ -\beta\,(\kappa_t^+ - \kappa_\infty)\,\mathrm{d}t \ + \ \iota_\mathrm{s}\,\mathrm{d}J_t^+ + \ \iota_\mathrm{c}\,\mathrm{d}J_t^- \quad , \quad \mathrm{d}\kappa_t^- \ = \ -\beta\,(\kappa_t^- - \kappa_\infty)\,\mathrm{d}t \ + \ \iota_\mathrm{c}\,\mathrm{d}J_t^+ + \ \iota_\mathrm{s}\,\mathrm{d}J_t^- \tag{10}$$

with $\iota_\mathrm{c}, \iota_\mathrm{s}, \beta, \kappa_\infty \geq 0$, and where J^+ (resp. J^-) is a càdlàg jump process that counts the number of buy market orders (resp. sell market orders) from environment traders since time 0, and the two processes cannot jump simultaneously. In other words, $J = J^+ - J^-$ jumps of ± 1 when N jumps and we have $\Delta J_t = \mathrm{sign}(\Delta N_t)$. This model boils down to the Poisson model in the case $\iota_\mathrm{c} = \iota_\mathrm{s} = \beta = 0$. The meaning of the parameters is rather clear: κ^+ and κ^- are mean reverting processes, and the parameters ι_s and ι_c respectively describe how a market buy order increases the instantaneous probability of buy (resp. sell) orders. More precisely, ι_s encodes both the splitting of meta-orders, and the fact that participants tend to follow market trends (which is called the herding effect). Empirically, it is found by Tòth et al. [33] that the main contribution comes from the splitting effect. The parameter ι_c describes opportunistic traders that sell (resp. buy) after a sudden rise (resp. fall) of the price.

Hawkes processes have been recently used in the literature to model the price. In particular, Bacry et al. [7] consider this model with $\nu = 1$, $\iota_\mathrm{s} = 0$, and deterministic jumps (i.e. μ is a Dirac mass). More recently, Bacry and Muzy [9] have proposed an four-dimensional Hawkes process to model the market buy and sell orders together with the up and down events on the price. In contrast, the model that we study here determines the price impact of an order in function of its size.

The main results are the following.

- The optimal execution problem can be solved explicitly in the MIH model and the optimal strategy has still a quite simple form, see Theorem 5.1. Of course, this result relies on the assumptions of linear price impact and exponential decay kernel, which are not in accordance with empirical facts, see for example Potters and Bouchaud [29] and Bouchaud et al. [12]. We mention here that it would be possible to keep an affine structure of the optimal strategy by considering complete monotone decay kernels as in Alfonsi and Schied [4]. However, we believe that the optimal strategy is interesting at least from a qualitative point of view, since it gives clear insights on how to react optimally to observed market orders and on the role of the different parameters of the model.

- Price Manipulation Strategies necessarily appear in the MIP model. They mainly arise from the fact that the strategy which consists in trading instantly half of the volume of each incoming market order in the opposite direction is profitable on average, since the price resilience of the market is modeled as deterministic and the jump rate of N is constant.
- In the MIH model, Price Manipulation Strategies also arise. Depending on the parameters of the model and on the size of each observed market order, one should trade instantly in the opposite direction as in the MIP model or in the same direction to take advantage of the self-excitation property of Hawkes processes. However, the Hawkes framework allows for a very specific equilibrium to take place, that we call the Mixed-market-Impact Hawkes Martingale (MIHM) model, where PMS disappear. When fitting the MIH to market data, one may then expect to get parameters close to the MIHM model.
- In the MIHM model, one has in particular $\iota_{\mathrm{s}} > \iota_{\mathrm{c}},\ \nu < 1$ and $\beta = \rho$, i.e. the self-excitation property of the order flow exactly compensates the price resilience induced by market makers. The resulting price process is a martingale even at high frequencies, and in this case we find that the optimal strategy and cost function are those of Obizhaeva and Wang [28]. The conditions of this model imply that if $\iota_{\mathrm{c}} = 0$, the norm α/β of the Hawkes kernel that symbolizes the endogeneity ratio of the market, see Filimonov and Sornette [17], should be equal to $1-\nu$, i.e. the proportion of market impact which is transient.
- The fact of reacting to the market orders of other traders with instantaneous market orders may be seen as artificial volume that could lead to market instability. We show that in the MIH framework, the conditions under which it is profitable for the strategic trader to react instantaneously to other trades is quite equivalent to the existence of PMS. This makes the connection between market stability and free profits.

4 The Mixed-market-Impact Poisson (MIP) model

We will use the following function in the sequel

$$L(r,\lambda,t) \ := \ r\int_0^t \frac{\exp(\lambda s)}{2+rs}\,\mathrm{d}s \ = \ \exp(-2\lambda/r)\ \left[\mathcal{E}\left(\frac{\lambda}{r}(2+rt)\right) - \mathcal{E}\left(\frac{2\lambda}{r}\right)\right], \tag{11}$$

where $\mathcal{E}(y) = -\int_{-y}^{+\infty} \frac{e^{-u}}{u}\,\mathrm{d}u$ is the exponential integral of y, in terms of Cauchy principal value if $y > 0$. Since we will only consider differences $\mathcal{E}(y) - \mathcal{E}(y')$ with either $y, y' > 0$ or $y, y' < 0$, we will only consider proper integrals. The function $\mathcal{E}$ is standard and is implemented in many packages such as the Boost C++ library. Thus, L can be evaluated as a closed formula.

4.1 The optimal strategy

The following theorem gives the optimal strategy and the value function of the problem in the Poisson model.

Theorem 4.1. *Let $\epsilon \in [0,1)$. The optimal admissible liquidating strategy X^* for the position x_0 in the Poisson model for N is*

$$\begin{aligned}
(1-\epsilon)\Delta X_0^* \ &= \ -\frac{(1-\epsilon)x_0 \ + \ [1+\rho T]\left(qD_0 - \frac{m_1}{\rho}\ \delta_0\right) \ - \ \frac{\nu m_1}{4}\rho T^2\delta_0}{2+\rho T} \\
(1-\epsilon)\Delta X_T^* \ &= \ \frac{qD_0 - (1-\epsilon)x_0}{2+\rho T} \ - \ \frac{m_1}{\rho}\times\left[\frac{1-\frac{\nu}{4}\rho^2T^2}{2+\rho T} \ + \ (1-\nu)\ln\left(1+\frac{\rho T}{2}\right) + \frac{\nu}{2}\ \rho T\right]\ \delta_0 \\
&\quad + \sum_{0<\tau<T} \frac{(1-\nu)\Delta N_\tau}{2+\rho(T-\tau)}
\end{aligned}$$

and, on $(0,T)$,

$$
\begin{aligned}
(1-\epsilon)\mathrm{d}X_t^* \quad = \quad & \left[\frac{1+\rho T\left(1+\frac{\nu}{4}\rho T\right)}{2+\rho T} - (1-\nu)\ln\left(\frac{2+\rho T}{2+\rho(T-t)}\right) - \frac{\nu}{2}\,\rho t \;-\; \frac{1+\frac{\nu}{2}\rho(T-t)}{2+\rho(T-t)}\right] m_1\delta_0\ \mathrm{d}t \\
& + \left[\frac{qD_0-(1-\epsilon)x_0}{2+\rho T} \;+\; \sum_{0<\tau<t}\frac{(1-\nu)\Delta N_\tau}{2+\rho(T-\tau)}\right]\rho\mathrm{d}t \\
& - \frac{1+\rho(T-t)}{2+\rho(T-t)} \times (1-\nu)\mathrm{d}N_t
\end{aligned}
$$

The corresponding cost function is given by $\mathcal{C}(x_0, D_0, S_0)$, *where*

$$
\begin{aligned}
q\times\mathcal{C}(x,d,z) \quad = \quad & -q(z+d)x \;+\; \left[\frac{1-\epsilon}{2+\rho T}+\frac{\epsilon}{2}\right]x^2 \;+\; \frac{\rho T}{2+\rho T}\left[qd-\mathcal{G}_0(T)\,\frac{\delta_0 m_1}{\rho}\right]x \\
& - \frac{1}{1-\epsilon}\times\frac{\rho T/2}{2+\rho T}\left[qd-\mathcal{G}_0(T)\,\frac{\delta_0 m_1}{\rho}\right]^2 - \frac{1}{1-\epsilon}\times\frac{\nu^2}{48}\,\rho^3T^3\,\left(\frac{\delta_0 m_1}{\rho}\right)^2 \\
& - \frac{(1-\nu)^2}{1-\epsilon}\times m_2\times\left[\frac{T}{2}-\frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right]\,\Sigma_0.
\end{aligned}
\tag{12}
$$

where $\mathcal{G}_0(T) = 1+\frac{\nu}{2}\rho T$. *In particular, the model admits price manipulation strategies unless* $m_1 = m_2 = 0$ *(i.e.* $N\equiv 0$*) and* $D_0 = 0$.

The proof of this theorem is given in Appendix A.3.1. When $m_1 = m_2 = 0$, there is no other orders and we get back the results in the Obizhaeva and Wang model [28]. In this case, the steady state is $D_0 = 0$ and there is no price manipulation strategy, see Alfonsi and Schied [3]. Otherwise, Poissonian market orders always create possible manipulation strategies, which we discuss now in detail.

First, we observe that the most important source of gains in (12) is in $O(T^3)$ when T is getting large. It appears when $\delta_0 \neq 0$, which means that there is a price tendency. There is a predictable imbalance in the order flow of which the strategic trader can take advantage. The corresponding arbitrages are the δ_0 terms in the strategy given in Theorem 4.1, and the associated expected gains are the negative terms proportional to δ_0^2 in the value function. Also, we observe that when $\delta_0 \neq 0$, the gain goes to infinity when $\rho \to 0$. This infinite gain can be reached by the following admissible strategy X^λ

$$
\Delta X_0^\lambda = \lambda\delta_0 \quad , \quad \mathrm{d}X_t^\lambda = 0 \quad \text{for } t\in(0,T) \quad , \quad \Delta X_T^\lambda = -\lambda\delta_0,
$$

for some $\lambda\in\mathbb{R}$. From (4), we have when $\rho = 0$, $C(X^\lambda) = -\lambda\delta_0\frac{N_T}{q}$ and therefore

$$
\mathbb{E}[C(X^\lambda)] = -\frac{\lambda\delta_0^2}{q}m_1T \underset{\lambda\to+\infty}{\longrightarrow} -\infty.
$$

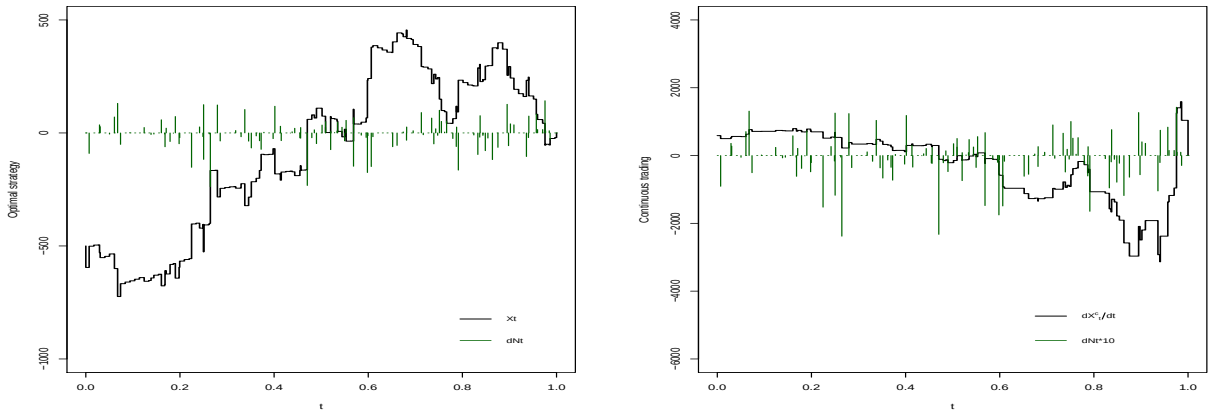
We now consider the "balanced case" where $\delta_0 = 0$. This means that the flows of buy and sell orders have the same distribution, which is a rather natural assumption. In this case, the negative cost grows in $O(T)$ and we have

$$
q\times\mathcal{C}(0,d,z) = -\frac{1}{1-\epsilon}\times\frac{\rho T/2}{2+\rho T}q^2d^2 - \frac{(1-\nu)^2}{1-\epsilon}\times m_2\times\left[\frac{T}{2}-\frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right]\,\Sigma_0.
$$

The term in d^2 is due to the fact that the strategic trader knows the initial temporary price shift and takes advantage of this information. The other term comes from the dynamic part of the strategy, in which the

strategic trader reacts instantaneously to price jumps to take advantage of their transient part. It vanishes when $\rho \to 0^+$ or $\nu \to 1^-$ since in both cases, the price impact becomes exclusively permanent.

Last, let us give some heuristic interpretation on the optimal strategy. When the strategic trader observes a market order at time τ, he immediately posts a market order of the opposite sign, with a proportion $\frac{1-\nu}{1-\epsilon} \times \frac{1+\rho(T-\tau)}{2+\rho(T-\tau)}$ of the volume. This allows him to take advantage of the deterministic resilience effect. He then compensates this block trade by modifying his continuous trading rate of $\frac{1-\nu}{1-\epsilon} \times \frac{\rho}{2+\rho(T-\tau)} \Delta N_\tau$ (with the sign opposite to his block trade), see Figure 1 for an illustration of this strategy. If moreover $\delta_0 \neq 0$, he takes advantage of the price trend by trading continuously at a time-dependent rate, and also adapts the initial jump of the strategy accordingly.



(a) $(X_t)_{0 \leq t \leq T}$

(b) $\mathrm{d}X_t^c/\mathrm{d}t$

Figure 1: Optimal strategy in the Poisson model for $q = 100,\ T = 1,\ \rho = 50,\ \epsilon = 0.1,\ \nu = 0.1,\ D_0 = 1,\ \kappa_0^+ = \kappa_0^- = 60,\ m_1 = 50,\ X_0 = -500$ and $\mu = \mathrm{Exp}(1/m_1)$, with the corresponding trajectory of (N_t).

4.2 Price manipulation strategies with model uncertainty

Theorem 4.1 shows that PMS exist when the market orders follow a compound Poisson process. However, this result holds under the assumption that the strategic trader knows exactly what the model parameters are. It is then natural to wonder if these PMS may disappear when he has a wrong estimation of some parameters. In this section, we focus on the case where the buy and sell orders are balanced ($\delta_0 = 0$), which means that there is no expected trend in the next future. A remarkable feature of the optimal strategy in this case is that it only depends on the process N, and does not depend directly on the law of the jumps and their intensity. Then, when applying the optimal strategy, only two quantities have to be known: qD_0 and ρ. In this paragraph, we consider a trader that wants to make profit without moving their position, and therefore uses the optimal liquidation strategy with $x_0 = 0$ shares. We assume without loss of generality that $S_0 = 0$ and denote by $\mathcal{C}_0(D_0)$ the cost of the optimal strategy of Theorem 4.1 in this case:

$$(1-\epsilon)q \times \mathcal{C}_0(D_0) = -\frac{\rho T/2}{2+\rho T}\, q^2 D_0^2 \;-\; (1-\nu)^2\, 2\kappa_0 m_2 \left[\frac{T}{2} - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right] \tag{13}$$

First, we assume that he knows ρ but not D_0. Then, we assume that he knows that $D_0 = 0$, but does not know the true value of ρ. In both cases, we show that the trader can still make profits on average. In particular, he can do this by using the basic strategy that consists in trading half of the volume (times $(1-\nu)/(1-\epsilon)$ if $\epsilon \neq \nu$) of each incoming market order in the opposite direction. This indicates that modeling the time stamps of market orders as a Poisson process is not compatible in our model with eliminating PMS.

Uncertainty on D_0

In this paragraph, we assume that $D_0 = d_0$ is deterministic and that the strategic trader applies the optimal strategy computed with a false value $\tilde{d}_0$. His strategy $\tilde{X}$ is then

$$(1-\epsilon)\Delta\tilde{X}_0 = -\frac{q\tilde{d}_0[1+\rho T]}{2+\rho T},\ (1-\epsilon)\Delta\tilde{X}_T = \frac{q\tilde{d}_0}{2+\rho T} + \sum_{0<\tau<T}\frac{(1-\nu)\Delta N_\tau}{2+\rho(T-\tau)} \tag{14}$$

$$(1-\epsilon)\mathrm{d}\tilde{X}_t = \left[\frac{q\tilde{d}_0}{2+\rho T} + \sum_{0<\tau<t}\frac{(1-\nu)\Delta N_\tau}{2+\rho(T-\tau)}\right]\rho\mathrm{d}t - \frac{1+\rho(T-t)}{2+\rho(T-t)} \times (1-\nu)\mathrm{d}N_t.$$

Proposition 4.1. *The expected cost of the strategy $\tilde{X}$ is given by*

$$\mathbb{E}\left[C(\tilde{X})\right] = \mathcal{C}_0(d_0) + \frac{1}{(1-\epsilon)q} \times \frac{\rho T/2}{2+\rho T} \times q^2(\tilde{d}_0 - d_0)^2,$$

and is nonpositive if, and only if $|\tilde{d}_0 - d_0| \leq \Delta d_0(T)$, where

$$\Delta d_0(T) = \sqrt{d_0^2 + \frac{(1-\nu)^2}{q^2} \times \frac{4\kappa_0 m_2(2+\rho T)}{\rho T}\left[\frac{T}{2} - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right]}.$$

The proof of this proposition is given in Appendix C.1. We obtain that PMS with deadline T disappear if the error in the estimation of d_0 exceeds $\Delta d_0(T)$. We can check that $T \mapsto \Delta d_0(T)$ is nondecreasing, such that $\Delta d_0(0) = |d_0|$ and $\Delta d_0(T) \underset{T\to\infty}{\sim} (1-\nu)\sqrt{2\kappa_0 m_2 T}/q$. When T is large, even if the estimation of d_0 is imprecise, it is still possible to get a PMS. More importantly, we observe that the choice $\tilde{d}_0 = 0$ always leads to a PMS for any deadline T and should be used if no better estimation is available.

Uncertainty on ρ

To study the uncertainty effects on the parameter ρ, we conduct a reasoning similar to the previous paragraph, assuming now that we apply the optimal strategy with a false value of $\tilde{\rho}$ of ρ. To simplify the calculations, we place ourselves in the case $d_0 = 0$ which is the asymptotic average value of D, since N is a martingale. The strategy $\tilde{X}$ of the strategic trader is then given by

$$\Delta\tilde{X}_0 = 0,\ (1-\epsilon)\Delta\tilde{X}_T = \sum_{0<\tau<T}\frac{(1-\nu)\Delta N_\tau}{2+\tilde{\rho}(T-\tau)},\ (1-\epsilon)\mathrm{d}\tilde{X}_t = \sum_{0<\tau<t}\frac{(1-\nu)\Delta N_\tau}{2+\tilde{\rho}(T-\tau)}\tilde{\rho}\mathrm{d}t - \frac{1+\tilde{\rho}(T-t)}{2+\tilde{\rho}(T-t)}(1-\nu)\mathrm{d}N_t. \tag{15}$$

This strategy $\tilde{X}$ would be optimal if $\tilde{\rho} = \rho$, with the expected cost $-\frac{(1-\nu)^2}{(1-\epsilon)q} \times 2\kappa_0 m_2\left[\frac{T}{2} - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right]$.

Proposition 4.2. *Let $r = \frac{\tilde{\rho}}{\rho} > 0$. The expected cost of the strategy $\tilde{X}$ is given by*

$$\mathbb{E}[C(\tilde{X})] = \frac{(1-\nu)^2}{(1-\epsilon)q} \times \frac{2\kappa_0 m_2}{\rho} \times f(r),$$

where $f(r) = -\frac{\rho T}{2} + \ln\left(1+\frac{\rho r T}{2}\right) - \left(1-\frac{1}{r}\right)^2 L(\rho r, -\rho, T) - \left(\frac{1}{r}+r-2\right) \times \frac{\exp(-\rho T)}{2+\rho r T} + \left(r-\frac{1}{r}\right) \times \frac{1}{2+\rho r T} + \frac{1}{r} - 1$. The function f is negative, nonincreasing on $(0,1]$ and nondecreasing on $[1,\infty)$, and $\tilde{X}$ is thus a PMS.

The proof of Proposition 4.2 is postponed to Appendix C.2. This results shows the robustness of the PMS obtained by the optimal execution problem with respect to the estimation of ρ. When taking $\tilde{\rho} = 0$, the strategy $\tilde{X}$ is the very basic one which consists in trading half of the volume times $(1-\nu)/(1-\epsilon)$ of each incoming market order in the opposite direction.

To be more quantitative, in the worst-case underestimation scenario ($r \to 0^+$), we get from (55) that $\mathbb{E}[C(\tilde{X})] \underset{T\to+\infty}{\sim} -\frac{(1-\nu)^2}{(1-\epsilon)q} \times 2\kappa_0 m_2 \times T/4$, which is only divided by two with respect to the case where ρ is known. Similarly, in the worst-case overestimation scenario ($r \to +\infty$), the asymptotic expected cost $\mathbb{E}[C(\tilde{X})] \underset{T\to+\infty}{\sim} -\frac{(1-\nu)^2}{(1-\epsilon)q} \times 2\kappa_0 m_2 \times \rho T^2/12$ obtained from (56) equals two thirds of the one when ρ is known.

4.3 Transaction costs

The last subsection demonstrates the robustness of the PMS in the Poisson model. However, such strategies may be no longer relevant if there are transaction costs. We now consider the effect of proportional transaction costs, that can arise in practice from the crossing of the bid-ask spread or from transaction fees imposed by the market organizer, for instance. Only in this subsection, we assume that any market order of signed volume v undergoes the additional cost $\gamma|v|$, with $\gamma \geq 0$ a constant. We still consider the balanced model with $\delta_0 = 0$ and, for simplicity, $d_0 = 0$. Also, we focus on the existence of PMS and assume $x_0 = 0$. Solving explicitly the optimal execution problem with proportional transaction costs is not obvious. However, it would be natural for the strategic trader to consider a strategy which is a linear interpolation between the optimal one when there is no transaction cost and the strategy that ignores the jumps of N, which is optimal when $\gamma \to +\infty$. This leads to consider the strategy X^θ computed with a process $N^\theta = \theta \times N$ instead of N, with $\theta \in [0,1]$. In the present context, this yields $X^\theta = \theta \times X^*$, where X^* is given by Theorem 4.1 when $x_0 = d_0 = \delta_0 = 0$. We can compute the expected value of this strategy (see Appendix C.3)

$$\begin{aligned}\mathbb{E}[C(X^\theta)] + \gamma||X^\theta|| &= -[2\theta(1-\theta)+\theta^2] \times \frac{(1-\nu)^2}{(1-\epsilon)q} \times 2\kappa_0 m_2 \left[\frac{T}{2} - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right] + \gamma||X^\theta|| \\ &= \theta \times \left\{\gamma||X^*|| + (\theta-2) \times \frac{(1-\nu)^2}{(1-\epsilon)q} \times 2\kappa_0 m_2 \left[\frac{T}{2} - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right]\right\}\end{aligned} \tag{16}$$

where we define $||X|| = \mathbb{E}\left[\int_0^T |\mathrm{d}X|_t\right]$. The bracket term is increasing with respect to θ and is minimal for $\theta = 0$. This leads to the following result.

