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Chapter 1
Justification of the Bending-Gradient theory through asymptotic expansions

Arthur Lebée*, Karam Sab

Abstract In a recent work, a new plate theory for thick plates was suggested where the static unknowns are those of the Kirchhoff-Love theory, to which six components are added representing the gradient of the bending moment [1]. This theory, called the Bending-Gradient theory, is the extension to multilayered plates of the Reissner-Mindlin theory which appears as a special case when the plate is homogeneous. This theory was derived following the ideas from Reissner [2] without assuming a homogeneous plate. However, it is also possible to give a justification through asymptotic expansions. In the present paper, the latter are applied one order higher than the leading order to a laminated plate following monoclinic symmetry. Using variational arguments, it is possible to derive the Bending-Gradient theory. This could explain the convergence when the thickness is small of the Bending-Gradient theory to the exact solution illustrated in [3]. However, the question of the edge-effects and boundary conditions remains open.

1.1 Introduction

The classical theory of plates, known also as Kirchhoff-Love plate theory is based on the assumption that the normal to the mid-plane of the plate remains normal after transformation. This theory is also the first order of the asymptotic expansion with respect to the thickness [4]. Thus, it presents a good theoretical justification and was soundly extended to the case of periodic plates [5, 6]. It enables to have a first-order estimate of the macroscopic deflection as well as local stress fields. In most applications the first-order deflection is accurate enough. However, this theory does not capture the local effect of shear forces on the microstructure because shear forces are one higher-order derivative of the bending moment in equilibrium equations ($Q_\alpha = M_{\alpha\beta,\beta}$).

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Because shear forces are part of the macroscopic equilibrium of the plate, their effect is also of great interest for engineers when designing structures. However, modeling properly the action of shear forces is still a controversial issue. Reissner [2] suggested a model for homogeneous plates based on a parabolic distribution of transverse shear stress through the thickness (Reissner-Mindlin theory). This model performs well for homogeneous plates and gives more natural boundary conditions than those of Kirchhoff-Love theory. Thus, it is appreciated by engineers and broadly used in applied mechanics. However, the direct extension of this model to laminated plates raised many difficulties.

Two main path were followed for deriving models suitable for laminated plates: axiomatic approaches and asymptotic approaches.

In asymptotic approaches, a plate model is derived directly from the full 3D formulation of the problem, assuming the thickness of the plate goes to zero. In these approaches, the asymptotic expansion method plays a central role. As already mentioned, the leading order leads to Kirchhoff-Love plate theory [4, 5, 6]. Hence one needs to seek higher orders for bringing out the effect of shear forces. However, in the cases of laminated plates, this procedure does not lead to Reissner-Mindlin plate theory [7, 8].

In axiomatic approaches, 3D fields are assumed *a priori* and a plate theory is derived using integration through the thickness and variational tools. The reader can refer to the following reviews [9, 10, 11, 12]. Most suggestions leading to Reissner-Mindlin-like theories show discontinuous transverse shear stress through the thickness or are limited to some geometric configurations (orthotropy or cylindrical bending for instance). In this field, these limitations even led to the suggestion of "layerwise" models which give more satisfying results but are much more numerically intense than Reissner-Mindlin theory [12, 13]. Finally, let us point out that the theory suggested by Reissner [2] is usually considered as an axiomatic approach since the parabolic transverse shear stress distribution of the stress was derived without asymptotic arguments. Consequently, some work took literally this distribution and applied it to laminated plates. Like in many unsuccessful axiomatic approaches this led to discontinuous displacement fields and raised an unjustified suspicion over the original work.

Revisiting the approach from Reissner [2] directly with laminated plates, Lebée and Sab [1, 3] showed that the transverse shear static variables which come out when the plate is heterogeneous are not shear forces Q_α but the full gradient of the bending moment $R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma}$. Using conventional variational tools, they derived a new plate theory – called Bending-Gradient theory – which is actually turned into Reissner-Mindlin theory when the plate is homogeneous. This new plate theory is seen by the authors as an extension of Reissner's theory to heterogeneous plates which preserves most of its simplicity. It was applied to the cylindrical bending of carbon fibers laminated plates and compared to exact solutions in [3]. Very good agreement for the transverse shear distribution as well as in-plane displacement was pointed out and convergence with the slenderness was observed.

Originally designed for laminated plates, the Bending-Gradient theory was also extended to in-plane periodic plates using averaging considerations such as Hill-

Mandel principle and successfully applied to sandwich panels [14, 15] as well as space frames [16].

Because the derivation of the Bending-Gradient theory followed the ideas from Reissner [2], one can argue that it is basically an axiomatic approach. However, it is the intention of the present paper to demonstrate that there is a close link between the derivation of the Bending-Gradient theory and the asymptotic expansion method. Since the Bending-Gradient is turned into the Reissner-Mindlin theory when the plate is homogeneous, this link will be also demonstrated for the original work from Reissner [2].

In order to derive the Bending-Gradient theory through asymptotic expansions, we first set in Section 1.2 the 3D problem, its symmetries and the asymptotic expansions framework. For the sake of simplicity we choose the constitutive material and the loadings of the plate such that the bending moment is fully uncoupled with the membrane stress. Then in Section 1.3 we perform the standard resolution of the auxiliary problems and conclude that bringing out transverse shear effects through this approach is not satisfying. Then in Section 1.4 we derive the Bending-Gradient theory using variational considerations.

1.2 The asymptotic expansion framework

In this section, the asymptotic expansion framework is set in the special case of a laminated plate. This procedure was established by Sanchez-Palencia [17] for linear dynamics of 3D continuum. It starts with the definition of the 3D problem of the laminated plate which is under consideration. Then this problem is scaled in order to separate the in-plane and the out-of-plane variables and we assume that the fields follow an expansion depending on a small parameter: the inverse of the plate slenderness. Finally, the equations are gathered for each order of this parameter.

1.2.1 Notations

Vectors and higher-order tensors, up to sixth order, are used in the following. When using short notation, several underlining styles are used: vectors are straight underlined, $\underline{u}$. Second order tensors are underlined with a tilde: $\underset{\sim}{M}$ and $\underset{\sim}{K}$. Third order tensors are underlined with a parenthesis: $\underset{\frown}{R}$ and $\underset{\frown}{\Gamma}$. Fourth order tensors are are doubly underlined with a tilde: $\underset{\approx}{D}$ and $\underset{\approx}{s}$. Sixth order tensors are doubly underlined with a parenthesis: $\underset{\frown\frown}{h}$ and $\underset{\frown\frown}{I}$. The full notation with indices is also used. Then we follow Einstein's notation on repeated indices. Furthermore, Greek indices $\alpha,\beta,\delta,\gamma = 1,2$ denotes in-plane dimensions and Latin indices $i,j,k,l = 1,2,3$, all three dimensions.

The transpose operation ${}^{T}\bullet$ is applied to any order tensors as follows: $({}^{T}a)_{\alpha\beta\ldots\psi\omega} = a_{\omega\psi\ldots\beta\alpha}$. Three contraction products are defined, the usual dot product ($\underline{a}\cdot\underline{b} = a_i b_i$), the double contraction product ($\underset{\sim}{a} : \underset{\sim}{b} = a_{ij}b_{ji}$) and a triple contraction product

($\underset{\sim}{a} : \underset{\sim}{a} = a_{\alpha\beta\gamma}a_{\gamma\beta\alpha}$). The derivation operator $\underline{\nabla}$ is also formally represented as a vector: $\underset{\sim}{a} \cdot \underline{\nabla} = a_{ij}\nabla_j = a_{ij,j}$ is the divergence and $\underset{\sim}{a} \otimes \underline{\nabla} = a_{ij}\nabla_k = a_{ij,k}$ is the gradient. Here $\otimes$ is the dyadic product.