Proposition 4.3. *Let* $\gamma_T^c = \frac{2(1-\nu)^2}{(1-\epsilon)q} \times 2\kappa_0 m_2 \left[\frac{T}{2} - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right] / ||X^*|| > 0$. *If* $\gamma < \gamma_T^c$, $X^{\theta_T^*}$ *with*

$$\theta_T^* = 1 - \frac{\gamma}{\gamma_T^c} \in (0,1]$$

is a PMS, and it is the optimal choice among all the strategies X^θ, $\theta \in [0,1]$. *If* $\gamma \geq \gamma_T^c$, *none of these strategies lead to a PMS.*

Therefore, having $\gamma \geq \gamma_T^c$ for any $T > 0$ is a necessary (but not sufficient) to exclude PMS. From a regulatory perspective, requiring this condition either by fixing the tick size or by adding transaction costs would be a way to restrict the PMS in the Poisson framework. Although γ_T^c cannot be determined analytically, we have the following result.

Lemma 4.1. *The critical level* γ_T^c *of transaction costs satisfies* $\underline{\gamma}_T \leq \gamma_T^c \leq \overline{\gamma}_T$, *with*

$$\underline{\gamma}_T = \frac{2(1-\nu) \times 2\kappa_0 m_2 \left[\frac{T}{2} - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right]/q}{2\kappa_0 m_1 \left[T - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right] + \sqrt{\frac{2\kappa_0 m_2(2+\rho T)}{\rho}} \times \left\{\frac{\pi}{2} - \arccos\left(\sqrt{\frac{\rho T}{2+\rho T}}\right)\right\} - \sqrt{\frac{\kappa_0 m_2 T}{2+\rho T}}}$$

$$\overline{\gamma}_T = \frac{2(1-\nu)\ m_2}{q\ m_1} \times \left(1 - \frac{T/2}{T - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)}\right).$$

Besides, we have $\underline{\gamma}_\infty = \gamma_\infty^c = \overline{\gamma}_\infty = \frac{(1-\nu)m_2}{qm_1}$, *thus* $\theta_\infty^* = 1 - \frac{\gamma q m_1}{(1-\nu)m_2}$ *if* $\gamma < \frac{(1-\nu)m_2}{qm_1}$.

The proof of this Lemma is given in Appendix C.3. The limits when $T \to +\infty$ gives some quantitative insights on the value of the critical threshold that are in line with heuristic arguments. If the liquidity level q is high or ν is close to one, the temporary price impact of market orders from environment traders is small and the strategic trader can make less profit by reacting to these orders. Also, the higher the ratio m_2/m_1, the higher the variance of the volume of incoming orders, the larger the arbitrage opportunities when big market orders are placed on the book. Interestingly, $\gamma^{\mathrm{c}}_{\infty}$ does depend on κ_0 that tunes the instantaneous traded volume.

4.4 Low-frequency asymptotics

In this section, we study the price process (P_t) in the absence of the strategic trader, and we are interested in the behaviour of P at a large time scale when buy and sell orders are balanced, i.e. $\delta_0 = 0$. To do so, we first analyze the process D and show that it is ergodic. To do so, we introduce the processes

$$\mathrm{d}D^+_t \;=\; -\rho\, D^+_t\, \mathrm{d}t \;+\; \frac{1-\nu}{q}\mathrm{d}N^+_t\;,\; \mathrm{d}D^-_t \;=\; -\rho\, D^-_t\, \mathrm{d}t \;+\; \frac{1-\nu}{q}\mathrm{d}N^-_t,$$

with $D^+_0 = \max(D_0, 0)$ and $D^-_0 = \max(-D_0, 0)$. These processes are independent, nonnegative and such that $D = D^+ - D^-$.

Proposition 4.4. *As t goes to infinity, D^+_t converges in distribution to an a.s. positive and finite random variable D^+_∞. The Laplace transform of the limit law is given by*

$$\forall u > 0, \quad \mathbb{E}[\exp(-uD^+_\infty)] \;=\; \exp\left(-\frac{\kappa_0}{\rho}\int_0^{(1-\nu)u/q} \frac{1-\psi_\mu(\theta)}{\theta}\,\mathrm{d}\theta\right) \tag{17}$$

where ψ_μ is the Laplace transform of μ. In particular, we have $\mathbb{E}[D^+_\infty] = \frac{1-\nu}{q} \times \frac{\kappa_0 m_1}{\rho}$ and $Var[D^+_\infty] = \frac{(1-\nu)^2}{q^2} \times \frac{\kappa_0 m_2}{2\rho}$.

The proof of this theorem is given in Appendix D.1. It gives immediately that D_t converges in law to a random variable D_∞ that satisfies $\mathbb{E}[D_\infty] = 0$ and $\mathrm{Var}[D_\infty] = \frac{(1-\nu)^2}{q^2} \times \frac{\kappa_0 m_2}{\rho}$.

To study the low-frequency asymptotics of the price process P, we have to rescale accordingly price and time. We then consider the sequence $P^{(n)}_t = P_{nt}/\sqrt{n}$ for $n \geq 1$. We have $P^{(n)}_t = S^{(n)}_t + D^{(n)}_t$, where we also define $S^{(n)}_t = S_{nt}/\sqrt{n}$ and $D^{(n)}_t = D_{nt}/\sqrt{n}$. On the one hand, the process $D^{(n)}$ converges to zero from the ergodic property of D. On the other hand, the process $S^{(n)}_t$ converges in law to $\frac{\nu}{q}\sqrt{2\kappa_0 m_2}\; W$, with W a standard Brownian motion. This can be easily obtained since the increments of $S^{(n)}_t$ are independent and stationary and $S^{(n)}_t$ converges in distribution toward a centered normal variable with variance $2\kappa_0 m_2 t \left(\frac{\nu}{q}\right)^2$.

Proposition 4.5. *For any $T > 0$, the sequence $(P^{(n)}_t, t \in [0,T])$ converges weakly to the Bachelier price process $(\tilde{P}_t, t \in [0,T])$ as n goes to infinity, where $\tilde{P}_t \;=\; \frac{\nu}{q}\sqrt{2\kappa_0 m_2}\; W_t$.*

It is interesting to notice that the MIP model is compatible with low-frequency dynamics that exclude arbitrages, even though it allows PMS at a short time scale. Figure 2 illustrates this convergence and shows that the relative importance of D with respect to S vanishes as the time scale increases.

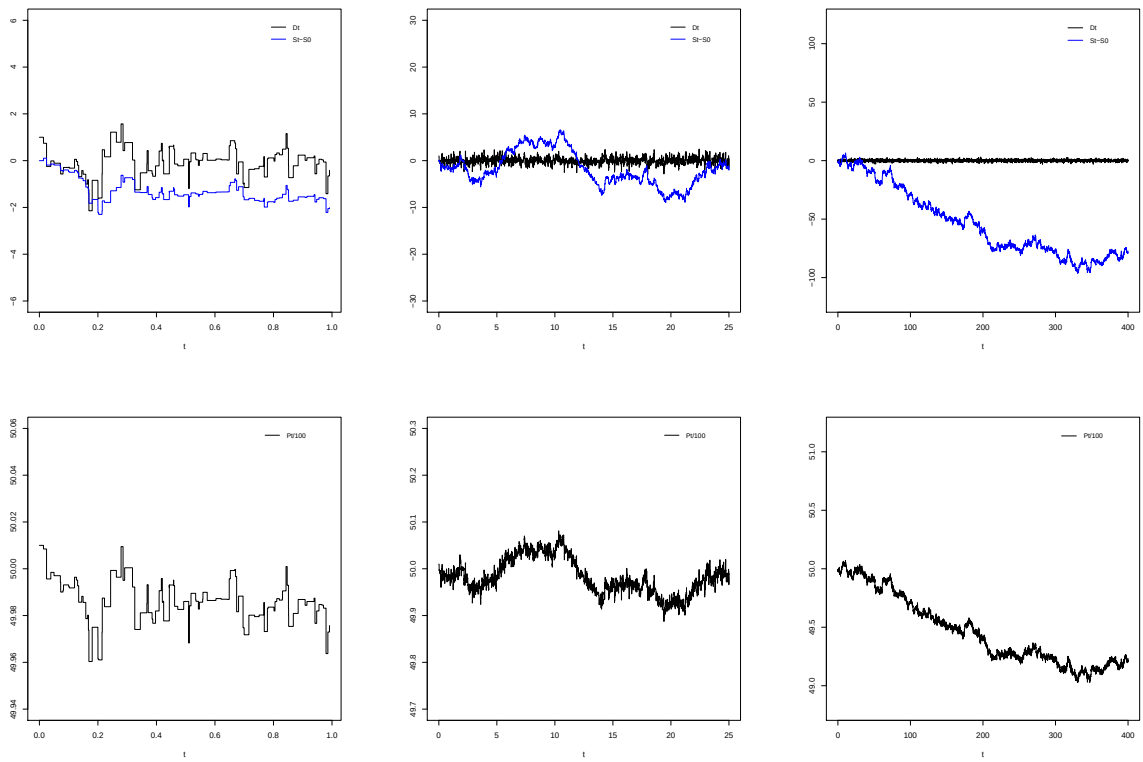


Figure 2: Simulated trajectories in the Poisson model with $q = 100,\ \rho = 50,\ D_0 = 1,\ S_0 = 5000,\ \kappa_0^+ = \kappa_0^- = 60,\ m_1 = 50,\ \nu = 0.3,\ \mu = \mathrm{Exp}(1/m_1)$. We plot D_t and $S_t - S_0$ (above), and $P_t/100$ (below), over the time scales: $T_2 = 1$ (left), $T_1 = 25$ (middle), $T_0 = 400$ (right). The scaling of the graphs with respect to the time scale is square root.

5 The Mixed-market-Impact Hawkes (MIH) model

We need to introduce some additional notations to present the main results on the optimal execution. Instead of working with κ_t^+ and κ_t^-, we will rather use the $\delta_t = \kappa_t^+ - \kappa_t^-$ and $\Sigma_t = \kappa_t^+ + \kappa_t^-$ that satisfy from (10)

$$\mathrm{d}\delta_t \ = \ -\beta\ \delta_t\ \mathrm{d}t \ + \ \alpha\ (\mathrm{d}J_t^+ - \mathrm{d}J_t^-) \quad , \quad \mathrm{d}\Sigma_t \ = \ -\beta\ (\Sigma_t - 2\kappa_\infty)\ \mathrm{d}t \ + \ (\alpha + 2\iota_\mathrm{c})\ (\mathrm{d}J_t^+ + \mathrm{d}J_t^-), \tag{18}$$

with $\alpha = \iota_\mathrm{s} - \iota_\mathrm{c}$. We note $(\tau_i)_{i\geq 1}$ the ordered random jump times of J (i.e. those of N) and set $\tau_0 = 0$. For $t \in [0, T]$, we also note $\chi_t = J_t^+ + J_t^-$ the total number of jumps that occurred between time 0 and time t. From (18), we have

$$\delta_t \ = \ \delta_0 \exp(-\beta t) \ + \ \alpha \sum_{l=1}^{\chi_t} \exp(-\beta(t - \tau_l))\Delta J_{\tau_l} \ = \ \delta_0 \ \exp(-\beta t) + \alpha \exp(-\beta t)\ \Theta_{\chi_t},$$

where we define $\Theta_0 = 0$ and

$$\Theta_i \ = \ \sum_{l=1}^{i} \exp(\beta\tau_l)\Delta J_{\tau_l} \ = \ \sum_{0<\tau\leq\tau_i} \exp(\beta\tau)\ \Delta J_\tau, i \geq 1.$$

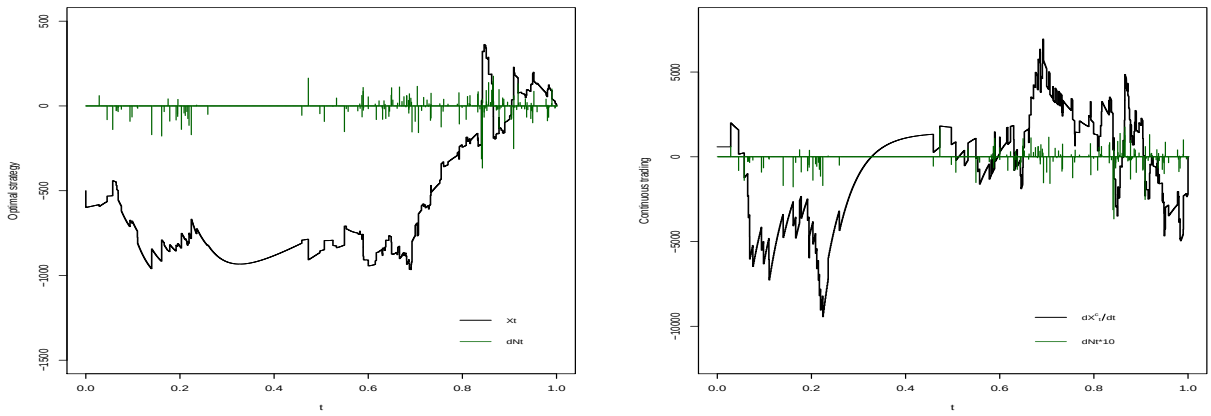
For $i \geq 0$ and $t \in [\tau_i, \tau_{i+1})$, we obtain that $\delta_t \exp(\beta t) = \delta_0 + \alpha\Theta_i$ only depends on t through the integer $i = \chi_t$. Last, we define the differentiable function $\zeta : \mathbb{R} \to \mathbb{R}^+$ by

$$\forall y \neq 0, \quad \zeta(y) \;=\; \frac{1-\exp(-y)}{y} \quad , \quad \zeta'(y) \;=\; \frac{(1+y)\exp(-y)-1}{y^2} \tag{19}$$

and $\zeta(0) = 1,\ \zeta'(0) = -1/2$. We have $\zeta' \leq 0$, $\zeta(y) \to +\infty$ as y goes to $-\infty$ and $\zeta(y) \to 0$ as y goes to $+\infty$. For $\lambda \in \mathbb{R}$ and $u \geq 0$, one also has $\frac{\mathrm{d}}{\mathrm{d}u}[u\zeta(\lambda u)] = \exp(-\lambda u)$ and $[1-\zeta(\lambda u)]/\lambda \to u/2$ as λ vanishes.

5.1 The optimal strategy

In this MIH model, the five state variables of the problem are X_t, D_t, S_t, δ_t and Σ_t. To describe the optimal strategy and the value function, we distinguish the cases $\alpha = \beta$ and $\alpha \neq \beta$. The general case $\alpha \neq \beta$ is stated in Theorem 5.1 below. When $\alpha = \beta$, some simplifications occur in the formulas. Theorem A.1 gives the optimal execution strategy and its cost in this case. The proof of these theorems is given in Appendix A.



(a) $(X_t)_{0 \leq t \leq T}$

(b) $\mathrm{d}X_t^{\mathrm{c}}/\mathrm{d}t$

Figure 3: Optimal strategy in the Hawkes model for $q = 100,\ T = 1,\ \rho = 50,\ \alpha = \beta = 20,\ \iota_{\mathrm{c}} = 0,\ \kappa_\infty = 10,\ \epsilon = 0.1,\ \nu = 0.1,\ D_0 = 1,\ \kappa_0^+ = \kappa_0^- = 60,\ m_1 = 50,\ X_0 = -500$ and $\mu = \mathrm{Exp}(1/m_1)$, with the corresponding trajectory of (N_t).

We introduce some notations, and define

$$\eta = \beta - \alpha.$$

We refer to (11) and (19) for the definitions of L and ζ, and we also set for $0 \leq s \leq u \leq T$

$$\phi_\eta(t) = \frac{1}{2(2+\rho(T-t))} \times \Big[1 + \exp(-\eta(T-t)) + \nu\rho(T-t)\zeta(\eta(T-t)) \\ + \frac{\beta}{\rho}\left[2 + \rho(T-t) \times \{1 + \zeta(\eta(T-t)) + \nu\rho[1-\zeta(\eta(T-t))]/\eta\}\right]\Big],$$

and

$$\begin{aligned}
\Phi_\eta(s,t) &= \frac{1}{2}\left(\frac{1}{\rho}+\frac{\nu}{\eta}\right)\times[\exp(-\beta s)-\exp(-\beta t)] \\
&\quad + \frac{\exp(-\beta T)}{2\rho}\times\left[1+\frac{\nu(\rho-2\beta)}{\eta}+\frac{\beta}{\eta}\left(1-\frac{\nu\rho}{\eta}\right)\right]\times[L(\rho,\beta,T-s)-L(\rho,\beta,T-t)] \\
&\quad + \frac{\exp(-\beta T)}{2\rho}\times\left[1-\frac{\nu\rho}{\eta}-\frac{\beta}{\eta}\left(1-\frac{\nu\rho}{\eta}\right)\right]\times[L(\rho,\alpha,T-s)-L(\rho,\alpha,T-t)].
\end{aligned}$$

Theorem 5.1. *Let $\epsilon\in[0,1)$. When $\eta=\beta-\alpha\neq 0$, the optimal strategy X^* is given by*

$$\begin{aligned}
(1-\epsilon)\Delta X_0^* &= -\frac{(1-\epsilon)x_0 \;+\; [1+\rho T]qD_0 \;-\; \frac{m_1}{2\rho}\times[2+\rho T\times\{1+\zeta(\eta T)+\nu\rho[1-\zeta(\eta T)]/\eta\}]\;\delta_0}{2+\rho T}, \\
(1-\epsilon)\Delta X_T^* &= \frac{qD_0-(1-\epsilon)x_0}{2+\rho T} \;+\; \frac{\delta_0 m_1}{2\rho}\times\left[\frac{2+\rho T\times\{1+\zeta(\eta T)+\nu\rho[1-\zeta(\eta T)]/\eta\}}{2+\rho T} \;-\; 2\rho\;\Phi_\eta(0,T)\right] \\
&\quad - \alpha m_1\left[\Theta_{\chi_T}\;\Phi_\eta(\tau_{\chi_T},T) \;+\; \sum_{i=1}^{\chi_T-1}\Theta_i\;\Phi_\eta(\tau_i,\tau_{i+1})\right] \;+\; \sum_{0<\tau\le T}\frac{(1-\nu)\;\Delta N_\tau}{2+\rho(T-\tau)} \\
&\quad + \frac{\alpha m_1}{2\rho}\times\sum_{0<\tau\le T}\frac{2+\rho(T-\tau)\times\{1+\zeta(\eta(T-\tau))+\nu\rho[1-\zeta(\eta(T-\tau))]/\eta\}}{2+\rho(T-\tau)}\;\Delta J_\tau \\
&\quad - \frac{m_1}{\rho}(\delta_0+\alpha\Theta_{\chi_T})\exp(-\beta T),
\end{aligned}$$

and, on $(0,T)$,

$$\begin{aligned}
(1-\epsilon)\mathrm{d}X_t^* &= \left[\frac{2+\rho T\times\{1+\zeta(\eta T)+\nu\rho[1-\zeta(\eta T)]/\eta\}}{2+\rho T} \;-\; 2\rho\;\Phi_\eta(0,t)\right]\;\frac{\delta_0 m_1}{2}\;\mathrm{d}t \\
&\quad - m_1\;\phi_\eta(t)\;\;(\delta_0+\alpha\Theta_{\chi_t})\exp(-\beta t)\;\mathrm{d}t \\
&\quad + \left[\frac{qD_0-(1-\epsilon)x_0}{2+\rho T} \;+\; \sum_{0<\tau\le t}\frac{(1-\nu)\Delta N_\tau}{2+\rho(T-\tau)}\right]\rho\mathrm{d}t \\
&\quad + \left[\sum_{0<\tau\le t}\frac{2+\rho(T-\tau)\times\{1+\zeta(\eta(T-\tau))+\nu\rho[1-\zeta(\eta(T-\tau))]/\eta\}}{2+\rho(T-\tau)}\;\Delta J_\tau\right]\;\frac{\alpha m_1}{2}\;\mathrm{d}t \\
&\quad - \left[\Theta_{\chi_t}\;\Phi_\eta(\tau_{\chi_t},t) \;+\; \sum_{i=1}^{\chi_t-1}\Theta_i\;\Phi_\eta(\tau_i,\tau_{i+1})\right]\rho\alpha m_1\;\mathrm{d}t \\
&\quad + \frac{1+\rho(T-t)}{2+\rho(T-t)}\left\{\frac{\alpha m_1}{\rho}\;\mathrm{d}J_t-(1-\nu)\;\mathrm{d}N_t\right\} \\
&\quad + \frac{\alpha m_1}{2\rho}(\nu\rho-\eta)\times\frac{\rho(T-t)\times[1-\zeta(\eta(T-t))]/\eta}{2+\rho(T-t)}\;\mathrm{d}J_t.
\end{aligned}$$

The value function function of the problem is then:

$$\begin{aligned}
q\times\mathcal{C}(t,x,d,z,\delta,\Sigma) &= -q(z+d)x \;+\; \left[\frac{1-\epsilon}{2+\rho(T-t)}+\frac{\epsilon}{2}\right]x^2 \;+\; \frac{\rho(T-t)}{2+\rho(T-t)}\left[qd-\;\mathcal{G}_\eta(T-t)\;\frac{\delta m_1}{\rho}\right]x \\
&\quad - \frac{1}{1-\epsilon}\times\frac{\rho(T-t)/2}{2+\rho(T-t)}\left[qd-\;\mathcal{G}_\eta(T-t)\;\frac{\delta m_1}{\rho}\right]^2 \;+\; \hat{c}_\eta(T-t)\;\left(\frac{\delta m_1}{\rho}\right)^2 \\
&\quad + e(T-t)\;\Sigma \;+\; g(T-t),
\end{aligned}$$

where for $u \in [0,T]$,

$$\begin{aligned}\mathcal{G}_\eta(u) &= \zeta(\eta u) + \nu\rho[1-\zeta(\eta u)]/\eta \\ \hat{c}_\eta(u) &= -\frac{1}{1-\epsilon} \times \left(1-\frac{\nu\rho}{\eta}\right)^2 \times \frac{\rho u \zeta(\eta u)}{8} \times [1+\exp(-\eta u) - 2\zeta(\eta u)]\,.\end{aligned}$$

The functions e and g are the unique solution of the differential equations (33) *and* (34) *with $e(0) = g(0) = 0$. When $\eta = 0$, the optimal strategy is given by Theorem A.1.*

Let us mention here that the functions e and g admit explicit forms by the mean of the exponential integral function, that are very cumbersome. They can be obtained by using a formal calculus software such as Mathematica. Since they do not play any role for the optimal strategy and require several pages to be displayed, we do not give these explicit formulas except in the case $\alpha = \beta$, see Theorem A.1.

It is worth to notice that the optimal strategy X^* is affine with respect to x_0, D_0, δ_0, J and N. This is due to the affine structure of the model and the quadratic costs. In particular, the reaction of the optimal strategy to the other trades does not depend on x_0. Also, the optimal strategy can be seen as the sum of the strategy (7) which is optimal when $D_0 = \delta_0 = 0$ and $J \equiv N \equiv 0$ and the optimal PMS which is given by Theorem 5.1 when $x_0 = 0$.

Let us make some comments on the optimal strategy and more precisely how the strategic trader reacts to the other orders. Quite similarly to the Poisson case, the trade that follows immediately an order is compensated by the continuous trading rate. Contrary to the Poisson model, this trade is not always contrary to the previous market order. Namely, in the case where $\eta = \nu\rho$, the strategic trader makes a trade in the opposite direction if $|\mathrm{d}N_t| > \frac{\alpha m_1}{\rho(1-\nu)}$, but trades in the same way otherwise. The same conclusion holds for any parameter value when $T - t \to 0$. We now consider the asymptotics when the trading horizon is large: in this case, it is likely to assume that $\eta > 0$ which is required to get stationary intensities κ^+ and κ^-, see Section 5.3. Then, when $T - t \to +\infty$ the jump part of the optimal strategy can be well approximated by

$$\frac{\alpha m_1}{2\rho}\left(1+\frac{\rho\nu}{\eta}\right)\mathrm{d}J_t - (1-\nu)\,\mathrm{d}N_t.$$

Therefore, the strategic trader makes a trade in the opposite direction if $|\mathrm{d}N_t| > \frac{\alpha m_1}{2\rho(1-\nu)}(1+\frac{\rho\nu}{\eta})$ and trades in the same direction otherwise. Heuristically, we can understand this behavior as follows when $\iota_c = 0$. In the MIH model, market orders all have the same excitation on the intensity. If a market buy order is relatively small, it may be a part of a big split order, and thus be followed by other buy orders that will make the price go up, and the strategic trader has interest to follow this trend. However, if a market buy order is relatively big, the price resilience effect is likely to dominate and the strategic trader has interest to trade in the opposite way, as in the Poisson model.

Last, it is interesting to notice that the optimal strategy only depends on (ι_s, ι_c) through $\alpha = \iota_s - \iota_c$. This key self-excitation parameter tunes the way that the strategic trader should react to other market orders.

5.2 The Mixed-Impact Hawkes Martingale (MIHM) model and manipulation strategies

Theorem 2.1 gives a necessary and sufficient condition on N to exclude Price Manipulation Strategies. Here, we apply this result to identify which parameters in the Hawkes model exclude PMS.

Proposition 5.1. *The MIH model does not admit PMS if, and only if the following conditions hold*

$$\beta = \rho,\ \alpha = (1-\nu)\rho,\ \mu = \mathit{Dirac}(m_1)\ \textit{(i.e. } N = m_1 \times J\textit{), and } qD_0 = \frac{m_1}{\rho}\delta_0 \tag{20}$$

or $\mu = \mathit{Dirac}(0)$ with $D_0 = 0$. In both these cases, the optimal execution strategy is given by (7).

To be precise, when $m_1 = 0$, $N \equiv 0$ and the MIH model does not depend any longer on the parameters α and β, that can then be fixed arbitrarily.

Proof. From Theorem 2.1, PMS are excluded if, and only if the price P is a martingale when $X \equiv 0$. In this case, we have from (1), (2), (3) and (18)

$$\mathrm{d}P_t = -\rho D_t dt + \frac{1}{q}\mathrm{d}N_t = \frac{1}{q}(\mathrm{d}N_t - \delta_t m_1 \mathrm{d}t) + \left(\frac{m_1}{q}\delta_t - \rho D_t\right)\mathrm{d}t.$$

Therefore, P is a martingale if, and only if $\frac{m_1}{\rho}\delta_t = qD_t$ $\mathbb{P}$-a.s., dt a.e. This condition is equivalent to $qD_0 = \frac{m_1}{\rho}\delta_0$ and $qdD_t = \frac{m_1}{\rho}d\delta_t$. From (3) and (18), the latter condition is equivalent to

$$\rho q D_t = \frac{m_1}{\rho}\beta\delta_t \text{ and } (1-\nu)dN_t = \alpha\frac{m_1}{\rho}\mathrm{d}J_t,$$

which leads to (20). Conversely, (20) implies $\frac{m_1}{\rho}\delta_t = qD_t$, and P is then a martingale. □

Remark 5.1. *When $\beta = \rho$, $\alpha = (1-\nu)\rho$, $\mu = Dirac(m_1)$, we get from the previous calculations that $d(\frac{m_1}{q}\delta_t - \rho D_t) = -\rho(\frac{m_1}{q}\delta_t - \rho D_t)dt$, and therefore $\frac{m_1}{q}\delta_t - \rho D_t$ converges exponentially to zero. The condition $qD_0 = \frac{m_1}{\rho}\delta_0$ simply means that the model starts from this steady state.*

One can also check directly that the optimal strategy and its cost given by Theorem 5.1 coincide with those of Theorem 2.1 when (20) holds. For clear reasons, we call Mixed-Impact Hawkes Martingale (MIHM) model the MIH model when (20) holds. Proposition 5.1 is very interesting since it gives what should be the parameters of the MIH model in a perfect market without PMS, and makes connection between the different model parameters. First, the condition $\beta = \rho$ means that the mean-reverting action of liquidity providers compensates the autocorrelation in the signs of the trades of liquidity takers ; we thus reach a conclusion similar to Bouchaud et al. [12]. The condition $\alpha = (1-\nu)\beta$ gives a link between the Hawkes kernel and the proportion $1-\nu$ of transient price impact. When $\iota_c = 0$, α/β represents the average number of child orders coming from one market order, and is thus equal to the proportion of endogenous orders (i.e. triggered by other orders) in the market. What we obtain here is that this ratio should be equal to $1-\nu$, which is a *a priori* different measure of endogeneity, since it gives the proportion of market impact that does not influence the low-frequency price (see Section 5.3). The positivity of α reflects the fact that the parameter ι_c tuning opportunistic trading should be small to avoid market instability. Last, it is interesting to notice that μ should be a Dirac mass, which means that market orders should be in principle all of the same size. This is a consequence of the modeling that assumes that any market order has the same excitation on the intensity, regardless of its size. However, if we admit that orders that have roughly the same size have a similar impact, the size of market orders should cluster around some values. Typically, the most common size should be the average size that moves the price of one tick.

Of course, in practice, it would be miraculous if the estimation of the MIH model to market data led to parameters satisfying exactly (20). Market frictions such as transaction costs (and the crossing of the bid-ask spread) or the latency to execute an order increase the execution costs (see Stoikov and Waeber [32]) and then limit possible PMS. Therefore, one may expect that an estimation of the MIH to market data gives parameters that are not too far away but different from the condition (20). Then, the optimal strategy given by Theorem 5.1 is useful and should give better perform than the one given by (7). The practical implementation of the MIH model on market data is left for future research.

The framework of the MIH model also gives some interesting insights for the characterization of the existence of short-time arbitrages. Let us introduce the following definition.

Definition 5.1. *We say that a market admits weak Price Manipulation Strategies (wPMS) if the cost of a liquidation strategy can be reduced by trading immediately after other market orders.*

Corollary 5.1. *In the MIH model, the market does not admit wPMS if, and only if,*

$$\beta = \rho,\ \alpha = (1-\nu)\rho \text{ and } \mu = Dirac(m_1) \tag{21}$$

or $\mu = Dirac(0)$.

Proof. The proof is quite straightforward from Theorem 5.1. The jump term

$$\frac{1+\rho(T-t)}{2+\rho(T-t)}\left\{\frac{\alpha m_1}{\rho}\ \mathrm{d}J_t - (1-\nu)\ \mathrm{d}N_t\right\} + \frac{\alpha m_1}{2\rho}(\nu\rho-\eta)\times\frac{\rho(T-t)\times[1-\zeta(\eta(T-t))]/\eta}{2+\rho(T-t)}\ \mathrm{d}J_t$$

cannot be almost surely equal to zero if μ is not a Dirac mass, and we necessarily have $\mu = \mathrm{Dirac}(m_1)$. When $m_1 > 0$, this leads to

$$\forall t \geq 0,\ (1+\rho t)\left(\frac{\alpha}{\rho} - (1-\nu)\right) + \frac{\alpha}{2\rho}(\nu\rho-\eta)\rho t\frac{1-\zeta(\eta t)}{\eta} = 0,$$

which precisely gives (21). The converse implication is obvious. □

By Remark 5.1, the condition $qD_0 = \frac{m_1}{\rho}\delta_0$ means that the model has reached its equilibrium, which is basically the case after some time. Therefore, the conditions that exclude wPMS and PMS in the MIH model are quite the same. This is an interesting link between two different point of views. The condition "no PMS" means that there is no free source of income. The condition "no wPMS" rather brings on market stability, since it excludes artificial trading volume coming from the response to other trades.

5.3 Law of the price process and low-frequency asymptotics

In this section, we are interested in price asymptotics when the strategic trader is absent, i.e. $X \equiv 0$. We first focus on the intensities κ^+ and κ^-. From (10), we have

$$\kappa_t^+ = \kappa_0^+\exp(-\beta t) + \kappa_\infty(1-\exp(-\beta t)) + \int_0^t \iota_\mathrm{s}\exp(-\beta(t-s))\mathrm{d}J_s^+ + \int_0^t \iota_\mathrm{c}\exp(-\beta(t-s))\mathrm{d}J_s^-$$
$$\kappa_t^- = \kappa_0^-\exp(-\beta t) + \kappa_\infty(1-\exp(-\beta t)) + \int_0^t \iota_\mathrm{c}\exp(-\beta(t-s))\mathrm{d}J_s^+ + \int_0^t \iota_\mathrm{s}\exp(-\beta(t-s))\mathrm{d}J_s^-.$$

We know that this process converges to a stationary law if $\iota_\mathrm{s} + \iota_\mathrm{c} = \alpha + 2\iota_\mathrm{c} < \beta$, see e.g. equation (6) of Bacry et al. [7]. Since $\delta_t = \delta_0\exp(-\beta t) + \alpha\int_0^t \exp(-\beta(t-s))\mathrm{d}J_s$ and $J_t - \int_0^t \delta_s \mathrm{d}s$ is a martingale, one has $\frac{\mathrm{d}}{\mathrm{d}t}\{\exp(\beta t)\mathbb{E}[\delta_t]\} = \alpha\exp(\beta t)\mathbb{E}[\delta_t]$, thus

$$\mathbb{E}[\delta_t] = \delta_0\exp(-(\beta-\alpha)t).$$

Similarly, $\Sigma_t = 2\kappa_\infty(1-\exp(-\beta t)) + \Sigma_0\exp(-\beta t) + (\alpha+2\iota_\mathrm{c})\int_0^t \exp(-\beta(t-s))(\mathrm{d}J_s^+ + \mathrm{d}J_s^-)$, which yields $\frac{\mathrm{d}}{\mathrm{d}t}\{\exp(\beta t)\mathbb{E}[\Sigma_t]\} = (\alpha+2\iota_\mathrm{c})\exp(\beta t)\mathbb{E}[\Sigma_t] + 2\beta\kappa_\infty\exp(\beta t)$. The resolution of this ODE gives $\mathbb{E}[\Sigma_t] = \Sigma_0 + 2\beta\kappa_\infty t$ if $\alpha + 2\iota_\mathrm{c} = \beta$, and

$$\mathbb{E}[\Sigma_t] = \Sigma_0\exp(-(\beta-\alpha-2\iota_\mathrm{c})t) + 2\overline{\kappa}[1-\exp(-(\beta-\alpha-2\iota_\mathrm{c})t)]$$

where

$$\overline{\kappa} := \frac{\kappa_\infty}{1-(\alpha+2\iota_\mathrm{c})/\beta}$$

otherwise. We then have the following result.

Proposition 5.2. *The process* (κ_t^+, κ_t^-) *converges to a stationary law if, and only if* $\alpha + 2\iota_\mathrm{c} < \beta$.

Remark 5.2. *In is interesting to look at the stationarity condition when the conditions $\alpha = (1-\nu)\beta,\ \beta = \rho$ required for the MIHM model hold. In this case, the stationarity condition becomes $\rho > (1-\nu)\rho + 2\iota_{\mathrm{c}}$, i.e.*

$$\iota_{\mathrm{c}} < \nu\rho/2.$$

The quantity of $\nu \times \rho$ can be seen as a stationarity margin left for cross-excitation in market orders. If this condition is violated, the price process is still a martingale but the mutual excitation between buy and sell orders makes the volatility increase exponentially.

We now study the low-frequency asymptotics of the price process P. As in the Poisson case, we consider the sequence $P_t^{(n)} = P_{nt}/\sqrt{n}$ for $n \geq 1$. We have $P_t^{(n)} = S_t^{(n)} + D_t^{(n)}$, where we also set $S_t^{(n)} = S_{nt}/\sqrt{n}$ and $D_t^{(n)} = D_{nt}/\sqrt{n}$. To study the behaviour of $D^{(n)}$, we need the following lemma that is proved in Appendix D.2.

Lemma 5.1. *When $\alpha + 2\iota_{\mathrm{c}} < \beta$, the expectation $\mathbb{E}[D_t^2]$ converges when $t \to +\infty$.*

Thanks to this lemma, $(D_{t_1}^{(n)}, \ldots, D_{t_k}^{(n)})$ converges to zero for the L^2 norm for any $0 \leq t_1 \leq \cdots \leq t_k$. This gives that the process $D^{(n)}$ converges to zero. We now focus on the convergence of $S_t^{(n)} = \frac{\nu}{q}\frac{N_{nt}^+ - N_{nt}^-}{\sqrt{n}}$. To do so, we apply Corollary 1 of Bacry et al. [8]. It gives the convergence in law of

$$\left(\sqrt{n}\left(\frac{1}{n}\begin{pmatrix} J_{nt}^+ \\ J_{nt}^- \end{pmatrix} - t\frac{\kappa_\infty}{1-\alpha/\beta}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), t \in [0,T]\right)$$

to $\left(\sqrt{\frac{\kappa_\infty}{1-\alpha/\beta}}\frac{\beta}{(\beta-\alpha)(\beta-(\alpha+2\iota_{\mathrm{c}}))}\begin{pmatrix} (\beta-\iota_{\mathrm{s}})B_t^1 - \iota_{\mathrm{c}}B_t^2 \\ -\iota_{\mathrm{c}}B_t^1 + (\beta-\iota_{\mathrm{s}})B_t^2 \end{pmatrix}, t \in [0,T]\right)$, where B is a two dimensional standard Brownian motion. Therefore, $\left(\frac{J_{nt}^+ - J_{nt}^-}{\sqrt{n}}, t \in [0,T]\right)$ converges in law to

$$\left(\sqrt{\frac{2\kappa_\infty}{1-\alpha/\beta}}\frac{1}{1-\frac{\alpha+2\iota_{\mathrm{c}}}{\beta}}W_t, t \in [0,T]\right),$$

where W is a standard Brownian motion. Let us now assume that the jumps are bounded, i.e.

$$\exists K > 0,\ \mu([0,K]) = 1. \tag{22}$$

Then, it is not difficult to adapt the proof of Bacry et al. [8] and in particular Lemma 7 of [8] to get that

$$\left(\sqrt{n}\left(\frac{1}{n}\begin{pmatrix} N_{nt}^+ \\ N_{nt}^- \end{pmatrix} - t\frac{\kappa_\infty m_1}{1-\alpha/\beta}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), t \in [0,T]\right)$$

converges in law to $\left(\sqrt{\frac{\kappa_\infty m_2}{1-\alpha/\beta}}\frac{\beta}{(\beta-\alpha)(\beta-(\alpha+2\iota_{\mathrm{c}}))}\begin{pmatrix} (\beta-\iota_{\mathrm{s}})B_t^1 - \iota_{\mathrm{c}}B_t^2 \\ -\iota_{\mathrm{c}}B_t^1 + (\beta-\iota_{\mathrm{s}})B_t^2 \end{pmatrix}, t \in [0,T]\right)$. This leads to the following result.

Proposition 5.3. *When $\alpha + 2\iota_{\mathrm{c}} < \beta$ and (22) holds, $(P_t^{(n)}, t \in [0,T])$ converges in law to $(\tilde{P}_t, t \in [0,T])$ with $\tilde{P}_t = \frac{\nu}{q}\frac{\sqrt{2\kappa_\infty m_2}}{\sqrt{1-\frac{\alpha}{\beta}}\left(1-\frac{\alpha+2\iota_{\mathrm{c}}}{\beta}\right)}W_t = \frac{\nu}{q}\sqrt{\frac{2\overline{\kappa}m_2}{\left(1-\frac{\alpha}{\beta}\right)\left(1-\frac{\alpha+2\iota_{\mathrm{c}}}{\beta}\right)}}W_t$.*

We get back the result of Proposition 4.5 in the MIP model when $\iota_{\mathrm{c}} = \iota_{\mathrm{s}} = 0$ and $\kappa_0 = \kappa_\infty$, and get again a Bachelier model at a low frequency. Figure 4 illustrates the convergence. It is interesting to notice that we have

$$\tilde{P}_t = \frac{1}{q}\frac{\sqrt{2\nu\kappa_\infty m_2}}{\nu - 2\frac{\iota_{\mathrm{c}}}{\rho}}W_t$$

in the MIHM model, and we get that the volatility explodes when $\iota_c \to \frac{\nu\rho}{2}$, which is in line with Remark 5.2. The volatility of $\tilde{P}$ is greater than $\frac{1}{q}\sqrt{\frac{2\kappa_\infty m_2}{\nu}}$ and goes to $+\infty$ when the permanent impact ν goes to 0, which is different from the Poisson case.

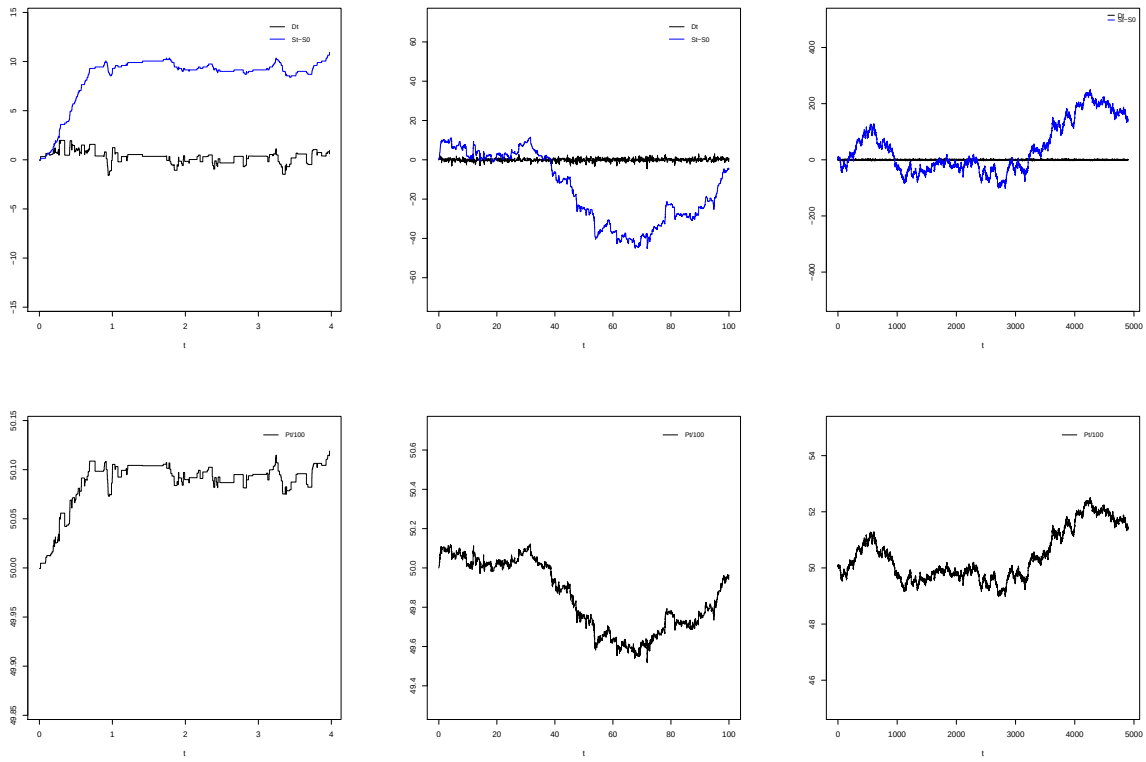


Figure 4: Simulated trajectories in the MIHM model with $q = 100$, $\rho = 50$, $D_0 = -0.05$, $S_0 = 5000$, $\alpha = 35$, $\beta = 50$, $\kappa_\infty = 6$, $\kappa_0^+ = 25$, $\kappa_0^- = 30$, $\iota_c = 1.5$, $m_1 = 50$, $\nu = 0.3$, $\mu = \text{Dirac}(m_1)$. We plot D_t and $S_t - S_0$ (above), and $P_t/100$ (below), over the time scales: $T_2 = 4$ (left), $T_1 = 100$ (middle), $T_0 = 4900$ (right). The scaling of the graphs with respect to the time scale is square root.