1.2.2 The 3D problem

The laminated plate occupies a domain $\Omega^t = \omega^L \times]-\frac{t}{2}, \frac{t}{2}[$ where ω^L is the middle surface of the plate (its typical size is L) and t its thickness. The boundary of the plate, $\partial\Omega^t$, is decomposed into three parts:

$$\begin{aligned} &\partial\Omega^t = \partial\Omega_{\text{lat}} \cup \partial\Omega_3^+ \cup \partial\Omega_3^- \\ \text{with} \quad &\partial\Omega_{\text{lat}} = \partial\omega^L \times]-\frac{t}{2}, \frac{t}{2}[\quad \text{and} \quad \partial\Omega_3^\pm = \omega^L \times \left\{\pm\frac{t}{2}\right\}. \end{aligned} \tag{1.1}$$

The plate is fully clamped on its lateral boundary, $\partial\Omega_{\text{lat}}$, and is submitted to the same distributed and purely transverse force $\underline{f} = f_3(x_1,x_2)\underline{e}_3$ both on its upper and lower boundaries $\partial\Omega_3^+$ and $\partial\Omega_3^-$.

The fourth-order stiffness tensor $\underset{\approx}{C}^t(x_3)$ characterizing the elastic properties of the constituent material at every point $\underline{x} = (x_1,x_2,x_3)$ of Ω^t is introduced. We assume the following monoclinic symmetry: $C^t_{3\alpha\beta\gamma} = C^t_{\alpha 333} = 0$. In addition, $\underset{\approx}{C}^t$ does not depend on (x_1,x_2) and is an even function of x_3 to ensure full uncoupling between in-plane and out-of-plane problems. Thus, the constitutive equation writes as:

$$\underset{\sim}{\sigma}^t(\underline{x}) = \underset{\approx}{C}^t(x_3) : \underset{\sim}{\varepsilon}^t(\underline{x}) \tag{1.2}$$

where $\underset{\sim}{\sigma}^t = \left(\sigma^t_{ij}(\underline{x})\right)$ is the stress tensor and $\underset{\sim}{\varepsilon}^t = \left(\varepsilon^t_{ij}(\underline{x})\right)$ is the strain tensor at point $\underline{x}$. The tensor $\underset{\approx}{C}^t$ follows the classical symmetries of linear elasticity and is positive definite.

The full 3D elastic problem, $\mathscr{P}^{3D}$, is to find in Ω^t a displacement vector field $\underline{u}^t$, a strain tensor field $\underset{\sim}{\varepsilon}^t$ and a stress tensor field $\underset{\sim}{\sigma}^t$ such that the static conditions ($SC^{3D,t}$):

$$SC^{3D,t} : \begin{cases} \underset{\sim}{\sigma}^t \cdot \underline{\nabla} = 0 \text{ on } \Omega^t & (1.3a) \\ \underset{\sim}{\sigma}^t \cdot (\pm\underline{e}_3) = \underline{f} \text{ on } \partial\Omega_3^\pm, & (1.3b) \end{cases}$$

the kinematic conditions ($KC^{3D,t}$):

$$KC^{3D,t} : \begin{cases} \underset{\sim}{\varepsilon}^t = \underline{u}^t \otimes^s \underline{\nabla} \text{ on } \Omega^t & (1.4a) \\ \underline{u}^t = 0 \text{ on } \partial\Omega_{\text{lat}} & (1.4b) \end{cases}$$

and the constitutive law (1.2) are satisfied. Here, $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is the orthonormal basis associated with coordinates (x_1,x_2,x_3) and $\bullet \otimes^s \underline{\nabla}$ denotes the symmetric part of the gradient operator.

1.2.2.1 Variational formulation of the 3D problem

The strain and stress energy density w^{3D} and w^{*3D} are respectively given by:

$$w^{3D}\left(\underset{\sim}{\varepsilon}\right) = \frac{1}{2}\underset{\sim}{\varepsilon} : \underset{\approx}{C}^{t} : \underset{\sim}{\varepsilon}, \qquad w^{*3D}\left(\underset{\sim}{\sigma}\right) = \frac{1}{2}\underset{\sim}{\sigma} : \underset{\approx}{S}^{t} : \underset{\sim}{\sigma} \tag{1.5}$$

They are related by the following Legendre-Fenchel transform:

$$w^{*3D}\left(\underset{\sim}{\sigma}\right) = \sup_{\underset{\sim}{\varepsilon}} \left\{ \underset{\sim}{\sigma} : \underset{\sim}{\varepsilon} - w^{3D}\left(\underset{\sim}{\varepsilon}\right) \right\} \tag{1.6}$$

The kinematic variational approach states that the strain solution $\underset{\sim}{\varepsilon}^{t}$ of $\mathscr{P}^{3D}$ is the one that minimizes P^{3D} among all kinematically compatible strain fields:

$$P^{3D}\left(\underset{\sim}{\varepsilon}^{t}\right) = \min_{\underset{\sim}{\varepsilon} \in KC^{3D,t}} \left\{ P^{3D}\left(\underset{\sim}{\varepsilon}\right) \right\} \tag{1.7}$$

where P^{3D} is the potential energy given by:

$$P^{3D}\left(\underset{\sim}{\varepsilon}\right) = \int_{\Omega^{t}} w^{3D}\left(\underset{\sim}{\varepsilon}\right) d\Omega^{t} - \int_{\omega^{L}} \left(\underset{\sim}{f} \cdot \underset{\sim}{u}^{+} + \underset{\sim}{f} \cdot \underset{\sim}{u}^{-} \right) d\omega^{L} \tag{1.8}$$

and $\underset{\sim}{u}^{\pm} = \underset{\sim}{u}(x_1, x_2, \pm t/2)$ are the 3D displacement fields on the upper and lower faces of the plate.

The static variational approach states that the stress solution $\underset{\sim}{\sigma}^{t}$ of $\mathscr{P}^{3D}$ is the one that minimizes P^{*3D} among all statically compatible stress fields:

$$P^{*3D}\left(\underset{\sim}{\sigma}^{t}\right) = \min_{\underset{\sim}{\sigma} \in SC^{3D,t}} \left\{ P^{*3D}\left(\underset{\sim}{\sigma}\right) \right\} \tag{1.9}$$

where P^{*3D} is the complementary potential energy given by:

$$P^{*3D}\left(\underset{\sim}{\sigma}\right) = \int_{\Omega^{t}} w^{*3D}\left(\underset{\sim}{\sigma}\right) d\Omega^{t} \tag{1.10}$$

1.2.2.2 Effect of symmetries

For the sake of simplicity, we chose the 3D plate problem such that only flexural part is involved and no membranal part.

The 3D problem $\mathscr{P}^{3D}$ is skew-symmetric through a planar symmetry with respect to the mid-plane of the plate (known also as "mirror symmetry" in laminates engineering) because $\underset{\approx}{C}^{t}$ is an even function of only x_3. This means that, when applying the transformation $x_3 \to -x_3$ the problem remains unchanged but the boundary condition (1.3b) changes its sign. Consequently the in-plane displacement $u^{t}_{\alpha}(x_1, x_2, x_3)$ is an odd function of x_3 and the out-of-plane displacement $u^{t}_{3}(x_1, x_2, x_3)$ is an even function of x_3. Similarly, the in-plane stress $\sigma^{t}_{\alpha\beta}(x_1, x_2, x_3)$ and transverse

compression $\sigma^t_{33}(x_1,x_2,x_3)$ are odd functions of x_3 and the transverse shear stress $\sigma^t_{\alpha 3}(x_1,x_2,x_3)$ is an even function of x_3.

In terms of resultants and averaged displacements, the integration through the thickness of u^t_α and $\sigma^t_{\alpha\beta}$ vanish and then the plate problem will be purely flexural. Of course, this result affects also the asymptotic expansion procedure and enables many simplifications.

1.2.3 Scaling

Once the 3D problem is set, we scale it for clearly separating the in-plane variables (which are related to macroscopic problems) and the out-of-plane variable (which is related to microscopic perturbations). Hence, L is the typical scale of the in-plane variables (*e.g.* the span and also the wavelength of the loadings). We introduce the following change of variable $Y_\alpha = L^{-1}x_\alpha$ for the in-plane variable where $Y_\alpha \in \omega$. The domain ω is the scaled mid-plane of the plate. Moreover we define $z = t^{-1}x_3$ for the out-of-plane variable, $z \in]-\frac{1}{2},\frac{1}{2}[$. Consequently, we define the small parameter as: $\eta = t/L$.