A Proof for the optimal control problem (Theorems 4.1, A.1 and 5.1)

A.1 Notations and methodology

In the general Hawkes model, the jump intensity of the process (N_t) is characterized by the càdlàg Markovian process (δ_t, Σ_t) defined by (18), taking values in $\mathbb{R} \times \mathbb{R}^+$, which is constant and equal to its initial value in the Poisson model. The state variable of the problem is then $(X_t, D_t, S_t, \delta_t, \Sigma_t)$, and the control is $X_t - x_0$, i.e. the variation of the position of the strategic trader, $(X_t)_{t\in[0,T]}$ being an admissible strategy as described in Definition 2.1. The control program is thus to minimize $\mathbb{E}\left[C(0, X)\right]$ over all admissible strategies, where the cost $C(t, X)$ of the strategy X between t and T is given by

$$C(t, X) \;=\; \int_{[t,T)} P_u \, \mathrm{d}X_u \;+\; \frac{1}{2q} \sum_{t \le \tau < T} (\Delta X_\tau)^2 \;-\; P_T X_T \;+\; \frac{1}{2q}\, X_T^2.$$

The final value at time $t = T$ is the cost of a market order of signed volume $\Delta X_T = -X_T$ (so that $X_{T^+} = X_T + \Delta X_T = 0$). At time t, the price P_t depends on D_t and S_t which in turn depend on $(X_u)_{u\in[0,t]}$. The value function of the problem is

$$\mathcal{C}(t,x,d,z,\delta,\Sigma) \;=\; \inf_{(X_u)_{u\in[t,T]}\ \text{adm.}} \mathbb{E}\left[C(t,X)\right]$$

with $X_t = x,\ D_t = d,\ S_t = z,\ \delta_t = \delta$ and $\Sigma_t = \Sigma$. In order to determine analytically the value function and the optimal control of the problem, we use the probabilistic formulation of the verification theorem. We determine *a priori* a continuously differentiable function $\mathcal{C}(t,x,d,z,\delta,\Sigma)$ and an admissible strategy X^* and we verify that

$$\Pi_t(X) \;:=\; \int_0^t P_u\ \mathrm{d}X_u \;+\; \frac{1}{2q}\sum_{0\le\tau<t}(\Delta X_\tau)^2 \;+\; \mathcal{C}(t,X_t,D_t,S_t,\delta_t,\Sigma_t) \tag{23}$$

is a submartingale for any admissible strategy X, and that $\Pi_t(X^*)$ is a martingale. We proceed in three steps:

1. We define a suitable function $\mathcal{C}$, and derive a set of ODEs on its coefficients which is a necessary condition for $\mathcal{C}$ to be the value function of the problem. This step is common to the Poisson model ($\alpha = \beta = 0,\ \delta_t \equiv \delta_0,\ \Sigma_t \equiv \Sigma_0$), and the cases ($0 < \alpha = \beta$) and ($\alpha \neq \beta$) for the Hawkes model.
2. We solve the set of ODEs by distinguishing the cases $\alpha = \beta$ and $\alpha \neq \beta$.
3. Using the results of the previous steps, we derive the strategy X^* such that $\Pi_t(X^*)$ is a martingale. Here, we must distinguish the cases $\alpha = \beta = 0,\ 0 < \alpha = \beta$ and $\alpha \neq \beta$.

The verification argument then yields that $\mathcal{C}(t,x,d,z,\delta,\Sigma)$ is the value function and that X^* is optimal. Without loss of generality, we can assume that $q = 1$ by using Remark 2.1.

A.2 Necessary conditions on the value function

We search a cost function $\mathcal{C}$ as a generic quadratic form of the variables x,d,z,δ,Σ with time-dependent coefficient (the variable z symbolizes the current value of the fundamental price S_t). As we see further, we need $\mathcal{C}$ to verify $\partial_x\mathcal{C} + (1-\epsilon)\partial_d\mathcal{C} + \epsilon\ \partial_z\mathcal{C} + d + z = 0$: it is thus necessary that $\mathcal{C}$ is a quadratic form of $(d-(1-\epsilon)x),\ (z-\epsilon x),\ \delta$ and Σ, plus a term $-(d+z)^2/2$. We define

$$\begin{aligned}\mathcal{C}(t,x,d,z,\delta,\Sigma) \;=\;& a(T-t)(d-(1-\epsilon)x)^2 \;+\; \frac{1}{2}(z-\epsilon x)^2 \;+\; (d-(1-\epsilon)x)(z-\epsilon x) \;-\; \frac{(d+z)^2}{2}\\ &+\; b(T-t)\ \delta\ (d-(1-\epsilon)x) \;+\; c(T-t)\ \delta^2 \;+\; e(T-t)\ \Sigma \;+\; g(T-t),\end{aligned} \tag{24}$$

with $a,b,c,e,g : \mathbb{R}^+ \to \mathbb{R}$ continuously differentiable functions. We choose the limit condition $\mathcal{C}(T,x,d,z,\delta,\Sigma) = -(d+z)x + x^2/2 = \frac{1}{2}(d+z-x)^2 - (d+z)^2/2$, which is the cost of a trade of signed volume $-x$. We thus have

$$a(0) = \frac{1}{2},\ b(0) = c(0) = e(0) = g(0) = 0.$$

Let us note that other terms should be added in equation (24) for $\mathcal{C}$ to be a generic quadratic form. The five terms

$$h_1(T-t)\ (d-(1-\epsilon)x) \;+\; h_2(T-t)\ \Sigma(d-(1-\epsilon)x) \;+\; h_3(T-t)\ \delta\Sigma \;+\; h_4(T-t)\ \delta \;+\; h_5(T-t)(z-\epsilon x)$$

have to be equal to zero since $\mathcal{C}(t,x,d,z,\delta,\Sigma) = \mathcal{C}(t,-x,-d,-z,-\delta,\Sigma)$ by using Remark 2.1 and the fact that the buy and sell orders play a symmetric role. For the term in Σ^2, we checked in prior calculations that it is necessarily associated to a zero coefficient. For $\Delta x \in \mathbb{R}$, we have

$$\mathcal{C}(t,x+\Delta x,d+(1-\epsilon)\Delta x,z+\epsilon\Delta x,\delta,\Sigma) - \mathcal{C}(t,x,d,z,\delta,\Sigma) \;=\; -(d+z)\times\Delta x \;-\; \frac{(\Delta x)^2}{2}. \tag{25}$$

The process $\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)$ is làdlàg, and with the notations of Remark 2.2, we have by using (25)

$$\begin{aligned}
\mathrm{d}\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t) &= \partial_t\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)\mathrm{d}t \;+\; \partial_x\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)\mathrm{d}X_t^{\mathrm{c}} \\
&\quad + \partial_d\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)\Big(- \rho D_t\mathrm{d}t + (1-\epsilon)\mathrm{d}X_t^{\mathrm{c}}\Big) \\
&\quad + \partial_z\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)\;\epsilon\mathrm{d}X_t^{\mathrm{c}} \\
&\quad - \beta\;\delta_t\;\partial_\delta\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)\;\mathrm{d}t - \beta(\Sigma_t - 2\kappa_\infty)\;\partial_\Sigma\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)\;\mathrm{d}t \\
&\quad + \Big[\mathcal{C}(t, X_t, D_{t^-} + (1-\nu)\Delta N_t, S_{t^-} + \nu\Delta N_t, \delta_{t^-} + \alpha(\Delta J_t^+ - \Delta J_t^-), \Sigma_{t^-} + (\alpha + 2\iota_{\mathrm{c}})(\Delta J_t^+ + \Delta J_t^-)) \\
&\qquad - \mathcal{C}(t, X_t, D_{t^-}, S_{t^-}, \delta_{t^-}, \Sigma_{t^-})\Big] \;-\; (D_t + S_t)\;\Delta X_t \;-\; \frac{(\Delta X_t)^2}{2}.
\end{aligned}$$

The definition of $\Pi(X)$ given by (23) yields $\mathrm{d}\Pi_t(X) = (D_t + S_t)\mathrm{d}X_t^{\mathrm{c}} + (D_t + S_t)\Delta X_t + (\Delta X_t)^2/2 + \mathrm{d}\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)$. We define the continuous finite variation process $(A_t^X)_{t\in(0,T)}$ such that $A_{0^+}^X = \mathcal{C}(0, X_{0^+}, D_{0^+}, S_{0^+}, \delta_0, \Sigma_0)$ and for $t \in (0, T)$

$$\begin{aligned}
\mathrm{d}A_t^X &= (D_t + S_t)\;\mathrm{d}X_t^{\mathrm{c}} \;+\; Z(t, X_t, D_t, S_t, \delta_t, \Sigma_t)\mathrm{d}t \\
&\quad + \partial_t\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)\mathrm{d}t \;+\; \partial_x\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)\mathrm{d}X_t^{\mathrm{c}} \\
&\quad + \partial_d\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)\Big(- \rho D_t\mathrm{d}t + (1-\epsilon)\mathrm{d}X_t^{\mathrm{c}}\Big) \\
&\quad + \partial_z\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)\;\epsilon\mathrm{d}X_t^{\mathrm{c}} \\
&\quad - \beta\;\delta_t\;\partial_\delta\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)\;\mathrm{d}t \;-\; \beta\;(\Sigma_t - 2\kappa_\infty)\;\partial_\Sigma\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)\;\mathrm{d}t,
\end{aligned}$$

where, for $V \sim \mu$,

$$\begin{aligned}
Z(t, x, d, z, \delta, \Sigma) &:= \frac{\Sigma + \delta}{2} \times \mathbb{E}\big[\mathcal{C}(t, x, d + (1-\nu)V, z + \nu V, \delta + \alpha, \Sigma + \alpha + 2\iota_{\mathrm{c}}) - \mathcal{C}(t, x, d, z, \delta, \Sigma)\big] \\
&\quad + \frac{\Sigma - \delta}{2} \times \mathbb{E}\big[\mathcal{C}(t, x, d - (1-\nu)V, z - \nu V, \delta - \alpha, \Sigma + \alpha + 2\iota_{\mathrm{c}}) - \mathcal{C}(t, x, d, z, \delta, \Sigma)\big].
\end{aligned}$$

Then, $\Pi(X) - A^X$ is a martingale (let us note that almost surely, $\mathrm{d}t$ -a.e. on $(0, T)$, $Z(t, X_t, D_{t^-}, S_{t^-}, \delta_{t^-}, \Sigma_{t^-}) = Z(t, X_t, D_t, S_t, \delta_t, \Sigma_t)$). This yields that $\Pi(X)$ is a submartingale (resp. a martingale) iff A^X is increasing (resp. constant). From (25), we obtain $\partial_x\mathcal{C}(t, x, d, z, \delta, \Sigma) + (1-\epsilon)\partial_d\mathcal{C}(t, x, d, z, \delta, \Sigma) + \epsilon\;\partial_z\mathcal{C}(t, x, d, z, \delta, \Sigma) + d + z = 0$, and then

$$\begin{aligned}
\mathrm{d}A_t^X &= \Big\{\partial_t\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t) - \rho\;D_t\;\partial_d\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t) + Z(t, X_t, D_t, S_t, \delta_t, \Sigma_t) \\
&\qquad - \beta\;\delta_t\;\partial_\delta\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t) \;-\; \beta\;(\Sigma_t - 2\kappa_\infty)\;\partial_\Sigma\mathcal{C}(t, X_t, D_t, S_t, \delta_t, \Sigma_t)\Big\}\mathrm{d}t. \qquad (26)
\end{aligned}$$

Given the quadratic nature of the problem, we search a process A^X of the form

$$\mathrm{d}A_t^X \;=\; \frac{\rho}{1-\epsilon}\mathrm{d}t \times \Big[j(T-t)(D_t - (1-\epsilon)X_t) \;-\; D_t \;+\; k(T-t)\;\delta_t\Big]^2, \qquad (27)$$

with $j, k : \mathbb{R}^+ \to \mathbb{R}$ continuously differentiable functions, in order to obtain an non-decreasing process A^X that can be constant for a specific strategy X^*. Let us note $Y_t := D_t - (1-\epsilon)X_t$, $\Xi_t := S_t - \epsilon X_t$, $y := d - (1-\epsilon)x$, $\xi := z - \epsilon x$. Since $d + z = y + \xi + x = \xi + \frac{d - \epsilon y}{1-\epsilon}$, we have

$$\begin{aligned}
\partial_t\mathcal{C}(t, x, d, z, \delta, \Sigma) &= -\dot{a}\;y^2 \;-\; \dot{b}\;\delta y \;-\; \dot{c}\;\delta^2 \;-\; \dot{e}\;\Sigma \;-\; \dot{g}, \\
-\rho d\;\partial_d\mathcal{C}(t, x, d, z, \delta, \Sigma) &= -\left(2\rho a + \frac{\rho\epsilon}{1-\epsilon}\right)\;dy \;+\; \frac{\rho}{1-\epsilon}\;d^2 \;-\; \rho b\;\delta d, \\
-\beta\delta\;\partial_\delta\mathcal{C}(t, x, d, z, \delta, \Sigma) &= -\beta b\;\delta y \;-\; 2\beta c\;\delta^2, \\
-\beta(\Sigma - 2\kappa_\infty)\;\partial_\Sigma\mathcal{C}(t, x, d, z, \delta, \Sigma) &= -\beta e\;\Sigma \;+\; 2\beta\kappa_\infty e,
\end{aligned}$$

$$\mathbb{E}\big[\mathcal{C}(t,x,d+(1-\nu)V,z+\nu V,\delta+\alpha,\Sigma+\alpha)-\mathcal{C}(t,x,d,z,\delta,\Sigma)\big]$$

$$\begin{aligned}
&= \; a\,[(1-\nu)^2 m_2 + 2(1-\nu)m_1\,y] \;+\; \frac{\nu^2}{2}m_2 \;+\; \nu m_1\,\xi \\
&\quad + \nu(1-\nu)m_2 \;+\; \nu m_1 y \;+\; (1-\nu)m_1\xi \;-\frac{1}{2}\left(m_2 + 2\,m_1\,\xi + \frac{2m_1}{1-\epsilon}\,d - \frac{2\epsilon m_1}{1-\epsilon}\,y\right) \\
&\quad + b\,[(1-\nu)m_1\,\delta \;+\; \alpha\,y \;+\; \alpha(1-\nu)m_1] \;+\; c\,[\alpha^2 \;+\; 2\alpha\,\delta] \;+\; (\alpha+2\iota_{\mathrm{c}})e,
\end{aligned}$$

and thus

$$\begin{aligned}
Z(t,x,d,z,\delta,\Sigma) \;&=\; \left(m_1 \times \left[2(1-\nu)a+\nu+\frac{\epsilon}{1-\epsilon}\right] \;+\; \alpha b\right)\,\delta y \;-\; \frac{m_1}{1-\epsilon}\,\delta d \\
&\quad + [(1-\nu)m_1 b \;+\; 2\alpha c]\;\delta^2 \\
&\quad + \left(m_2 \times \left[(1-\nu)^2 a + \nu(1-\nu/2) - \frac{1}{2}\right] + \alpha(1-\nu)m_1 b + \alpha^2 c + (\alpha+2\iota_{\mathrm{c}})e\right)\;\Sigma,
\end{aligned}$$

where we consider $\mathcal{C}$ as a function of the variables t,x,d,z,δ,Σ as in equation (26), and substitute $d-(1-\epsilon)x$ by y and $z-\epsilon x$ by ξ in the results. We then make the change of variables $(x,d,z,\delta,\Sigma)\to(y,d,\xi,\delta,\Sigma)$, and we identify each term of equations (26) and (27):

(Eq. dy): $\quad -\left(2\rho a + \frac{\rho\epsilon}{1-\epsilon}\right) = -\frac{2\rho}{1-\epsilon}j.$

(Eq. y^2): $\quad -\dot{a} \;=\; \frac{\rho}{1-\epsilon}j^2.$

(Eq. dy) yields $j=(1-\epsilon)a+\frac{\epsilon}{2}$. We input this relation in (Eq. y^2) and we have $\dot{j}=(1-\epsilon)\dot{a}=-\rho j^2$ thus $j(u)=\frac{1}{2+\rho u}$ since $j(0)=(1-\epsilon)a(0)+\frac{\epsilon}{2}=\frac{1}{2}$. This yields $a(u)=\frac{1}{1-\epsilon}\left(\frac{1}{2+\rho u}-\frac{\epsilon}{2}\right)$ with (Eq. dy).

(Eq. δy): $\quad -\,\dot{b} \;-\; \beta b \;+\; \alpha b \;+\; m_1 \times \left[2(1-\nu)a+\nu+\frac{\epsilon}{1-\epsilon}\right] \;=\; \frac{2\rho}{1-\epsilon}jk.$

(Eq. δd): $\quad -\,\rho b \;-\; \frac{m_1}{1-\epsilon} \;=\; -\frac{2\rho}{1-\epsilon}k,$

which yields $k(u) \;=\; \frac{1-\epsilon}{2}\;b(u) \;+\; \frac{m_1}{2\rho}$. Plugging equation (29) in (Eq. δy), we have $\dot{b}=-(\beta-\alpha)b-\frac{2\rho}{1-\epsilon}j\left(\frac{1-\epsilon}{2}b+\frac{m_1}{2\rho}\right)+m_1\left[2(1-\nu)a+\nu+\frac{\epsilon}{1-\epsilon}\right]$, and since $j/(1-\epsilon) \;=\; a+\epsilon/[2(1-\epsilon)]$, we have

$$\dot{b}(u) \;=\; \left[-(\beta-\alpha)-\frac{\rho}{2+\rho u}\right]b(u) \;+\; \frac{m_1}{1-\epsilon}\times\frac{1+\nu\rho u}{2+\rho u}.$$

(Eq. δ^2): $\quad -\,\dot{c} \;-\; 2\beta c \;+\; 2\alpha c \;+\; (1-\nu)m_1 b \;=\; \frac{\rho}{1-\epsilon}\;k^2.$

(Eq. Σ): $\quad -\,\dot{e} \;-\; \beta e \;+\; (\alpha+2\iota_{\mathrm{c}})e \;+\; m_2 \times \left[(1-\nu)^2 a+\nu(1-\nu/2)-\frac{1}{2}\right] \;+\; \alpha(1-\nu)m_1 b \;+\; \alpha^2 c \;=\; 0.$

We have $2(1-\epsilon)\times\left[(1-\nu)^2 a+\nu(1-\nu/2)-\frac{1}{2}\right]=2(1-\nu)^2/(2+\rho u)-(1-\nu)^2\epsilon+\nu(2-\nu)(1-\epsilon)-(1-\epsilon)$, thus

$$\dot{e}(u) \;=\; -(\beta-\alpha-2\iota_{\mathrm{c}})e(u) \;+\; \alpha(1-\nu)m_1 b(u) \;+\; \alpha^2 c(u) \;+\; \frac{(1-\nu)^2\;m_2}{1-\epsilon}\times\left[\frac{1}{2+\rho u}-\frac{1}{2}\right]$$

(Eq. constant): $\quad -\,\dot{g} \;+\; 2\beta\kappa_\infty e \;=\; 0.$

We obtain two conditions on the coefficients of the process A^X

$$j(u) \;=\; \frac{1}{2+\rho u}, \tag{28}$$

$$k(u) \;=\; \frac{1-\epsilon}{2}\;b(u) \;+\; \frac{m_1}{2\rho}, \tag{29}$$

and the following set of necessary conditions on the coefficients of $\mathcal{C}$

$$
\begin{aligned}
a(u) &= \frac{1}{1-\epsilon}\left(\frac{1}{2+\rho u}-\frac{\epsilon}{2}\right), && (30)\\
\dot{b}(u) &= \left[-(\beta-\alpha)-\frac{\rho}{2+\rho u}\right] b(u) \;+\; \frac{m_1}{1-\epsilon}\times\frac{1+\nu\rho u}{2+\rho u}, && (31)\\
\dot{c}(u) &= -2(\beta-\alpha)\; c(u) \;+\; (1-\nu)m_1\; b(u) \;-\; \frac{\rho}{1-\epsilon}\; k(u)^2, && (32)\\
\dot{e}(u) &= -(\beta-\alpha-2\iota_{\mathrm{c}})e(u) \;+\; \alpha(1-\nu)m_1 b(u) \;+\; \alpha^2 c(u) \;+\; \frac{(1-\nu)^2\; m_2}{1-\epsilon}\times\left[\frac{1}{2+\rho u}-\frac{1}{2}\right], && (33)\\
\dot{g}(u) &= 2\beta\kappa_\infty\; e(u), && (34)\\
b(0) &= c(0) \;=\; e(0) \;=\; g(0) \;=\; 0.
\end{aligned}
$$

The resolution of this set of equations determines entirely the function $\mathcal{C}(t,x,d,z,\delta,\Sigma)$ defined in (24). This is the purpose of the next step of this proof. Let us note that at this stage, we already know that the system given by Equations (28) to (34) admits a unique solution, and that the function $\mathcal{C}$ which solves the system is the value function of the problem by using the verification argument.

A.3 Resolution of the system of ODEs

First of all, we use Equation (30) to simplify the function $\mathcal{C}$. The constant term (w.r.t. the time variable t) in equation (24) is $\frac{1}{2}(z-\epsilon x)^2 \;+\; (d-(1-\epsilon)x)(z-\epsilon x) \;-\; \frac{(d+z)^2}{2} \;=\; -zx-\frac{d^2}{2}-\epsilon dx+\left[\frac{\epsilon}{2}+\frac{\epsilon}{2}(1-\epsilon)\right]x^2$, thus the sum of $a(T-t)(d-(1-\epsilon)x)^2$ and this constant term can be rewritten as

$$
-(z+d)x \;+\; \left[\frac{1-\epsilon}{2+\rho(T-t)}+\frac{\epsilon}{2}\right]x^2 \;-\; \frac{1}{1-\epsilon}\times\frac{\rho(T-t)/2}{2+\rho(T-t)}\; d^2 \;+\; \frac{\rho(T-t)}{2+\rho(T-t)}\; dx. \qquad (35)
$$

A.3.1 Case $\alpha=\beta$ (Theorems 4.1 and A.1)

In the case $\alpha=\beta$, equation (31) becomes $\dot{b}(u) \;=\; -\frac{\rho}{2+\rho u}b(u) \;+\; \frac{m_1}{1-\epsilon}\times\frac{1+\nu\rho u}{2+\rho u}$. We thus search a solution of the form $b(u)=\frac{\tilde{b}(u)}{2+\rho u}$, which yields $\dot{\tilde{b}}(u)=\frac{m_1}{1-\epsilon}\times(1+\nu\rho u)$ and we obtain

$$
b(u) \;=\; \frac{m_1 u}{1-\epsilon}\times\frac{1+\frac{\nu}{2}\rho u}{2+\rho u},
$$

with the initial condition $b(0)=0$. Equation (29) gives $k(u)=\frac{m_1}{2\rho}\times\left(\frac{\rho u+\frac{\nu}{2}\rho^2u^2}{2+\rho u}+1\right)$ and

$$
k(u) \;=\; \frac{m_1}{\rho}\times\frac{1+\rho u+\frac{\nu}{4}\rho^2u^2}{2+\rho u}. \qquad (36)
$$

Equation (32) can be rewritten as

$$
\dot{c}(u) \;=\; \frac{m_1^2}{1-\epsilon}\times\left[\frac{(1-\nu)\left(u+\frac{\nu}{2}\rho u^2\right)}{2+\rho u} \;-\; \frac{1}{\rho}\left(\frac{1+\rho u+\frac{\nu}{4}\rho^2u^2}{2+\rho u}\right)^2\right].
$$

We have $u+\frac{\nu}{2}\rho u^2 \;=\; \frac{\nu}{2}u(2+\rho u)+(1-\nu)u$, thus $\frac{u+\frac{\nu}{2}\rho u^2}{2+\rho u} \;=\; \frac{\nu}{2}u+\frac{1-\nu}{\rho}\times\left(1-\frac{2}{2+\rho u}\right)$. Moreover,

$$
\begin{aligned}
\frac{\rho^2u^2}{4} &= \frac{1}{4}(2+\rho u)^2-(2+\rho u)+1,\\
1+\rho u+\frac{\nu}{4}\rho^2u^2 &= \frac{\nu}{4}(2+\rho u)^2+(1-\nu)(2+\rho u)-(1-\nu),
\end{aligned}
$$

which yields $(1+\rho u+\frac{\nu}{4}\rho^2u^2)^2 \;=\; (1-\nu)^2[(2+\rho u)^2-2(2+\rho u)+1]+\left[\frac{\nu}{2}\left(1-\frac{\nu}{2}\right)(1+\rho u)+\frac{\nu^2}{16}\rho^2u^2\right](2+\rho u)^2$ by isolating the terms in $(1-\nu)^2$. We obtain

$$\dot{c}(u) \;=\; -\frac{m_1^2}{1-\epsilon}\left[\frac{(1-\nu)^2}{\rho}\times\frac{1}{(2+\rho u)^2} \;+\; \frac{\nu}{\rho}\times\left(\frac{1}{2}-\frac{\nu}{4}+\frac{\nu}{4}\rho u+\frac{\nu}{16}\rho^2u^2\right)\right].$$

Using the final condition $c(0)=0$, we have

$$c(u) \;=\; -\;\frac{(1-\nu)^2}{1-\epsilon}\times\frac{m_1^2}{\rho^2}\times\left[\frac{1}{2}-\frac{1}{2+\rho u}\right] \;-\; \frac{\nu m_1^2}{\rho^2(1-\epsilon)}\times\left[\left(\frac{1}{2}-\frac{\nu}{4}\right)\rho u+\frac{\nu}{8}\rho^2u^2+\frac{\nu}{48}\rho^3u^3\right].$$

Equation (33) with $\alpha=\beta$ gives $\dot{e}(u) \;=\; 2\iota_{\mathrm{c}}e(u)+\beta(1-\nu)m_1b(u) \;+\; \beta^2c(u) \;+\; \frac{(1-\nu)^2\;m_2}{1-\epsilon}\times\left[\frac{1}{2+\rho u}-\frac{1}{2}\right]$. One has

$$\begin{aligned}
\beta(1-\nu)m_1b(u) \;&=\; \frac{1-\nu}{1-\epsilon}\times\frac{\beta m_1^2}{\rho}\times\frac{\rho u+\frac{\nu}{2}\rho^2u^2}{2+\rho u}\\
&=\; \frac{(1-\nu)^2}{1-\epsilon}\times\frac{2\beta m_1^2}{\rho}\times\left[\frac{1}{2}-\frac{1}{2+\rho u}\right] \;+\; \frac{\beta\nu m_1^2}{\rho(1-\epsilon)}\times\left(\frac{1}{2}-\frac{\nu}{2}\right)\times\rho u,\\
\beta^2c(u) \;&=\; -\;\frac{(1-\nu)^2}{1-\epsilon}\times\frac{\beta^2m_1^2}{\rho^2}\times\left[\frac{1}{2}-\frac{1}{2+\rho u}\right] \;-\; \frac{\beta^2\nu m_1^2}{\rho^2(1-\epsilon)}\times\left(\frac{1}{2}-\frac{\nu}{2}\right)\times\rho u\\
&\qquad -\;\frac{\beta^2\nu m_1^2}{\rho^2(1-\epsilon)}\times\left[\frac{\nu}{4}\rho u+\frac{\nu}{8}\rho^2u^2+\frac{\nu}{48}\rho^3u^3\right].
\end{aligned}$$

We set $e(u)=\tilde{e}(u)\exp(2\iota_{\mathrm{c}}u)$ and we have

$$\begin{aligned}
\dot{\tilde{e}}(u) \;&=\; \frac{(1-\nu)^2}{1-\epsilon}\times\left(\frac{\beta m_1^2(2\rho-\beta)}{\rho^2}-m_2\right)\times\left[\frac{\exp(-2\iota_{\mathrm{c}}u)}{2}-\frac{\exp(-2\iota_{\mathrm{c}}u)}{2+\rho u}\right]\\
&\quad+\;\frac{\beta\nu(1-\nu)m_1^2}{2\rho(1-\epsilon)}\times\left(1-\frac{\beta}{\rho}\right)\times\rho u\exp(-2\iota_{\mathrm{c}}u) \;-\; \frac{\beta^2\nu^2m_1^2}{4\rho^2(1-\epsilon)}\times\left[\rho u+\frac{1}{2}\rho^2u^2+\frac{1}{12}\rho^3u^3\right]\exp(-2\iota_{\mathrm{c}}u).
\end{aligned}$$

For $k\in\mathbb{N}$ and $u\geq 0$, we define $I_k(u)=\exp(2\iota_{\mathrm{c}}u)\int_0^u s^k\exp(-2\iota_{\mathrm{c}}s)\mathrm{d}s$. We first assume $\iota_{\mathrm{c}}>0$. An integration by parts yields $I_k(u)=\frac{k}{2\iota_{\mathrm{c}}}I_{k-1}(u)-\frac{u^k}{2\iota_{\mathrm{c}}}$ for $k\geq 1$, and since $I_0(u)=\frac{\exp(2\iota_{\mathrm{c}}u)-1}{2\iota_{\mathrm{c}}}$, one has

$$\forall k\in\mathbb{N},\;\forall u\geq 0,\quad I_k(u)=\frac{k!}{(2\iota_{\mathrm{c}})^{k+1}}\times\left[\exp(2\iota_{\mathrm{c}}u)-\sum_{j=0}^{k}\frac{(2\iota_{\mathrm{c}})^j}{j!}u^j\right],$$

and we obtain

$$\begin{aligned}
e(u) \;&=\; -\;\frac{(1-\nu)^2}{1-\epsilon}\times\left(m_2-\frac{\beta m_1^2(2\rho-\beta)}{\rho^2}\right)\times\left[\frac{I_0(u)}{2} \;-\; \frac{\exp(2\iota_{\mathrm{c}}u)}{\rho}\;L(\rho,-2\iota_{\mathrm{c}},u)\right]\\
&\quad+\;\frac{\beta\nu(1-\nu)m_1^2}{2\rho^2(1-\epsilon)}\times\left(1-\frac{\beta}{\rho}\right)\times\rho^2I_1(u) \;-\; \frac{\beta^2\nu^2m_1^2}{4\rho^3(1-\epsilon)}\times\left[\rho^2I_1(u)+\frac{1}{2}\rho^3I_2(u)+\frac{1}{12}\rho^4I_3(u)\right].
\end{aligned}\tag{37}$$

We verify that for $k\in\mathbb{N}$ and $u\geq 0$, $\int_0^u I_k(s)\mathrm{d}s=\frac{I_{k+1}(u)}{k+1}$. Besides, an integration by parts yields $\frac{1}{\rho}\int_0^u\exp(2\iota_{\mathrm{c}}s)L(\rho,-2\iota_{\mathrm{c}},s)\mathrm{d}s=\frac{1}{2\iota_{\mathrm{c}}\rho}\times\left[\exp(2\iota_{\mathrm{c}}u)L(\rho,-2\iota_{\mathrm{c}},u)-\ln\left(1+\frac{\rho u}{2}\right)\right]$. We plug these results in equation (34) to get

$$\begin{aligned}
g(u)=-2\beta\kappa_\infty\times\frac{(1-\nu)^2}{1-\epsilon}\times\left(m_2-\frac{\beta m_1^2(2\rho-\beta)}{\rho^2}\right)\left\{\frac{I_1(u)}{2}-\frac{1}{2\iota_{\mathrm{c}}\rho}\times\left[\exp(2\iota_{\mathrm{c}}u)L(\rho,-2\iota_{\mathrm{c}},u)-\ln\left(1+\frac{\rho u}{2}\right)\right]\right\}\\
+\;\frac{\beta^2\kappa_\infty\nu(1-\nu)m_1^2}{2\rho^3(1-\epsilon)}\times\left(1-\frac{\beta}{\rho}\right)\times\rho^3I_2(u)-\frac{\beta^3\kappa_\infty\nu^2m_1^2}{4\rho^4(1-\epsilon)}\times\left[\rho^3I_2(u)+\frac{1}{3}\rho^4I_3(u)+\frac{1}{24}\rho^5I_4(u)\right].
\end{aligned}\tag{38}$$

In the case $\iota_c = 0$, one simply has $I_k(u) = \frac{u^{k+1}}{k+1}$ and $\frac{1}{\rho}\int_0^u \exp(2\iota_c s)L(\rho, -2\iota_c, s)\mathrm{d}s = \left(u + \frac{2}{\rho}\right) \times \frac{1}{\rho}\ln\left(1 + \frac{\rho u}{2}\right) - \frac{u}{\rho}$, and the functions e and g can be deduced easily from (37) and (38).

To determine the value functions $\mathcal{C}$ of Theorems 4.1 ($\beta = 0$) and A.1 ($\beta > 0$), we recall equation (35) and we observe that

$$b(u) = \frac{1}{1-\epsilon} \times \frac{\rho u}{2+\rho u} \times \frac{m_1}{\rho}\, \mathcal{G}_0(u), \tag{39}$$

$$\begin{aligned} c(u) &= -\frac{m_1^2}{\rho^2(1-\epsilon)} \times \frac{\rho u/2}{2+\rho u} \times \left[(1-\nu)^2 + \nu\left(1 - \frac{\nu}{2}\right)(2+\rho u) + \frac{\nu^2}{4}\rho u(2+\rho u)\right] - \frac{\nu^2 m_1^2}{48\rho^2(1-\epsilon)} \times \rho^3 u^3 \\ &= -\frac{1}{1-\epsilon} \times \frac{\rho u/2}{2+\rho u} \times \frac{m_1^2}{\rho^2}\, \mathcal{G}_0(u)^2 - \frac{\nu^2 m_1^2}{48\rho^2(1-\epsilon)} \times \rho^3 u^3, \end{aligned} \tag{40}$$

with

$$\mathcal{G}_0(u) := 1 + \frac{\nu}{2}\rho u.$$

Thus,

$$-\frac{1}{1-\epsilon} \times \frac{\rho u/2}{2+\rho u}\, d^2 + \frac{\rho u}{2+\rho u}\, dx + b(u)\, \delta(d - (1-\epsilon)x) + c(u)\, \delta^2$$

is equal to

$$\frac{\rho u}{2+\rho u}\left[d - \mathcal{G}_0(u)\frac{\delta m_1}{\rho}\right] x - \frac{1}{1-\epsilon} \times \frac{\rho u/2}{2+\rho u}\left[d - \mathcal{G}_0(u)\frac{\delta m_1}{\rho}\right]^2 - \frac{1}{1-\epsilon} \times \frac{\nu^2}{48}\, \rho^3(u)^3 \left(\frac{\delta m_1}{\rho}\right)^2.$$

This yields the expressions of the value function in Theorems 4.1 and A.1.

A.3.2 Case $\alpha \neq \beta$ (Theorem 5.1)

We note $\eta = \beta - \alpha$ so that $\eta \neq 0$ in this section. To solve equation (31), we search a solution of the form $b(u) = \tilde{b}(u) \times \exp(-\eta u)/(2+\rho u)$. This yields $\dot{\tilde{b}}(u) = \frac{m_1}{1-\epsilon} \times (1 + \nu\rho u) \times \exp(\eta u)$, and since $\tilde{b}(0) = 2b(0) = 0$,

$$\tilde{b}(u) = \frac{m_1}{1-\epsilon} \times \left[\frac{\exp(\eta u) - 1}{\eta} + \nu\rho \int_0^u s\exp(\eta s)\, \mathrm{d}s\right].$$

We obtain

$$b(u) = \frac{m_1 u}{1-\epsilon} \times \frac{\zeta(\eta u) + \nu\rho[1 - \zeta(\eta u)]/\eta}{2+\rho u} = \frac{1}{1-\epsilon} \times \frac{\rho u}{2+\rho u} \times \frac{m_1}{\rho}\, \mathcal{G}_\eta(u), \tag{41}$$

where ζ is defined in (19) and

$$\mathcal{G}_\eta(u) := \zeta(\eta u) + \nu\rho[1 - \zeta(\eta u)]/\eta.$$

Equation (29) then gives

$$k(u) = \frac{m_1}{2\rho} \times \frac{2 + \rho u \times \{1 + \zeta(\eta u) + \nu\rho[1 - \zeta(\eta u)]/\eta\}}{2+\rho u}. \tag{42}$$

In order to avoid cumbersome calculations, we use a prior determination of the function c, and we check that it satisfies $c(0) = 0$ and that it is a solution of the ODE (32). We define

$$c(u) := -\frac{1}{1-\epsilon} \times \frac{\rho u/2}{2+\rho u} \times \frac{m_1^2}{\rho^2}\, \mathcal{G}_\eta(u)^2 - \frac{m_1^2}{8(1-\epsilon)\rho} \times \left(1 - \frac{\nu\rho}{\eta}\right)^2 \times u\zeta(\eta u) \times [1 + \exp(-\eta u) - 2\zeta(\eta u)]. \tag{43}$$

We have indeed $c(0) = 0$. Since

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}\left[u\mathcal{G}_\eta(u)\right] &= \exp(-\eta u) + \nu\rho \times u\zeta(\eta u) \;=\; 1 + \nu\rho u - \eta u\mathcal{G}_\eta(u), \\
\frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{u\mathcal{G}_\eta(u)^2}{2+\rho u}\right] &= -\frac{2(1+\rho u)\mathcal{G}_\eta(u)^2}{(2+\rho u)^2} + \frac{2\,\mathcal{G}_\eta(u)[1+\nu\rho u]}{2+\rho u} - 2\eta \times \frac{u\mathcal{G}_\eta(u)^2}{2+\rho u},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}\Big[u\zeta(\eta u) \times [1 + \exp(-\eta u) - 2\zeta(\eta u)]\Big] &= 2\left[\zeta(\eta u) - \exp(-\eta u)\right]^2 \;=\; 2\left[(1+\eta u)\zeta(\eta u) - 1\right]^2, \\
\left(1 - \frac{\nu\rho}{\eta}\right) \times [(1+\eta u)\zeta(\eta u) - 1] &= (1+\eta u)\mathcal{G}_\eta(u) - (1+\nu\rho u),
\end{aligned}$$

we obtain

$$\begin{aligned}
\dot{c}(u) &= \frac{2\eta}{1-\epsilon} \times \frac{\rho u/2}{2+\rho u} \times \frac{m_1^2}{\rho^2}\,\mathcal{G}_\eta(u)^2 - \frac{m_1^2}{(1-\epsilon)\rho} \times \left[\frac{1+\nu\rho u}{2+\rho u}\,\mathcal{G}_\eta(u) - \frac{1+\rho u}{(2+\rho u)^2}\,\mathcal{G}_\eta(u)^2\right] \\
&\quad - \frac{m_1^2}{4(1-\epsilon)\rho} \times \Big[(1+\eta u)\mathcal{G}_\eta(u) - (1+\nu\rho u)\Big]^2 \\
&= \frac{2\eta}{1-\epsilon} \times \frac{\rho u/2}{2+\rho u} \times \frac{m_1^2}{\rho^2}\,\mathcal{G}_\eta(u)^2 - \nu m_1 \times \frac{1}{1-\epsilon} \times \frac{\rho u}{2+\rho u} \times \frac{m_1}{\rho}\,\mathcal{G}_\eta(u) \\
&\quad - \frac{m_1^2}{4(1-\epsilon)\rho} \times \frac{(2+\rho u)^2 - 4 - 4\rho u}{(2+\rho u)^2}\,\mathcal{G}_\eta(u)^2 - \frac{m_1^2}{4(1-\epsilon)\rho} \times \left[2\eta u + \eta^2 u^2\right]\,\mathcal{G}_\eta(u)^2 \\
&\quad - \frac{m_1^2}{4(1-\epsilon)\rho} \times \left[\frac{4}{2+\rho u} - 2(1+\eta u)(1+\nu\rho u)\right]\,\mathcal{G}_\eta(u) - \frac{m_1^2}{4(1-\epsilon)\rho} \times [1+\nu\rho u]^2 \\
&= \frac{2\eta}{1-\epsilon} \times \frac{\rho u/2}{2+\rho u} \times \frac{m_1^2}{\rho^2}\,\mathcal{G}_\eta(u)^2 + \left(\frac{1}{2} - \nu\right) m_1 b(u) - \frac{\rho(1-\epsilon)}{4}\,b^2(u) - \frac{m_1^2}{4(1-\epsilon)\rho} \\
&\quad + \frac{\eta m_1^2}{4(1-\epsilon)\rho} \times \left\{\left[2u(1+\nu\rho u) + 2\frac{\nu\rho}{\eta}u\right]\mathcal{G}_\eta(u) - (2u+\eta u^2)\mathcal{G}_\eta(u)^2 - 2\frac{\nu\rho}{\eta}u - \frac{\nu^2\rho^2}{\eta}u^2\right\}.
\end{aligned}$$

The term $\left[2u(1+\nu\rho u) + 2\frac{\nu\rho}{\eta}u\right]\mathcal{G}_\eta(u) - (2u+\eta u^2)\mathcal{G}_\eta(u)^2 - 2\frac{\nu\rho}{\eta}u - \frac{\nu^2\rho^2}{\eta}u^2$ is equal to

$$\begin{aligned}
&u\zeta(\eta u) \times \left\{2\left(1 - \frac{\nu\rho}{\eta}\right)\left[1 - \frac{\nu\rho}{\eta} + \frac{\nu\rho}{\eta}(1+\eta u - 1 - \eta u)\right] - \left(1 - \frac{\nu\rho}{\eta}\right)^2 (2+\eta u)\zeta(\eta u)\right\} \\
&\quad + \frac{\nu\rho}{\eta}\left[2u(1+\nu\rho u) + 2\frac{\nu\rho}{\eta}u\right] - \frac{\nu^2\rho^2}{\eta^2}(2u+\eta u^2) - 2\frac{\nu\rho}{\eta}u - \frac{\nu^2\rho^2}{\eta}u^2 \\
&= \left(1 - \frac{\nu\rho}{\eta}\right)^2 \times u\zeta(\eta u) \times [1 + \exp(-\eta u) - 2\zeta(\eta u)].
\end{aligned}$$

This yields using Equations (29) and (43)

$$\begin{aligned}
\dot{c}(u) &= -2\eta c(u) + \left(\frac{1}{2} - \nu\right) m_1 b(u) - \frac{\rho(1-\epsilon)}{4}\,b^2(u) - \frac{m_1^2}{4(1-\epsilon)\rho} \\
&= -2\eta c(u) + (1-\nu)\,m_1 b(u) - \frac{\rho}{1-\epsilon}\,k(u)^2,
\end{aligned}$$

and we eventually obtain that the function c defined in (43) is the unique solution of (32) that satisfies $c(0) = 0$. For the functions e and g, we recall here that they admit explicit but very cumbersome formulas that can be obtained by using a formal calculus software.

A.4 Determination of the optimal strategy

The final step of the proof is to determine the strategy X^* such that $\Pi(X^*)$ is a martingale, or equivalently such that A^{X^*} is constant. Equations (27) and (28) yield

$$\begin{aligned} \mathrm{d}A_t^X &= \frac{\rho}{1-\epsilon}\mathrm{d}t \times \left[\frac{D_t - (1-\epsilon)X_t}{2+\rho(T-t)} \;-\; D_t \;+\; k(T-t)\;\delta_t\right]^2 \\ &= \frac{\rho/(1-\epsilon)}{[2+\rho(T-t)]^2}\;\mathrm{d}t \times \Big[(1-\epsilon)X_t \;+\; [1+\rho(T-t)]\;D_t \;-\; [2+\rho(T-t)]\;k(T-t)\;\delta_t\Big]^2. \end{aligned}$$

Thus, A^{X^*} is constant on $(0,T)$ if, and only if

$$\text{a.s. , } \mathrm{d}t \text{ -a.e. on } (0,T)\;, \quad (1-\epsilon)X_t^* = -\;[1+\rho(T-t)]\;D_t^* \;+\; [2+\rho(T-t)]\;k(T-t)\;\delta_t, \tag{44}$$

where $D = D^*$ when the strategy X^* is used by the strategic trader. Then, we characterize the strategy X^* on $[0,T]$ with the three following steps:

- The initial jump ΔX_0^* of the strategy is such that (X^*, D^*) satisfies equation (44) at time $t = 0^+$.
- The strategy X^* on $(0,T)$ is obtained by differentiating equation (44).
- The final jump $\Delta X_T^* = -X_T^*$ closes the position of the strategic trader at time T.

We now use this resolution protocol separately in the Poisson model $\alpha = \beta = 0$, and in the Hawkes model $\max(|\alpha|, \beta) > 0$ where we distinguish the two sub-cases $\alpha = \beta > 0$ and $\alpha \neq \beta$.