Based on this change of variables, the fourth-order elasticity tensor can be rewritten as:

$$\underset{\approx}{C}^t(x_3) = \underset{\approx}{\mathbb{C}}\left(t^{-1}x_3\right) = \underset{\approx}{\mathbb{C}}(z) \tag{1.11}$$

where $\underset{\approx}{\mathbb{C}}$ is a function of z. In the following, double-stroke fonts denote fields which are only function of the local variable z (*i.e.* localization fields).

The distributed forces are classically scaled the following way (see: [4, 5, 18]):

$$\underline{f}(x_1,x_2) = \eta^2 \frac{F_3(Y_1,Y_2)}{2}\underline{e}_3 \tag{1.12}$$

Similarly, in the following, fields with capital letters are only function of (Y_1,Y_2) (*i.e.* macroscopic fields).

Furthermore, from the fields of the 3D problem $(\underline{u}^t,\underset{\approx}{\varepsilon}^t,\underset{\approx}{\sigma}^t)$ we define the non-dimensional fields $(\underline{u},\underset{\approx}{\varepsilon},\underset{\approx}{\sigma})$ as follows:

$$\begin{cases} \underline{u}^t\ (x_1,x_2,x_3) = L\underline{u}\ (x_1/L,x_2/L,x_3/t) = L\underline{u}\ (Y_1,Y_2,z) \\ \underset{\approx}{\varepsilon}^t\ (x_1,x_2,x_3) = \ \underset{\approx}{\varepsilon}\ (x_1/L,x_2/L,x_3/t) = \ \underset{\approx}{\varepsilon}\ (Y_1,Y_2,z) \\ \underset{\approx}{\sigma}^t\ (x_1,x_2,x_3) = \ \underset{\approx}{\sigma}\ (x_1/L,x_2/L,x_3/t) = \ \underset{\approx}{\sigma}\ (Y_1,Y_2,z) \end{cases} \tag{1.13}$$

The derivation rule for those functions is:

$$\begin{aligned} \underline{\nabla} &= \left(\frac{d}{dx_1},\frac{d}{dx_2},\frac{d}{dx_3}\right) \\ &= L^{-1}\left(\frac{\partial}{\partial Y_1},\frac{\partial}{\partial Y_2},0\right) + t^{-1}\left(0,0,\frac{\partial}{\partial z}\right) = L^{-1}\underline{\nabla}_Y + t^{-1}\underline{\nabla}_z. \end{aligned} \tag{1.14}$$

We will also use the variational formulation of the 3D problem. Hence we provide here the scaled variational formulation. The set of statically compatible fields can be rewritten as:

$$SC^{3D}: \begin{cases} \underset{\sim}{\sigma} \cdot \underline{\nabla}_{(Y,z)} = 0 \text{ on } \Omega, & (1.15a) \\ \underset{\sim}{\sigma} \cdot (\pm e_3) = \dfrac{\eta^2}{2} F_3 e_3 \quad \text{on } \partial\Omega_3^{\pm}, & (1.15b) \end{cases}$$

where $\underline{\nabla}_{(Y,z)} = \underline{\nabla}_Y + \frac{1}{\eta}\,\underline{\nabla}_z$. The kinematically compatible fields becomes (KC^{3D}):

$$KC^{3D}: \begin{cases} \underset{\sim}{\varepsilon} = \underline{u} \otimes^s \underline{\nabla}_{(Y,z)} \text{ on } \Omega, & (1.16a) \\ \underline{u} = 0 \quad \text{on } \partial\omega \times]-\dfrac{1}{2}, +\dfrac{1}{2}[& (1.16b) \end{cases}$$

Then the potential energy rewrittes as:

$$P^{3D}\left(\underset{\sim}{\varepsilon}\right) = tL^2 \int_\omega \left(\left\langle w^{3D}\left(\underset{\sim}{\varepsilon}\right) \right\rangle - \eta \frac{u_3^+ + u_3^-}{2} F_3 \right) d\omega \tag{1.17}$$

where $\langle \bullet \rangle$ is the integration through the thickness: $\langle \bullet \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} \bullet \, dz$. The complementary energy becomes also:

$$P^{*3D}\left(\underset{\sim}{\sigma}\right) = tL^2 \int_\omega \left\langle w^{*3D}\left(\underset{\sim}{\sigma}\right) \right\rangle d\omega \tag{1.18}$$

Now, $\underset{\approx}{\mathbb{C}}$, ω and F_3 being fixed, the homogenization problem is to find a consistent approximation of the solution of the 3D problem $\mathscr{P}^{3D}$ (1.2-1.3-1.4) assuming η is small.

1.2.4 Expansion

The asymptotic expansion method [17, 19] will be used to provide a formal justification of the Bending-Gradient theory. The starting point of the method is to assume that the solution to (1.2-1.3-1.4) can be written as a series in power of η in the following form:

$$\begin{cases} \underline{u} = \eta^{-1}\underline{u}^{-1} + \eta^0 \underline{u}^0 + \eta^1 \underline{u}^1 + \cdots \\ \underset{\sim}{\varepsilon} = \qquad\qquad \eta^0 \underset{\sim}{\varepsilon}^0 + \eta^1 \underset{\sim}{\varepsilon}^1 + \cdots \\ \underset{\sim}{\sigma} = \qquad\qquad \eta^0 \underset{\sim}{\sigma}^0 + \eta^1 \underset{\sim}{\sigma}^1 + \cdots \end{cases} \tag{1.19}$$

where $p = -1, 0, 1, 2...$ and $\underline{u}^p, \underset{\sim}{\varepsilon}^p$ and $\underset{\sim}{\sigma}^p$ are functions of (Y_1, Y_2, z) which follow the same parity as the 3D solution (Section 1.2.2.2). The series are started from the order η^0 for $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$, and from the order η^{-1} for $\underline{u}$. Then, the expansion (1.19)

– taking into account the change of variable – must be inserted in the equations (1.2-1.3-1.4) and all the terms of the same order η^p must be identified.

1.2.4.1 Statically admissible fields

The 3D equilibrium equation, $\underset{\sim}{\sigma}^t \cdot \underline{\nabla} = 0$ on Ω^t, becomes:

$$L\left(\underset{\sim}{\sigma}^t \cdot \underline{\nabla}\right) = \eta^{-1}\left(\underset{\sim}{\sigma}^0 \cdot \underline{\nabla}_z\right) + \eta^0\left(\underset{\sim}{\sigma}^0 \cdot \underline{\nabla}_Y + \underset{\sim}{\sigma}^1 \cdot \underline{\nabla}_z\right) + \cdots = 0.$$

Identifying all the terms of the above series to be zero, it is found :

$$\sigma^0_{i3,3} = 0 \tag{1.20}$$

for the order η^{-1},

$$\sigma^p_{i\alpha,\alpha} + \sigma^{p+1}_{i3,3} = 0 \tag{1.21}$$

for the order η^p with $p \geq 0$. The derivation $\bullet_{,i}$ is performed without ambiguity with respect to (Y_1, Y_2, z). The boundary condition, $\underset{\sim}{\sigma}^t \cdot \underline{e}_3 = \pm \underline{f}$ on $\partial\Omega_3^{\pm}$, gives the following equations:

$$\sigma^p_{i3}\left(Y_1, Y_2, \pm\frac{1}{2}\right) = 0 \tag{1.22}$$

for the order $p \geq 0$ and $p \neq 2$. When $p = 2$ we have:

$$\sigma^2_{\alpha 3}\left(Y_1, Y_2, \pm\frac{1}{2}\right) = 0 \quad \text{and} \quad \sigma^2_{33}\left(Y_1, Y_2, \pm\frac{1}{2}\right) = \pm\frac{1}{2}F_3\left(Y_1, Y_2\right) \tag{1.23}$$

1.2.4.2 Kinematically compatible fields

From the compatibility equation, it is found that the strain rate field can be written as:

$$\underset{\sim}{\varepsilon}^t = L\underline{u} \otimes^s \underline{\nabla} = \eta^{-2}\underset{\sim}{\varepsilon}^{-2} + \eta^{-1}\underset{\sim}{\varepsilon}^{-1} + \eta^0\underset{\sim}{\varepsilon}^0 + \cdots \tag{1.24}$$

with:

$$\varepsilon^{-2}_{\alpha\beta} = 0, \quad \varepsilon^{-2}_{\alpha 3} = \frac{1}{2}u^{-1}_{\alpha,3} \quad \text{and} \quad \varepsilon^{-2}_{33} = u^{-1}_{3,3} \tag{1.25}$$

and for all $p \geq -1$:

$$\varepsilon^p_{\alpha\beta} = \frac{1}{2}\left(u^p_{\alpha,\beta} + u^p_{\beta,\alpha}\right), \quad \varepsilon^p_{\alpha 3} = \frac{1}{2}\left(u^{p+1}_{\alpha,3} + u^p_{3,\alpha}\right) \quad \text{and} \quad \varepsilon^p_{33} = u^{p+1}_{3,3} \tag{1.26}$$