A.4.1 Case of the Poisson model (Theorem 4.1)

Equations (36) and (44) give the following condition on the strategy X^*

$$\text{a.s., } \mathrm{d}t \text{ -a.e. on } (0,T), \quad (1-\epsilon)X_t^* = -\;[1+\rho(T-t)]\;D_t^* \;+\; \frac{m_1}{\rho} \times \left[1+\rho(T-t)+\frac{\nu}{4}\rho^2(T-t)^2\right]\;\delta_t \tag{45}$$

The initial jump of X^* at $t = 0$ is such that (45) is verified for $t = 0^+$:

$$\begin{aligned} (1-\epsilon)(x_0 + \Delta X_0^*) &= -\;[1+\rho T]\,(D_0 + (1-\epsilon)\Delta X_0^*) \;+\; \frac{m_1}{\rho} \times \left[1+\rho T + \frac{\nu}{4}\rho^2 T^2\right]\;\delta_0 \\ \Leftrightarrow\; (1-\epsilon)\Delta X_0^* &= -\;\frac{(1-\epsilon)x_0 \;+\; [1+\rho T]\left(D_0 - \frac{m_1}{\rho}\;\delta_0\right) \;-\; \frac{\nu m_1}{4\rho}\;\rho^2 T^2 \delta_0}{2+\rho T}. \end{aligned} \tag{46}$$

In this case, $\delta_t \equiv \delta_0$ is constant and we obtain by differentiating (45)

$$\begin{aligned} (1-\epsilon)\mathrm{d}X_t^* &= \rho D_t^* \mathrm{d}t \;-\; [1+\rho(T-t)]\;\mathrm{d}D_t^* \;-\; m_1 \times \left[1+\frac{\nu}{2}\rho(T-t)\right]\;\delta_0\;\mathrm{d}t \\ \Leftrightarrow\; (1-\epsilon)\mathrm{d}X_t^* &= \rho D_t^*\;\mathrm{d}t \;-\; \frac{m_1\left[1+\frac{\nu}{2}\rho(T-t)\right]}{2+\rho(T-t)}\;\delta_0\;\mathrm{d}t \;-\; \frac{1+\rho(T-t)}{2+\rho(T-t)} \times (1-\nu)\mathrm{d}N_t. \end{aligned}$$

This yields $\mathrm{d}D_t^* = -\;\frac{m_1\left[1+\frac{\nu}{2}\rho(T-t)\right]}{2+\rho(T-t)}\;\delta_0\;\mathrm{d}t \;+\; \frac{(1-\nu)\mathrm{d}N_t}{2+\rho(T-t)}$, and we determine D_t^* explicitly on $(0,T)$:

$$D_{0^+}^* \;=\; D_0 + (1-\epsilon)\Delta X_0^* \;=\; \frac{D_0 - (1-\epsilon)x_0}{2+\rho T} \;+\; \frac{m_1}{\rho} \times \frac{1+\rho T\left(1+\frac{\nu}{4}\rho T\right)}{2+\rho T}\;\delta_0, \tag{47}$$

which gives

$$D_t^* = \frac{D_0 - (1-\epsilon)x_0}{2+\rho T} + \frac{m_1}{\rho} \times \left[\frac{1+\rho T\left(1+\frac{\nu}{4}\rho T\right)}{2+\rho T} - (1-\nu)\ln\left(\frac{2+\rho T}{2+\rho(T-t)}\right) - \frac{\nu}{2}\,\rho t \right] \delta_0 + \sum_{0<\tau\leq t} \frac{(1-\nu)\Delta N_\tau}{2+\rho(T-\tau)}.$$

We obtain for $t \in (0,T)$

$$\begin{aligned}(1-\epsilon)\mathrm{d}X_t^* &= \left[\frac{1+\rho T\left(1+\frac{\nu}{4}\rho T\right)}{2+\rho T} - (1-\nu)\ln\left(\frac{2+\rho T}{2+\rho(T-t)}\right) - \frac{\nu}{2}\,\rho t - \frac{1+\frac{\nu}{2}\rho(T-t)}{2+\rho(T-t)} \right] m_1\delta_0\,\mathrm{d}t \\ &\quad + \left[\frac{D_0-(1-\epsilon)x_0}{2+\rho T} + \sum_{0<\tau\leq t} \frac{(1-\nu)\Delta N_\tau}{2+\rho(T-\tau)} \right] \rho \mathrm{d}t \\ &\quad - \frac{1+\rho(T-t)}{2+\rho(T-t)} \times (1-\nu)\mathrm{d}N_t.\end{aligned}$$

Equation (45) finally yields

$$\begin{aligned}(1-\epsilon)\Delta X_T^* &= -(1-\epsilon)X_T^* \\ &= D_{T^-}^* - \frac{m_1}{\rho}\,\delta_0 \\ &= \frac{D_0-(1-\epsilon)x_0}{2+\rho T} - \frac{m_1}{\rho} \times \left[\frac{1-\frac{\nu}{4}\rho^2T^2}{2+\rho T} + (1-\nu)\ln\left(1+\frac{\rho T}{2}\right) + \frac{\nu}{2}\,\rho T \right] \delta_0 \\ &\quad + \sum_{0<\tau<T} \frac{(1-\nu)\Delta N_\tau}{2+\rho(T-\tau)}.\end{aligned}$$

A.4.2 Case of the Hawkes model (Theorems A.1 and 5.1)

Lemma A.1. *Let $\phi : [0,T] \to \mathbb{R}$ be a measurable function, and for $0 \leq s \leq t \leq T$, $\Phi(s,t) := \int_s^t \phi(u)\exp(-\beta u)\,\mathrm{d}u$. We then have for all $t \in [0,T]$*

$$\int_0^t \phi(u)\,\delta_u\,\mathrm{d}u = \delta_0\,\Phi(0,t) + \alpha\,\Theta_{\chi_t}\,\Phi(\tau_{\chi_t},t) + \alpha \sum_{i=1}^{\chi_t-1} \Theta_i\,\Phi(\tau_i,\tau_{i+1})$$

Proof. The proof is straightforward since for $u \in [\chi_t, t]$, $\delta_u = \delta_0\,\exp(-\beta u) + \alpha\exp(-\beta u)\,\Theta_{\chi_t}$ and for $i \in \{0,\cdots,\chi_t-1\}$ and $u \in [\tau_i,\tau_{i+1})$, $\delta_u = \delta_0\,\exp(-\beta u) + \alpha\exp(-\beta u)\,\Theta_i$. □

Case $\alpha = \beta > 0$ (Theorem A.1)

In this sub-case, we have $\alpha = \beta$ as in the Poisson model and Equation (45) still holds. Moreover, Equation (46) also holds since the process $(\delta_t)_{t\geq 0}$ and the constant process equal to δ_0 coincide at time $t = 0$. This yields the initial jump ΔX_0^* of the strategy X^*. Now, in the case $\alpha = \beta > 0$, (δ_t) is a càdlàg process of

dynamics $\mathrm{d}\delta_t = -\beta\ \delta_t\ \mathrm{d}t + \beta\ \mathrm{d}J_t$. We thus obtain by differentiating (45)

$$\begin{aligned}
(1-\epsilon)\mathrm{d}X_t^* &= \rho D_t^* \mathrm{d}t \ - \ [1+\rho(T-t)]\ \mathrm{d}D_t^* \ - \ m_1 \times \left[1+\frac{\nu}{2}\rho(T-t)\right]\ \delta_t\ \mathrm{d}t \\
&\quad + \frac{m_1}{\rho} \times \left[1+\rho(T-t)+\frac{\nu}{4}\rho^2(T-t)^2\right]\ \mathrm{d}\delta_t \\
\Leftrightarrow\ (1-\epsilon)\mathrm{d}X_t^* &= \rho D_t^*\ \mathrm{d}t \ - \ m_1\ \phi_0(t)\ \delta_t\ \mathrm{d}t \\
&\quad + \frac{1+\rho(T-t)}{2+\rho(T-t)}\left\{\frac{\beta m_1}{\rho}\ \mathrm{d}J_t - (1-\nu)\ \mathrm{d}N_t\right\} \ + \ \frac{\beta\nu m_1}{4\rho} \times \frac{\rho^2(T-t)^2}{2+\rho(T-t)}\ \mathrm{d}J_t,
\end{aligned}$$

where the explicit value of δ_t is given by $\delta_t = \delta_0\ \exp(-\beta t) + \beta \sum_{0<\tau\leq t} \exp(-\beta(t-\tau))\ \Delta J_\tau$, and we define for $0 \leq t \leq T$,

$$\phi_0(t)\ :=\ \frac{1+\frac{\nu}{2}\rho(T-t)+\frac{\beta}{\rho}\left[1+\rho(T-t)+\frac{\nu}{4}\rho^2(T-t)^2\right]}{2+\rho(T-t)}.$$

We now determine D_t^*: for $t \in (0,T)$,

$$\begin{aligned}
\mathrm{d}D_t^* &= -\rho D_t^* \mathrm{d}t \ + \ (1-\epsilon)\mathrm{d}X_t^* \ + \ (1-\nu)\mathrm{d}N_t \\
&= \ - \ m_1\ \phi_0(t)\ \delta_t\ \mathrm{d}t \ + \ \frac{(1-\nu)\ \mathrm{d}N_t}{2+\rho(T-t)} \ + \ \frac{\beta m_1}{\rho} \times \frac{1+\rho(T-t)+\frac{\nu}{4}\rho^2(T-t)^2}{2+\rho(T-t)}\ \mathrm{d}J_t.
\end{aligned}$$

We have

$$\begin{aligned}
D_{0+}^* &= D_0 + (1-\epsilon)\Delta X_0^* \ = \ \frac{D_0-(1-\epsilon)x_0}{2+\rho T} \ + \ \frac{m_1}{\rho} \times \frac{1+\rho T\left(1+\frac{\nu}{4}\rho T\right)}{2+\rho T}\ \delta_0 \\
\int_{(0,t]} \mathrm{d}D_u^* &= \ - \ m_1 \int_{(0,t]} \phi_0(u)\ \delta_u\ \mathrm{d}u \ + \ \sum_{0<\tau\leq t} \frac{(1-\nu)\ \Delta N_\tau}{2+\rho(T-\tau)} \\
&\quad + \frac{\beta m_1}{\rho} \times \sum_{0<\tau\leq t} \frac{1+\rho(T-\tau)+\frac{\nu}{4}\rho^2(T-\tau)^2}{2+\rho(T-\tau)}\ \Delta J_\tau.
\end{aligned}$$

Let us focus on the calculation of $\int_0^t \phi_0(u)\ \delta_u\ \mathrm{d}u$. We define $\Phi_0(s,t) = \int_s^t \phi_0(u)\exp(-\beta u)\ \mathrm{d}u$ for $0 \leq s \leq t \leq T$. Lemma A.1 yields for $t \in [0,T]$

$$\int_0^t \phi_0(u)\ \delta_u\ \mathrm{d}u \ = \ \delta_0\ \Phi_0(0,t) \ + \ \beta\ \Theta_{\chi_t}\ \Phi_0\left(\tau_{\chi_t},t\right) \ + \ \beta \sum_{i=1}^{\chi_t-1} \Theta_i\ \Phi_0(\tau_i,\tau_{i+1})$$

The function Φ_0 must then be determined. For $t \in [0,T]$, we have

$$\frac{\beta}{\rho}\left[1+\rho(T-t)+\frac{\nu}{4}\rho^2(T-t)^2\right] = \frac{\beta\nu}{4\rho}\ \rho(T-t)\ [2+\rho(T-t)] + \frac{\beta}{\rho}\left(1-\frac{\nu}{2}\right)[2+\rho(T-t)] - (1-\nu)\frac{\beta}{\rho},$$

which gives

$$\phi_0(t) \ = \ \left[\frac{\beta}{\rho}+\frac{\nu}{2}\left(1-\frac{\beta}{\rho}\right)\right] \ + \ \frac{(1-\nu)\left(1-\frac{\beta}{\rho}\right)}{2+\rho(T-t)} \ + \ \frac{\beta\nu}{4\rho} \times \rho(T-t).$$

Since $\int_s^t \frac{\exp(-\beta u)}{2+\rho(T-u)}\mathrm{d}u \ = \ \exp(-\beta T)\int_{T-t}^{T-s} \frac{\exp(\beta u)}{2+\rho u}\mathrm{d}u$, this yields (see (11) for the definition of L)

$$\begin{aligned}
\Phi_0(s,t) &= \left[\frac{\beta}{\rho}+\frac{\nu}{2}\left(\frac{1}{2}-\frac{\beta}{\rho}\right)\right] \times \frac{\exp(-\beta s)-\exp(-\beta t)}{\beta} \\
&\quad + (1-\nu)\left(1-\frac{\beta}{\rho}\right) \times \frac{\exp(-\beta T)}{\rho} \times [L(\rho,\beta,T-s) - L(\rho,\beta,T-t)] \\
&\quad + \frac{\nu}{4}\left[(T-s)\exp(-\beta s) - (T-t)\exp(-\beta t)\right].
\end{aligned}$$

We thus have for $t \in (0,T)$

$$\begin{aligned}
D_t^* &= \frac{D_0-(1-\epsilon)x_0}{2+\rho T} + \frac{\delta_0 m_1}{\rho} \times \left[\frac{1+\rho T\left(1+\frac{\nu}{4}\rho T\right)}{2+\rho T} - \rho\ \Phi_0(0,t)\right] \\
&\quad - \beta m_1 \left[\Theta_{\chi_t}\ \Phi_0\left(\tau_{\chi_t},t\right) + \sum_{i=1}^{\chi_t-1}\Theta_i\ \Phi_0(\tau_i,\tau_{i+1})\right] \\
&\quad + \sum_{0<\tau\le t}\frac{(1-\nu)\ \Delta N_\tau}{2+\rho(T-\tau)} + \frac{\beta m_1}{\rho}\times\sum_{0<\tau\le t}\frac{1+\rho(T-\tau)+\frac{\nu}{4}\rho^2(T-\tau)^2}{2+\rho(T-\tau)}\ \Delta J_\tau,
\end{aligned}$$

and we eventually obtain the strategy X^* on $(0,T)$

$$\begin{aligned}
(1-\epsilon)\mathrm{d}X_t^* &= \left[\frac{1+\rho T\left(1+\frac{\nu}{4}\rho T\right)}{2+\rho T} - \rho\ \Phi_0(0,t)\right]\ m_1\delta_0\ \mathrm{d}t\ -\ m_1\ \phi_0(t)\ \left(\delta_0+\beta\Theta_{\chi_t}\right)\exp(-\beta t)\ \mathrm{d}t \\
&\quad + \left[\frac{D_0-(1-\epsilon)x_0}{2+\rho T} + \sum_{0<\tau\le t}\frac{(1-\nu)\Delta N_\tau}{2+\rho(T-\tau)}\right]\rho\mathrm{d}t \\
&\quad + \left[\sum_{0<\tau\le t}\frac{1+\rho(T-\tau)+\frac{\nu}{4}\rho^2(T-\tau)^2}{2+\rho(T-\tau)}\ \Delta J_\tau - \rho\Theta_{\chi_t}\ \Phi_0\left(\tau_{\chi_t},t\right) - \rho\sum_{i=1}^{\chi_t-1}\Theta_i\ \Phi_0(\tau_i,\tau_{i+1})\right]\beta m_1\mathrm{d}t \\
&\quad + \frac{1+\rho(T-t)}{2+\rho(T-t)}\left\{\frac{\beta m_1}{\rho}\ \mathrm{d}J_t - (1-\nu)\ \mathrm{d}N_t\right\} + \frac{\beta\nu m_1}{4\rho}\times\frac{\rho^2(T-t)^2}{2+\rho(T-t)}\ \mathrm{d}J_t, \qquad (48)
\end{aligned}$$

and the final jump at time T (using equation (45))

$$\begin{aligned}
(1-\epsilon)\Delta X_T^* &= -(1-\epsilon)X_T^* \\
&= D_{T^-}^* - \frac{m_1}{\rho}\ \delta_T \\
&= \frac{D_0-(1-\epsilon)x_0}{2+\rho T} + \frac{\delta_0 m_1}{\rho}\times\left[\frac{1+\rho T\left(1+\frac{\nu}{4}\rho T\right)}{2+\rho T} - \rho\ \Phi_0(0,T)\right] \\
&\quad - \beta m_1\left[\Theta_{\chi_T}\ \Phi_0\left(\tau_{\chi_T},T\right) + \sum_{i=1}^{\chi_T-1}\Theta_i\ \Phi_0(\tau_i,\tau_{i+1})\right] \\
&\quad + \sum_{0<\tau\le T}\frac{(1-\nu)\ \Delta N_\tau}{2+\rho(T-\tau)} + \frac{\beta m_1}{\rho}\times\sum_{0<\tau\le T}\frac{1+\rho(T-\tau)+\frac{\nu}{4}\rho^2(T-\tau)^2}{2+\rho(T-\tau)}\ \Delta J_\tau \\
&\quad - \frac{m_1}{\rho}\left(\delta_0+\beta\Theta_{\chi_T}\right)\exp(-\beta T). \qquad (49)
\end{aligned}$$

Theorem A.1. *Let $\epsilon \in [0,1)$. For $0 \le s \le t \le T$, we define*

$$\begin{aligned}
\phi_0(t) &= \left[\frac{\beta}{\rho}+\frac{\nu}{2}\left(1-\frac{\beta}{\rho}\right)\right] + \frac{(1-\nu)\left(1-\frac{\beta}{\rho}\right)}{2+\rho(T-t)} + \frac{\beta\nu}{4\rho}\times\rho(T-t) \\
\Phi_0(s,t) &= \left[\frac{\beta}{\rho}+\frac{\nu}{2}\left(\frac{1}{2}-\frac{\beta}{\rho}\right)\right]\times\frac{\exp(-\beta s)-\exp(-\beta u)}{\beta} \\
&\quad + (1-\nu)\left(1-\frac{\beta}{\rho}\right)\times\frac{\exp(-\beta T)}{\rho}\times\left[L(\rho,\beta,T-s)-L(\rho,\beta,T-u)\right] \\
&\quad + \frac{\nu}{4}\left[(T-s)\exp(-\beta s)-(T-u)\exp(-\beta u)\right]
\end{aligned}$$

In the case $\alpha = \beta$, the optimal strategy X^* in the MIH model is given by (46), (48) *and* (49). *The value function function of the problem is then*

$$\begin{aligned} q \times \mathcal{C}(t,x,d,z,\delta,\Sigma) &= -q(z+d)x \;+\; \left[\frac{1-\epsilon}{2+\rho(T-t)} + \frac{\epsilon}{2}\right] x^2 \;+\; \frac{\rho(T-t)}{2+\rho(T-t)}\left[qd - \mathcal{G}_0(T-t)\;\frac{\delta m_1}{\rho}\right] x \\ &\quad - \frac{1}{1-\epsilon} \times \frac{\rho(T-t)/2}{2+\rho(T-t)}\left[qd - \mathcal{G}_0(T-t)\;\frac{\delta m_1}{\rho}\right]^2 - \frac{1}{1-\epsilon} \times \frac{\nu^2}{48}\;\rho^3(T-t)^3\;\left(\frac{\delta m_1}{\rho}\right)^2 \\ &\quad + e(T-t)\;\Sigma \;+\; g(T-t), \end{aligned}$$

where for $u \in [0,T]$, $\mathcal{G}_0(u) = 1 + \frac{\nu}{2}\rho u$ and e and g are given by (37) *and* (38).

Case $\alpha \neq \beta$ (Theorem 5.1)

We use Equations (42) and (44) to obtain the following characterization of the strategy X^*: a.s., dt-a.e. on $(0,T)$,

$$(1-\epsilon)X_t^* = -\;[1+\rho(T-t)]\;D_t^* \;+\; \frac{m_1}{2\rho} \times [2+\rho(T-t) \times \{1+\zeta(\eta(T-t)) + \nu\rho[1-\zeta(\eta(T-t))]/\eta\}]\;\delta_t. \tag{50}$$

The initial jump of X^* at $t=0$ is such that (50) is verified for $t = 0^+$:

$$\begin{aligned} (1-\epsilon)(x_0 + \Delta X_0^*) &= -\;[1+\rho T]\,(D_0 + (1-\epsilon)\Delta X_0^*) \;+\; \frac{m_1}{2\rho} \times [2+\rho T \times \{1+\zeta(\eta T) + \nu\rho[1-\zeta(\eta T)]/\eta\}]\;\delta_0 \\ \Leftrightarrow\; (1-\epsilon)\Delta X_0^* &= -\;\frac{(1-\epsilon)x_0 \;+\; [1+\rho T]D_0 \;-\; \frac{m_1}{2\rho} \times [2+\rho T \times \{1+\zeta(\eta T) + \nu\rho[1-\zeta(\eta T)]/\eta\}]\;\delta_0}{2+\rho T}. \end{aligned}$$

In the case $\alpha \neq \beta$, we have $\mathrm{d}\delta_t = -\beta\;\delta_t\;\mathrm{d}t + \alpha\;\mathrm{d}J_t$. We differentiate Equation (50)

$$\begin{aligned} (1-\epsilon)\mathrm{d}X_t^* &= \rho D_t^* \mathrm{d}t \;-\; [1+\rho(T-t)]\;\mathrm{d}D_t^* \;-\; \frac{m_1}{2} \times [1+\exp(-\eta(T-t)) + \nu\rho(T-t)\zeta(\eta(T-t))]\;\delta_t\;\mathrm{d}t \\ &\quad + \frac{m_1}{2\rho} \times [2+\rho(T-t) \times \{1+\zeta(\eta(T-t)) + \nu\rho[1-\zeta(\eta(T-t))]/\eta\}]\;\mathrm{d}\delta_t, \end{aligned}$$

which yields

$$\begin{aligned} (1-\epsilon)\mathrm{d}X_t^* &= \rho D_t^*\;\mathrm{d}t \;-\; m_1\;\phi_\eta(t)\;\delta_t\;\mathrm{d}t \;+\; \frac{1+\rho(T-t)}{2+\rho(T-t)}\left\{\frac{\alpha m_1}{\rho}\;\mathrm{d}J_t - (1-\nu)\;\mathrm{d}N_t\right\} \\ &\quad + \frac{\alpha m_1}{2\rho} \times \frac{\rho(T-t) \times \{\zeta(\eta(T-t)) - 1 + \nu\rho[1-\zeta(\eta(T-t))]/\eta\}}{2+\rho(T-t)}\;\mathrm{d}J_t, \end{aligned}$$

where for $t \in [0,T]$

$$\phi_\eta(t) \;:=\; \frac{1}{2} \times \frac{1+\exp(-\eta(T-t)) + \nu\rho(T-t)\zeta(\eta(T-t)) + \frac{\beta}{\rho}\,[2+\rho(T-t) \times \{1+\zeta(\eta(T-t)) + \nu\rho[1-\zeta(\eta(T-t))]/\eta\}]}{2+\rho(T-t)}$$

and $\delta_t = \delta_0\;\exp(-\beta t) + \alpha \sum_{0<\tau\leq t} \exp(-\beta(t-\tau))\;\Delta J_\tau$. For $t \in (0,T)$,

$$\begin{aligned} \mathrm{d}D_t^* &= -\rho D_t^* \mathrm{d}t \;+\; (1-\epsilon)\mathrm{d}X_t^* \;+\; (1-\nu)\mathrm{d}N_t \\ &= -\;m_1\;\phi_\eta(t)\;\delta_t\;\mathrm{d}t \\ &\quad + \frac{(1-\nu)\;\mathrm{d}N_t}{2+\rho(T-t)} \;+\; \frac{\alpha m_1}{2\rho} \times \frac{2+\rho(T-t) \times \{1+\zeta(\eta(T-t)) + \nu\rho[1-\zeta(\eta(T-t))]/\eta\}}{2+\rho(T-t)}\;\mathrm{d}J_t, \end{aligned}$$

and we have

$$D^*_{0+} \;=\; D_0 + (1-\epsilon)\Delta X^*_0 \;=\; \frac{D_0 - (1-\epsilon)x_0}{2+\rho T} \;+\; \frac{m_1}{2\rho} \times \frac{2+\rho T \times \{1+\zeta(\eta T) + \nu\rho[1-\zeta(\eta T)]/\eta\}}{2+\rho T}\;\delta_0$$

$$\begin{aligned}\int_{(0,t]} \mathrm{d}D^*_u \;&=\; -\,m_1 \int_{(0,t]} \phi_\eta(u)\;\delta_u\;\mathrm{d}u \;+\; \sum_{0<\tau\le t} \frac{(1-\nu)\;\Delta N_\tau}{2+\rho(T-\tau)} \\ &\quad + \frac{\alpha m_1}{2\rho} \times \sum_{0<\tau\le t} \frac{2+\rho(T-\tau)\times\{1+\zeta(\eta(T-\tau)) + \nu\rho[1-\zeta(\eta(T-\tau))]/\eta\}}{2+\rho(T-\tau)}\;\Delta J_\tau.\end{aligned}$$

We define $\Phi_\eta(s,t) := \int_s^t \phi_\eta(u)\exp(-\beta u)\;\mathrm{d}u$ for $0 \le s \le t \le T$. Lemma A.1 yields for $t \in [0,T]$

$$\int_0^t \phi_\eta(u)\;\delta_u\;\mathrm{d}u \;=\; \delta_0\;\Phi_\eta(0,t) \;+\; \alpha\;\Theta_{\chi_t}\;\Phi_\eta\left(\tau_{\chi_t},t\right) \;+\; \alpha\;\sum_{i=1}^{\chi_t-1}\Theta_i\;\Phi_\eta(\tau_i,\tau_{i+1}).$$

To determine the function Φ_η, we write

$$\begin{aligned}\exp(-\eta(T-t)) \times \exp(-\beta t) \;&=\; \exp(-\beta T) \times \exp(\alpha(T-t)), \\ (T-t)\zeta(\eta(T-t)) \times \exp(-\beta t) \;&=\; \frac{\exp(-\beta T)}{\eta} \times [\exp(\beta(T-t)) - \exp(\alpha(T-t))].\end{aligned}$$

Thus, $\phi_\eta(t) \times \exp(\beta(T-t))$ is equal to

$$\frac{\beta}{2}\left(\frac{1}{\rho}+\frac{\nu}{\eta}\right)\times\exp(\beta(T-t)) + \left[\frac{1}{2}+\frac{\nu(\rho-2\beta)}{2\eta}+\frac{\beta}{2\eta}\left(1-\frac{\nu\rho}{\eta}\right)\right]\frac{\exp(\beta(T-t))}{2+\rho(T-t)} + \left[\frac{1}{2}-\frac{\nu\rho}{2\eta}-\frac{\beta}{2\eta}\left(1-\frac{\nu\rho}{\eta}\right)\right]\frac{\exp(\alpha(T-t))}{2+\rho(T-t)},$$

which yields for $0 \le s \le t \le T$,

$$\begin{aligned}\Phi_\eta(s,t) \;&=\; \frac{1}{2}\left(\frac{1}{\rho}+\frac{\nu}{\eta}\right) \times [\exp(-\beta s) - \exp(-\beta t)] \\ &\quad + \frac{\exp(-\beta T)}{2\rho} \times \left[1 + \frac{\nu(\rho-2\beta)}{\eta} + \frac{\beta}{\eta}\left(1-\frac{\nu\rho}{\eta}\right)\right] \times [L(\rho,\beta,T-s) - L(\rho,\beta,T-t)] \\ &\quad + \frac{\exp(-\beta T)}{2\rho} \times \left[1 - \frac{\nu\rho}{\eta} - \frac{\beta}{\eta}\left(1-\frac{\nu\rho}{\eta}\right)\right] \times [L(\rho,\alpha,T-s) - L(\rho,\alpha,T-t)].\end{aligned}$$

We obtain the expression of D^*_t for $t \in (0,T)$

$$\begin{aligned}D^*_t \;&=\; \frac{D_0-(1-\epsilon)x_0}{2+\rho T} \;+\; \frac{\delta_0 m_1}{2\rho} \times \left[\frac{2+\rho T \times \{1+\zeta(\eta T)+\nu\rho[1-\zeta(\eta T)]/\eta\}}{2+\rho T} \;-\; 2\rho\;\Phi_\eta(0,t)\right] \\ &\quad - \alpha m_1 \left[\Theta_{\chi_t}\;\Phi_\eta\left(\tau_{\chi_t},t\right) \;+\; \sum_{i=1}^{\chi_t-1}\Theta_i\;\Phi_\eta(\tau_i,\tau_{i+1})\right] \;+\; \sum_{0<\tau\le t}\frac{(1-\nu)\;\Delta N_\tau}{2+\rho(T-\tau)} \\ &\quad + \frac{\alpha m_1}{2\rho} \times \sum_{0<\tau\le t} \frac{2+\rho(T-\tau)\times\{1+\zeta(\eta(T-\tau))+\nu\rho[1-\zeta(\eta(T-\tau))]/\eta\}}{2+\rho(T-\tau)}\;\Delta J_\tau.\end{aligned}$$

The strategy X^* on $(0,T)$ is given by

$$
\begin{aligned}
(1-\epsilon)\mathrm{d}X_t^* \;=\; & \left[\frac{2+\rho T\times\{1+\zeta(\eta T)+\nu\rho[1-\zeta(\eta T)]/\eta\}}{2+\rho T} \;-\; 2\rho\;\Phi_\eta(0,t)\right]\;\frac{\delta_0 m_1}{2}\;\mathrm{d}t \\
& -\; m_1\;\phi_\eta(t)\;\;(\delta_0+\alpha\Theta_{\chi_t})\exp(-\beta t)\;\mathrm{d}t \\
& +\left[\frac{D_0-(1-\epsilon)x_0}{2+\rho T} \;+\; \sum_{0<\tau\le t}\frac{(1-\nu)\Delta N_\tau}{2+\rho(T-\tau)}\right]\rho\mathrm{d}t \\
& +\left[\sum_{0<\tau\le t}\frac{2+\rho(T-\tau)\times\{1+\zeta(\eta(T-\tau))+\nu\rho[1-\zeta(\eta(T-\tau))]/\eta\}}{2+\rho(T-\tau)}\;\Delta J_\tau\right]\;\frac{\alpha m_1}{2}\;\mathrm{d}t \\
& -\left[\Theta_{\chi_t}\;\Phi_\eta\left(\tau_{\chi_t},t\right) \;+\; \sum_{i=1}^{\chi_t-1}\Theta_i\;\Phi_\eta(\tau_i,\tau_{i+1})\right]\rho\alpha m_1\;\mathrm{d}t \\
& +\frac{1+\rho(T-t)}{2+\rho(T-t)}\left\{\frac{\alpha m_1}{\rho}\;\mathrm{d}J_t-(1-\nu)\;\mathrm{d}N_t\right\} \\
& +\frac{\alpha m_1}{2\rho}\times\frac{\rho(T-t)\times\{\zeta(\eta(T-t))-1+\nu\rho[1-\zeta(\eta(T-t))]/\eta\}}{2+\rho(T-t)}\;\mathrm{d}J_t,
\end{aligned}
$$

and the final jump at time T (using equation (50))

$$
\begin{aligned}
(1-\epsilon)\Delta X_T^* \;=\; & -(1-\epsilon)X_T^* \\
=\; & D_{T^-}^*-\frac{m_1}{\rho}\;\delta_T \\
=\; & \frac{D_0-(1-\epsilon)x_0}{2+\rho T} \;+\; \frac{\delta_0 m_1}{2\rho}\times\left[\frac{2+\rho T\times\{1+\zeta(\eta T)+\nu\rho[1-\zeta(\eta T)]/\eta\}}{2+\rho T} \;-\; 2\rho\;\Phi_\eta(0,T)\right] \\
& -\;\alpha m_1\left[\Theta_{\chi_T}\;\Phi_\eta\left(\tau_{\chi_T},T\right) \;+\; \sum_{i=1}^{\chi_T-1}\Theta_i\;\Phi_\eta(\tau_i,\tau_{i+1})\right] \;+\; \sum_{0<\tau\le T}\frac{(1-\nu)\;\Delta N_\tau}{2+\rho(T-\tau)} \\
& +\frac{\alpha m_1}{2\rho}\times\sum_{0<\tau\le T}\frac{2+\rho(T-\tau)\times\{1+\zeta(\eta(T-\tau))+\nu\rho[1-\zeta(\eta(T-\tau))]/\eta\}}{2+\rho(T-\tau)}\;\Delta J_\tau \\
& -\;\frac{m_1}{\rho}\left(\delta_0+\alpha\Theta_{\chi_T}\right)\exp(-\beta T).
\end{aligned}
$$

B Proof of Theorem 2.1

Let X be an admissible strategy. We introduce the following processes: $S_t^N = S_0 + \frac{\nu}{q}(N_t - N_0)$, $S_t^X = \frac{\epsilon}{q}(X_t - X_0)$,

$$
\mathrm{d}D_t^N = -\rho D_t^N\mathrm{d}t + \frac{1-\nu}{q}\mathrm{d}N_t \text{ and } \mathrm{d}D_t^X = -\rho D_t^X\mathrm{d}t + \frac{1-\epsilon}{q}\mathrm{d}X_t,
$$

with $D_0^N = D_0$ and $D_0^X = 0$. Thus, we have $S = S^N + S^X$, $D = D^N + D^X$ and thus $P = P^N + P^X$, where $P^N = S^N + D^N$ and $P^X = S^X + D^X$. From (4), we have

$$
C(X) = \int_{[0,T)} P_u^N\;\mathrm{d}X_u - P_T^N X_T + C^{\mathrm{OW}}(X),
$$

where

$$
C^{\mathrm{OW}}(X) = \int_{[0,T)} P_u^X\;\mathrm{d}X_u \;+\; \frac{1}{2q}\sum_{0\le\tau<T}(\Delta X_\tau)^2 \;-\; P_T^X X_T \;+\; \frac{1}{2q}\;X_T^2
$$

is a deterministic function of X that corresponds to the cost when $N \equiv 0$, which is the Obizhaeva and Wang model. We now make an integration by parts as in Remark 2.3 and get that

$$\int_{[0,T)} P_u^N \, \mathrm{d}X_u - P_T^N X_T = -\int_{[0,T)} X_u \, \mathrm{d}P_u^N.$$

When P^N is a martingale, this term has a null expectation. Therefore, the optimal execution strategy is the same as in the Obizhaeva and Wang model, see Gatheral, Schied and Slynko [20], Example 2.12, and there is no PMS. Otherwise, we can find $0 \leq s < t \leq T$ such that $\mathbb{E}[P_t^N|\mathcal{F}_s]$ and P_s^N are not almost surely equal. In this case, we consider the strategy $X_u = \mathbb{E}[P_t^N - P_s^N|\mathcal{F}_s]\mathbf{1}_{u\in(s,t]}$ that is a round-trip, i.e. $X_0 = X_{T+} = 0$. We then get

$$\mathbb{E}\left[-\int_{[0,T)} X_u \, \mathrm{d}P_u^N\right] = -\mathbb{E}[(P_t^N - P_s^N)\mathbb{E}[P_t^N - P_s^N|\mathcal{F}_s]] = -\mathbb{E}[\mathbb{E}[P_t^N - P_s^N|\mathcal{F}_s]^2] < 0.$$

Since $C^{\mathrm{OW}}(cX) = c^2 C^{\mathrm{OW}}(X)$, we can find c small enough such that $E[C(cX)] = -c\mathbb{E}[\mathbb{E}[P_t^N - P_s^N|\mathcal{F}_s]^2] + c^2 C^{\mathrm{OW}}(X) < 0$, and therefore cX is a PMS.

C Proofs for results on PMS in the Poisson model

The first three subsections of this appendix contain the proofs of the results given in Section 4.2. In the first two subsections, N is a martingale since $\delta_0 = 0$. To clarify the calculations, we use Remark 2.4 and assume without loss of generality that

$$\epsilon = \nu = 0 \text{ and } q = 1. \tag{51}$$

C.1 Proof of Proposition 4.1 (uncertainty on D_0)

Let $\tilde{D}_t = \tilde{d}_0 \exp(-\rho t) + \frac{1}{q}\int_0^t \exp(-\rho(t-s))\mathrm{d}N_s + \frac{1}{q}\int_0^t \exp(-\rho(t-s))\mathrm{d}\tilde{X}_s = D_t + (d_0 - \tilde{d}_0)\exp(-\rho t)$. We compute the expected value of the cost $C(\tilde{X})$ and get from Remark 2.4 and (51)

$$\begin{aligned}
\mathbb{E}\left[C(\tilde{X})\right] &= \mathbb{E}\left[\int_0^{T+} \tilde{D}_t \, \mathrm{d}\tilde{X}_t\right] + (d_0 - \tilde{d}_0)\,\mathbb{E}\left[\int_0^{T+} \exp(-\rho t)\, \mathrm{d}\tilde{X}_t\right] \\
&= -\,\tilde{d}_0^{\,2}\,\frac{\rho T/2}{2+\rho T} - 2\kappa_0 m_2\left[\frac{T}{2} - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right] + (d_0 - \tilde{d}_0)\,\mathbb{E}\left[\int_0^{T+} \exp(-\rho t)\, \mathrm{d}\tilde{X}_t\right],
\end{aligned}$$

since the first expectation is the optimal cost in the model starting from $\tilde{d}_0$. The integration up to $T+$ means that it takes into account the last trade at time T. We then have to calculate the last expectation. Since N is a martingale, we get from (14)

$$\begin{aligned}
\mathbb{E}\left[\int_0^{T+} \exp(-\rho t)\, \mathrm{d}\tilde{X}_t\right] &= -\tilde{d}_0\frac{1+\rho T}{2+\rho T} + \frac{\rho \tilde{d}_0}{2+\rho T}\int_{(0,T)} \exp(-\rho t)\, \mathrm{d}t + \exp(-\rho T)\frac{1}{2+\rho T}\,\tilde{d}_0 \\
&= -\tilde{d}_0\frac{\rho T}{2+\rho T},
\end{aligned}$$

which eventually gives $\mathbb{E}\left[C(\tilde{X})\right] = \frac{\rho T/2}{2+\rho T}\,[\tilde{d}_0^{\,2} - 2d_0\tilde{d}_0] - 2\kappa_0 m_2\left[\frac{T}{2} - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right]$.

C.2 Proof of Proposition 4.2 (uncertainty on ρ)

Let $\tilde{X}$ be as (15) and (51) hold. By Remark 2.4, the expected value of this strategy is

$$\begin{aligned}\mathbb{E}\left[C(\tilde{X})\right] &= \mathbb{E}\left[\int_{[0,T)}\left\{\int_{[0,t]}\exp(-\rho(t-s))\mathrm{d}N_s + \int_{[0,t]}\exp(-\rho(t-s))\ \mathrm{d}\tilde{X}_s\right\}\mathrm{d}\tilde{X}_t\right] + \frac{1}{2}\ \mathbb{E}\left[\sum_{0\le\tau<T}(\Delta\tilde{X}_\tau)^2\right] \\ &+\mathbb{E}\left[\left\{\int_{[0,T]}\exp(-\rho(T-t))\ \mathrm{d}N_t + \int_{[0,T]}\exp(-\rho(T-t))\ \mathrm{d}\tilde{X}_t\right\}\times\Delta\tilde{X}_T\right] + \ \frac{1}{2}\mathbb{E}\left[\Delta\tilde{X}_T^2\right].\end{aligned}$$

For the definitions of the functions L and $\mathcal{E}$, we refer to (11). In the balanced model $\delta_0 = 0$, N is a martingale. We also recall that $\mathbb{E}[\Delta N_\tau \Delta N_{\tau'}] = 0$ for $\tau \neq \tau'$ and $\mathbb{E}[\int_0^T g(s)dN_s \int_0^T h(s)dN_s] = 2\kappa_0 m_2 \int_0^T g(s)h(s)ds]$ for $g, h : [0,T] \to \mathbb{R}$ piecewise continuous. Using this, we get

$$\begin{aligned}\mathbb{E}\left[\sum_{0\le\tau<T}(\Delta\tilde{X}_\tau)^2\right] &= 2\kappa_0 m_2\int_0^T\left(\frac{1+\tilde{\rho}(T-t)}{2+\tilde{\rho}(T-t)}\right)^2\mathrm{d}t = 2\kappa_0 m_2\int_0^T\left(1-\frac{2}{2+\tilde{\rho}t}+\frac{1}{[2+\tilde{\rho}t]^2}\right)\mathrm{d}t \\ &= \frac{2\kappa_0 m_2}{\rho}\times\left\{\rho T\ +\ \frac{\rho}{\tilde{\rho}}\times\left[\frac{1}{2}-\frac{1}{2+\tilde{\rho}T}-2\ \ln\left(1+\frac{\tilde{\rho}T}{2}\right)\right]\right\},\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\ \left[\int_{[0,T)}\left(\int_{[0,t]}\exp(-\rho(t-s))\ \mathrm{d}N_s\right)\ \mathrm{d}\tilde{X}_t\right] \\ &= \tilde{\rho}\ \mathbb{E}\left[\int_{[0,T)}\left(\int_{[0,t]}\exp(-\rho(t-s))\ \mathrm{d}N_s\right)\ \left(\int_{[0,t]}\frac{\mathrm{d}N_s}{2+\tilde{\rho}(T-s)}\right)\mathrm{d}t\right]\ -\ \mathbb{E}\left[\sum_{0\le\tau<T}\frac{1+\tilde{\rho}(T-\tau)}{2+\tilde{\rho}(T-\tau)}\ (\Delta N_\tau)^2\right] \\ &= 2\kappa_0 m_2\ \left\{\tilde{\rho}\ \int_0^T\left(\int_0^t\frac{\exp(-\rho(t-s))}{2+\tilde{\rho}(T-s)}\mathrm{d}s\right)\mathrm{d}t\ -\ T\ +\ \int_0^T\frac{\mathrm{d}t}{2+\tilde{\rho}t}\right\} \\ &= 2\kappa_0 m_2\ \left\{\tilde{\rho}\ \int_0^T\left(\exp(\rho(T-t))\int_{T-t}^T\frac{\exp(-\rho s)}{2+\tilde{\rho}s}\mathrm{d}s\right)\mathrm{d}t\ -\ T\ +\ \frac{1}{\tilde{\rho}}\ln\left(1+\frac{\tilde{\rho}T}{2}\right)\right\} \\ &= \frac{2\kappa_0 m_2}{\rho}\ \left\{\left(1+\frac{\rho}{\tilde{\rho}}\right)\ln\left(1+\frac{\tilde{\rho}T}{2}\right)\ -\ L(\tilde{\rho},-\rho,T)\ -\ \rho T\right\},\end{aligned} \tag{52}$$

$$\mathbb{E}\left[(\Delta\tilde{X}_T)^2\right] = \mathbb{E}\left[\left(\int_0^T\frac{\mathrm{d}N_t}{2+\tilde{\rho}(T-t)}\right)^2\right] = 2\kappa_0 m_2\int_0^T\frac{\mathrm{d}t}{[2+\tilde{\rho}(T-t)]^2} = \frac{2\kappa_0 m_2}{\rho}\times\frac{\rho}{\tilde{\rho}}\times\left\{\frac{1}{2}-\frac{1}{2+\tilde{\rho}T}\right\},$$

$$\begin{aligned}\mathbb{E}\left[\left(\int_{[0,T]}\exp(-\rho(T-t))\ \mathrm{d}N_t\right)\Delta\tilde{X}_T\right] &= \mathbb{E}\left[\left(\int_{[0,T]}\exp(-\rho(T-t))\ \mathrm{d}N_t\right)\left(\int_{[0,T]}\frac{\mathrm{d}N_t}{2+\tilde{\rho}(T-t)}\right)\right] \\ &= 2\kappa_0 m_2\int_0^T\frac{\exp(-\rho t)}{2+\tilde{\rho}t}\mathrm{d}t = \frac{2\kappa_0 m_2}{\rho}\times\frac{\rho}{\tilde{\rho}}\times L(\tilde{\rho},-\rho,T).\end{aligned} \tag{53}$$

To calculate the other expectations, we need the following identity

$$\int_0^T\exp(-\rho t)\frac{1}{[2+\tilde{\rho}t]^2}\mathrm{d}t = \frac{1}{\tilde{\rho}}\left(\frac{1}{2}-\frac{\exp(-\rho T)}{2+\tilde{\rho}T}-\frac{\rho}{\tilde{\rho}}L(\tilde{\rho},-\rho,T)\right),$$

that comes from a straightforward integration by parts. We then get

$$
\begin{aligned}
&\mathbb{E}\left[\int_{[0,T)}\left(\int_{[0,t]}\exp(-\rho(t-s))\ \mathrm{d}\tilde{X}_s\right)\ \mathrm{d}\tilde{X}_t\right]\\
&=\tilde{\rho}\ \mathbb{E}\left[\int_{[0,T)}\left(\int_{[0,t]}\exp(-\rho(t-s))\ \mathrm{d}\tilde{X}_s\right)\ \left(\int_{[0,t]}\ \frac{\mathrm{d}N_s}{2+\tilde{\rho}(T-s)}\right)\ \mathrm{d}t\ \right]\\
&=-\tilde{\rho}\ \mathbb{E}\left[\int_{[0,T)}\left(\int_{[0,t]}\exp(-\rho(t-s))\ \frac{1+\tilde{\rho}(T-s)}{2+\tilde{\rho}(T-s)}\ \mathrm{d}N_s\right)\ \left(\int_{[0,t]}\ \frac{\mathrm{d}N_s}{2+\tilde{\rho}(T-s)}\right)\ \mathrm{d}t\right]\\
&\quad+\ \tilde{\rho}^2\ \mathbb{E}\left[\int_{[0,T)}\left(\int_{[0,t]}\exp(-\rho(t-s))\ \left(\int_{[0,s]}\ \frac{\mathrm{d}N_r}{2+\tilde{\rho}(T-r)}\right)\ \mathrm{d}s\right)\ \left(\int_{[0,t]}\ \frac{\mathrm{d}N_s}{2+\tilde{\rho}(T-s)}\right)\ \mathrm{d}t\right]\\
&=2\kappa_0 m_2\tilde{\rho}\ \int_0^T\left(\int_0^t\exp(-\rho(t-s))\left\{\int_0^s\ \frac{\tilde{\rho}\mathrm{d}r}{[2+\tilde{\rho}(T-r)]^2}-\frac{1+\tilde{\rho}(T-s)}{[2+\tilde{\rho}(T-s)]^2}\right\}\ \mathrm{d}s\right)\ \mathrm{d}t\\
&=\frac{2\kappa_0 m_2}{\rho}\tilde{\rho}\int_0^T[1-\exp(-\rho(T-s))]\left\{\frac{1}{[2+\tilde{\rho}(T-s)]^2}-\frac{1}{2+\tilde{\rho}T}\right\}ds\\
&=\frac{2\kappa_0 m_2}{\rho}\ \left\{\frac{\rho}{\tilde{\rho}}\times L(\tilde{\rho},-\rho,T)\ +\ \left(\frac{\tilde{\rho}}{\rho}-1\right)\times\frac{1-\exp(-\rho T)}{2+\tilde{\rho}T}\ -\ \frac{\tilde{\rho}T}{2+\tilde{\rho}T}\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_{[0,T]}\exp(-\rho(T-t))\ \mathrm{d}\tilde{X}_t\right)\Delta\tilde{X}_T\right]\\
&=\tilde{\rho}\ \mathbb{E}\left[\left(\int_{[0,T]}\exp(-\rho(T-t))\left(\int_{[0,t]}\ \frac{\mathrm{d}N_s}{2+\tilde{\rho}(T-s)}\right)\mathrm{d}t\right)\left(\int_{[0,T]}\ \frac{\mathrm{d}N_t}{2+\tilde{\rho}(T-t)}\right)\right]\\
&\quad-\ \mathbb{E}\left[\left(\int_{[0,T]}\exp(-\rho(T-t))\ \frac{1+\tilde{\rho}(T-t)}{2+\tilde{\rho}(T-t)}\ \mathrm{d}N_t\right)\left(\int_{[0,T]}\ \frac{\mathrm{d}N_t}{2+\tilde{\rho}(T-t)}\right)\right]\\
&=2\kappa_0 m_2\ \tilde{\rho}\ \int_0^T\exp(-\rho(T-t))\ \left(\int_0^t\ \frac{\mathrm{d}s}{[2+\tilde{\rho}(T-s)]^2}\right)\mathrm{d}t-\ 2\kappa_0 m_2\ \int_0^T\exp(-\rho t)\ \frac{1+\tilde{\rho}t}{[2+\tilde{\rho}t]^2}\ \mathrm{d}t\\
&=2\kappa_0 m_2\ \int_0^T\exp(-\rho t)\ \left\{\frac{1}{[2+\tilde{\rho}t]^2}-\frac{1}{2+\tilde{\rho}T}\right\}\mathrm{d}t\\
&=2\kappa_0 m_2\ \left\{\frac{1}{\tilde{\rho}}\left(\frac{1}{2}-\frac{\exp(-\rho T)}{2+\tilde{\rho}T}\right)\ -\ \frac{\rho}{\tilde{\rho}^2}\times L(\tilde{\rho},-\rho,T)\ -\ \frac{1}{\rho}\times\frac{1-\exp(-\rho T)}{2+\tilde{\rho}T}\right\}\\
&=\frac{2\kappa_0 m_2}{\rho}\ \left\{\frac{\rho}{2\tilde{\rho}}\ +\ \left(1-\frac{\rho}{\tilde{\rho}}\right)\times\frac{\exp(-\rho T)}{2+\tilde{\rho}T}\ -\ \frac{1}{2+\tilde{\rho}T}\ -\ \left(\frac{\rho}{\tilde{\rho}}\right)^2\times L(\tilde{\rho},-\rho,T)\right\}.
\end{aligned}
$$

Gathering all the terms, we eventually get $\mathbb{E}[C(\tilde{X})]=\frac{2\kappa_0 m_2}{\rho}\times f(\tilde{\rho}/\rho)$.

We now differentiate the function f of Proposition 4.2. By differentiating under the integral symbol and integrating by parts, we can check that

$$
\frac{\mathrm{d}}{\mathrm{d}r}L(\rho r,-\rho,T)=\frac{1}{r}-2\frac{\exp(-\rho T)}{r(2+r\rho T)}-\frac{2}{r^2}L(\rho r,-\rho,T).
$$

We write

$$
\begin{aligned}
f'(r)=&\frac{\rho T}{2+r\rho T}+2L(\rho r,-\rho,T)\left[\frac{(r-1)^2}{r^3}-\frac{r-1}{r^2}\right]-\frac{(r-1)^2}{r^2}\times\frac{\mathrm{d}}{\mathrm{d}r}L(\rho r,-\rho,T)-\left(\frac{r-1}{r}+\frac{r-1}{r^2}\right)\frac{\exp(-\rho T)}{2+r\rho T}\\
&+\left(r-1-\frac{r-1}{r}\right)\frac{\rho T\exp(-\rho T)}{[2+r\rho T]^2}+\left(1+\frac{1}{r}-\frac{r-1}{r^2}\right)\frac{1}{2+r\rho T}-\left(r-1+\frac{r-1}{r}\right)\frac{\rho T}{[2+r\rho T]^2}-\frac{1}{r}+\frac{r-1}{r^2},
\end{aligned}
$$

and then

$$
\begin{aligned}
f'(r) &= \frac{\exp(-\rho T)(r-1)}{r^4(2+r\rho T)^2}A(r), \text{ with} \\
A(r) &= -2\exp(\rho T)(2+r\rho T)^2 L(\rho r, -\rho, T) \\
&\quad + r\Big[4(\exp(\rho T)-1) + [2-2\rho T+\exp(\rho T)(4\rho T-2)]r + [\exp(\rho T)(1+(1-\rho T)^2)-2]r^2\Big].
\end{aligned} \tag{54}
$$

To check the monotonicity property of f, it is sufficient to check that A is nonnegative for $r>0$. To do so, we use that

$$
\begin{aligned}
L(\rho r, -\rho, T) &= \frac{\rho r}{2}\int_0^T \exp(-\rho s)ds - \frac{(\rho r)^2}{2}\int_0^T \frac{s\exp(-\rho s)}{2+\rho r s}ds \\
&= \frac{r}{2}(1-\exp(-\rho T)) - \frac{(\rho r)^2}{2(2+\rho r T)}\int_0^T s\exp(-\rho s)ds - \frac{(\rho r)^3}{2(2+\rho r T)}\int_0^T s(T-s)\frac{\exp(-\rho s)}{2+\rho r s}ds \\
&\le \frac{r}{2}(1-\exp(-\rho T)) - \frac{r^2}{2(2+\rho r T)}[1-(1+\rho T)\exp(-\rho T)] - \frac{r^3}{2(2+\rho r T)^2}[(2+\rho T)\exp(-\rho T)+\rho T-2],
\end{aligned}
$$

since $\frac{-1}{2+\rho r s} \le \frac{-1}{2+\rho r T}$ for $s\in[0,T]$. This gives $-\frac{2}{r}\exp(\rho T)(2+r\rho T)^2 L(\rho r,-\rho,T)$

$$
\ge -(\exp(\rho T)-1)[4+4\rho T r+(\rho T)^2 r^2] + (\exp(\rho T)-(1+\rho T))r(2+\rho r T) + r^2[2+\rho T+\exp(\rho T)(\rho T-2)],
$$

which precisely gives

$$
\frac{A(r)}{r} \ge 0.
$$

Let us now study the limits of f as $r\to 0^+$ and $r\to+\infty$. We have the following expansion $L(\rho r,-\rho,T) \underset{r\to 0+}{=} \frac{r}{2}[1-\exp(-\rho T)] - \frac{r^2}{4}[1-(1+\rho T)\exp(-\rho T)] + O(r^3)$ that gives

$$
f(r) \underset{r\to 0^+}{\longrightarrow} -\frac{\rho T}{4} + \frac{1-\exp(-\rho T)}{4} =: f_0(\rho T). \tag{55}
$$

When $r\to+\infty$, we use that $\mathcal{E}\left(-\frac{2}{r}\right) \underset{r\to+\infty}{=} \gamma + \ln\left(\frac{2}{r}\right) + o(1)$ where γ is the Euler constant to get $\ln\left(1+\frac{\rho r T}{2}\right) - \left(1-\frac{1}{r}\right)^2 L(\rho r,-\rho,T) = \ln(\rho T)+\gamma-\mathcal{E}(-\rho T)+o(1)$, and thus

$$
f(r) \underset{r\to+\infty}{\longrightarrow} -\frac{\rho T}{2} + \ln(\rho T) - \mathcal{E}(-\rho T) + \gamma + \frac{1-\exp(-\rho T)}{\rho T} - 1 =: f_\infty(\rho T). \tag{56}
$$

We now study the limits f_0 and f_∞ as functions of ρT. We have $f_0(0^+)=0$, $f_0'(y) = -\frac{1-\exp(-y)}{4} < 0$ for $y>0$, thus f_0 is negative and decreasing. Since

$$
\frac{\mathrm{d}}{\mathrm{d}y}[\mathcal{E}(-y)] = \frac{\exp(-y)}{y} \quad , \quad \mathcal{E}(-y) = \gamma + \ln(y) - y + \frac{y^2}{4} + \underset{y\to 0^+}{O}(y^3),
$$

we get $y^2 f_\infty'(y) = \exp(-y) - 1 + y - y^2/2 < 0$ for $y>0$, and $f_\infty(0^+)=0$. Therefore, f_∞ is negative and decreasing as well, and we have the results of Proposition 4.2.

C.3 Transaction costs (proof of Lemma 4.1)

We first show (16). We recall that X^θ is the optimal strategy when $N = N^\theta$. We then define $\tilde{D}$ as the solution of $\mathrm{d}\tilde{D}_t = -\rho\tilde{D}_t\mathrm{d}t + \frac{1-\nu}{q}\mathrm{d}N_t^\theta + \frac{1-\epsilon}{q}\mathrm{d}X_t$ with $\tilde{D}_0 = D_0 = 0$ We then have

$$D_t = \tilde{D}_t + \frac{(1-\nu)(1-\theta)}{q}\int_0^t \exp(-\rho(t-s))\mathrm{d}N_s.$$

We set $\tilde{X} = \frac{1-\epsilon}{(1-\nu)\theta}X^\theta$. From Remark (2.4), we get

$$\begin{aligned}\mathbb{E}[C(X^\theta)] =& \frac{(1-\nu)^2}{(1-\epsilon)q}\theta(1-\theta)\mathbb{E}\left[\int_{[0,T)}\int_0^t \exp(-\rho(t-s))\mathrm{d}N_s\mathrm{d}\tilde{X}_t - \int_0^T \exp(-\rho(T-s))dN_s\tilde{X}_T\right] \\ &- \frac{(1-\nu)^2}{(1-\epsilon)q}\times 2\kappa_0(\theta^2 m_2)\left[\frac{T}{2} - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right].\end{aligned}$$

By using (52) and (53) with $\tilde{\rho} = \rho$, we get that the expectation above is equal to

$$2\kappa_0 m_2\left\{\frac{2}{\rho}\ln\left(1+\frac{\tilde{\rho}T}{2}\right) - T\right\},$$

which gives (16).

Proof of Lemma 4.1. We recall that X^* is the optimal strategy given by Theorem 4.1 when $d_0 = x_0 = \delta_0 = 0$. We will show the following bounds:

$$2\kappa_0 m_1\left[T - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right] \le \frac{1-\epsilon}{1-\nu}\times||X^*||$$

$$\frac{1-\epsilon}{1-\nu}\times||X^*|| \le 2\kappa_0 m_1\left[T - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right] + \sqrt{\frac{2\kappa_0 m_2(2+\rho T)}{\rho}}\times\left\{\frac{\pi}{2} - \arccos\left(\sqrt{\frac{\rho T}{2+\rho T}}\right)\right\} - \sqrt{\frac{\kappa_0 m_2 T}{2+\rho T}} \tag{57}$$

We have

$$\begin{aligned}\frac{1-\epsilon}{1-\nu}\times||X^*|| &= 2\,\mathbb{E}\left[\int_0^T \frac{1+\rho(T-t)}{2+\rho(T-t)}\,\mathrm{d}N_t^+\right] + \rho\,\mathbb{E}\left[\int_0^T\left|\int_0^t \frac{\mathrm{d}N_s}{2+\rho(T-s)}\right|\mathrm{d}t\right] + \mathbb{E}\left[\left|\int_0^T \frac{\mathrm{d}N_t}{2+\rho(T-t)}\right|\right] \\ &= 2\kappa_0 m_1\int_0^T \frac{1+\rho t}{2+\rho t}\,\mathrm{d}t + \rho\int_0^T h(t)\,\mathrm{d}t + h(T) \\ &= 2\kappa_0 m_1\left[T - \frac{1}{\rho}\ln\left(1+\frac{\rho T}{2}\right)\right] + \rho\int_0^T h(t)\,\mathrm{d}t + h(T),\end{aligned}$$

where we set $h(t) = \mathbb{E}\left[\left|\int_0^t \frac{\mathrm{d}N_s}{2+\rho(T-s)}\right|\right]$, for $t\in[0,T]$. By Jensen's inequality, we have

$$h(t) \le \sqrt{\mathbb{E}\left[\left(\int_0^t \frac{\mathrm{d}N_s}{2+\rho(T-s)}\right)^2\right]} = \sqrt{2\kappa_0 m_2}\times\sqrt{\int_{T-t}^T \frac{\mathrm{d}s}{[2+\rho s]^2}} = \sqrt{\frac{2\kappa_0 m_2}{\rho(2+\rho T)}}\times\sqrt{\frac{\rho t}{2+\rho(T-t)}}.$$

This yields to (57) by using that $\int_0^y \sqrt{\frac{x}{1-x}}\,\mathrm{d}x = \frac{\pi}{2} - \arccos(\sqrt{y}) - \sqrt{y(1-y)}$ for $y\in[0,1)$ and

$$\rho\int_0^T h(t)\,\mathrm{d}t \le \rho\sqrt{\frac{2\kappa_0 m_2}{\rho(2+\rho T)}}\times\int_0^T\sqrt{\frac{\rho t}{2+\rho T-\rho t}}\,\mathrm{d}t = \sqrt{\frac{2\kappa_0 m_2(2+\rho T)}{\rho}}\times\int_0^{\rho T/(2+\rho T)}\sqrt{\frac{x}{1-x}}\,\mathrm{d}x.$$

D Proofs for low-frequency asymptotics

D.1 Proof of Proposition 4.4

From $D_t^+ = D_0^+ \exp(-\rho t) + \frac{1-\nu}{q}\int_0^t \exp(-\rho(t-s))\,\mathrm{d}N_s^+$ and Campbell's formula, see e.g. formula (3.8) in Kingman [26], we get for $u > 0$,

$$\begin{aligned}\mathbb{E}\left[e^{-uD_t^+}\right] &= e^{-uD_0^+\exp(-\rho t)}\exp\left(-\int_0^t\int_{\mathbb{R}}\left[1-\exp\left(-\frac{1-\nu}{q}uy\exp(-\rho(t-s))\right)\right]\mu(\mathrm{d}y)\,\kappa_0\,\mathrm{d}s\right)\\ &= e^{-uD_0^+\exp(-\rho t)}\exp\left(-\kappa_0\int_0^t\left[1-\psi_\mu\left(\frac{1-\nu}{q}u\exp(-\rho s)\right)\right]\mathrm{d}s\right),\end{aligned}$$

by Fubini's theorem, where $\psi_\mu(w) = \int_0^\infty \exp(-wy)\mu(\mathrm{d}y)$ is the Laplace transform of the law of the jump amplitudes. Now, we observe that $1-\psi_\mu\left(\frac{1-\nu}{q}u\exp(-\rho s)\right) \underset{s\to+\infty}{=} \frac{1-\nu}{q}um_1\exp(-\rho s) + o(e^{-\rho s})$ is integrable, which gives

$$\lim_{t\to\infty}\left[e^{-uD_t^+}\right] = \exp\left(-\kappa_0\int_0^\infty\left[1-\psi_\mu\left(\frac{1-\nu}{q}u\exp(-\rho t)\right)\right]\mathrm{d}t\right).$$

The change of variables $\theta = \frac{1-\nu}{q}u\exp(-\rho t)$ leads to (17), and the integral is well defined since $1-\psi_\mu(\theta) = m_1\theta + \underset{\theta\to0^+}{o}(\theta)$. The right hand side of (17) is clearly continuous at zero, which gives the convergence in law by Lévy's continuity theorem.

D.2 Proof of Lemma 5.1

To calculate $\mathbb{E}[D_t^2]$, we first have to determine first $\mathbb{E}[\delta_t^2]$ and $\mathbb{E}[D_t\delta_t]$. We have $\mathrm{d}\delta_t^2 = -2\beta\delta_t^2dt + 2\alpha\mathrm{d}J_t + \alpha^2(\mathrm{d}J_t)^2$ and thus $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\delta_t^2] = -2\beta\mathbb{E}[\delta_t^2] + 2\alpha\mathbb{E}[\delta_t] + \alpha^2\mathbb{E}[\Sigma_t]$, which gives

$$\mathbb{E}[\delta_t^2] = \delta_0^2\exp(-2\beta t) + \int_0^t(2\alpha\mathbb{E}[\delta_s] + \alpha^2\mathbb{E}[\Sigma_s])\exp(2\beta(s-t))ds.$$

Similarly, we have $\mathrm{d}(\delta_tD_t) = -(\rho+\beta)\delta_tD_tdt + \frac{1-\nu}{q}\mathrm{d}N_t\delta_t + \alpha D_t\mathrm{d}J_t + \frac{1-\nu}{q}\alpha|\mathrm{d}N_t|$, thus

$$\mathbb{E}[\delta_tD_t] = \delta_0D_0\exp(-(\rho+\beta-\alpha)t) + \int_0^t\left[\frac{1-\nu}{q}m_1\mathbb{E}[\delta_s^2] + \frac{1-\nu}{q}\alpha\mathbb{E}[\Sigma_s]\right]\exp((\beta+\rho-\alpha)(s-t))ds.$$

Last, we have $\mathrm{d}D_t^2 = -2\rho D_t^2dt + 2\frac{1-\nu}{q}D_t\mathrm{d}N_t + \left(\frac{1-\nu}{q}\right)^2(\mathrm{d}N_t)^2$, and

$$\mathbb{E}[D_t^2] = D_0^2\exp(-2\rho t) + \int_0^t\left(2\frac{1-\nu}{q}m_1\mathbb{E}[D_s\delta_s] + \left(\frac{1-\nu}{q}\right)^2m_2\mathbb{E}[\Sigma_s]\right)\exp(2\rho(s-t))ds.$$

When $\alpha + 2\iota_c < \beta$, we know that $\mathbb{E}[\delta_t]$ and $\mathbb{E}[\Sigma_t]$ converge when $t\to+\infty$. It is clear from the above formulas that $E[\delta_t^2]$, $\mathbb{E}[\delta_tD_t]$ and then $\mathbb{E}[D_t^2]$ also converge when $t\to+\infty$.

E Numerical tests

In this appendix, we test the results of Theorems 4.1, A.1 and Proposition 5.1 by comparing the closed formulas that we obtained for the expected cost of the optimal strategy to the corresponding values given by Monte Carlo simulations, for several sets of parameters. This validates both the formulas and our implementation of the optimal strategy.

Model	Poisson		Hawkes $\alpha = \beta$		MIHM	
ν	0	0.1	0	0.1	0	0.1
β	0	0	40	55	50	50
κ_∞	0	0	1	1	0.2	2
ι_c	0	0	0.2	0	0.03	0.3
Expected cost:						
Monte Carlo	-1185 ± 5	-1001 ± 5	-1228 ± 39	-20505 ± 2015	146 ± 11	140 ± 10
Closed formula	-1184	-1001	-1224	-21220	143	143

Table 1: First set of Monte Carlo tests with: $q = 100, T = 1, \rho = 50, S_0 = 0, \kappa_0^+ = 25, \kappa_0^- = 30, m_1 = 50, \epsilon = 0.1, X_0 = -500$ for the three models and $\mu = \text{Exp}(1/m_1), D_0 = 0.1$ except for MIHM. For the Monte Carlo simulations, we take $n = 40000$ paths and a discretization step $\Delta t = 0.0001$. The confidence intervals given in the table are the asymptotic Gaussian intervals of level 95%.

Model	Poisson		Hawkes $\alpha = \beta$		MIHM	
ν	0	0.3	0	0.3	0	0.3
β	0	0	1	0.6	0.8	0.8
κ_∞	0	0	0.01	0.005	0.005	0.2
ι_c	0	0	0	0.08	0.002	0.06
Expected cost:						
Monte Carlo	-1983 ± 11	-2292 ± 24	-885 ± 64	-3324 ± 147	1213 ± 21	1209 ± 20
Closed formula	-1988	-2284	-896	-3215	1221	1221

Table 2: Second set of Monte Carlo tests with: $q = 80, T = 10, \rho = 0.8, S_0 = 0, \kappa_0^+ = 1.5, \kappa_0^- = 1.1, m_1 = 85, \epsilon = 0.4, X_0 = 700$ for the three models and $\mu(\{k \times q\}) = (1 - q/m_1)^{k-1} q/m_1$ for $k \in \mathbb{N}\backslash\{0\}$, $D_0 = 0.1$ except for MIHM. For the Monte Carlo simulations, we take $n = 40000$ paths and a discretization step $\Delta t = 0.001$. The confidence intervals given in the table are the asymptotic Gaussian intervals of level 95%.

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