The boundary condition over $\partial\Omega_{\text{lat}}$ leads to:

$$\forall p \geq -1 \quad \text{and} \quad \forall\left(Y_1, Y_2\right) \in \partial\omega, \quad \underline{u}^p = 0. \tag{1.27}$$

1.3 Explicit or cascade resolution

Now that the asymptotic expansion framework is set, we detail the explicit resolution which is classically performed (see [5, 7] for instance). Basically it starts with the derivation of low order displacements which do not gerenate local strain but are related to purely macroscopic displacement fields. Then the zeroth-order equations are gathered. They enable the definition of the first auxiliary problem and the construction of the well-known Kirchhoff-Love macroscopic plate model. Then the first-order is solved the same way. Of course it would be possible to carry on the process any order higher.

1.3.1 Low order displacement fields

The assumption (1.19) provides the following equations:

$$\underset{\sim}{\varepsilon}^{-2} = 0, \tag{1.28}$$

and

$$\underset{\sim}{\varepsilon}^{-1} = 0, \tag{1.29}$$

From (1.28) it is deduced that $\underline{u}^{-1}$ is a rigid-body velocity field in z. Moreover, the in-plane displacement has zero average because of the symmetry condition (Section 1.2.2.2). Hence:

$$\underline{u}^{-1} = U_3^{-1}(Y_1, Y_2)\,\underline{e}_3. \tag{1.30}$$

Using (1.29) and the boundary conditions (1.27) it can be found that $\underline{u}^0$ has the following form:

$$\underline{u}^0 = \begin{pmatrix} -zU_{3,1}^{-1} \\ -zU_{3,2}^{-1} \\ U_3^0 \end{pmatrix}, \tag{1.31}$$

with the boundary conditions:

$$U_3^{-1} = U_{3,\alpha}^{-1} n_\alpha = U_3^0 = 0 \;\; \forall (Y_1, Y_2) \in \partial\omega. \tag{1.32}$$

where $\underline{n}$ is the outer normal to $\partial\omega$. Note that, since U_3^{-1} is null over $\partial\omega$, its tangential derivative will be also null over $\partial\omega$, hence only the normal gradient $U_{3,\alpha}^{-1} n_\alpha$ is required to be explicitly set to zero in this boundary condition.

1.3.2 Zeroth-order plate model (Kirchhoff-Love)

1.3.2.1 Zeroth-order auxiliary Problem

Gathering equilibrium equation for order -1, compatibility equation, boundary conditions and constitutive equations of order 0 we get the zeroth-order auxiliary problem for $z \in [-\frac{1}{2}, \frac{1}{2}]$:

$$\begin{cases} \sigma^0_{i3,3} = 0 & (1.33a) \\ \sigma^0_{ij} = \mathbb{C}_{ijkl}\varepsilon^0_{kl} & (1.33b) \\ \varepsilon^0_{\alpha\beta} = zK^{-1}_{\alpha\beta}, \quad \varepsilon^0_{\alpha 3} = \frac{1}{2}\left(u^1_{\alpha,3} + U^0_{3,\alpha}\right) \quad \text{and} \quad \varepsilon^0_{33} = u^1_{3,3} & (1.33c) \\ \sigma^0_{i3}\left(z = \pm\frac{1}{2}\right) = 0 & (1.33d) \end{cases}$$

where we define the lowest-order curvature as:

$$K^{-1}_{\alpha\beta} = -U^{-1}_{3,\alpha\beta} \tag{1.34}$$

Solving this problem does not raise difficulty. Using short notation, the displacement field writes as:

$$\underset{\sim}{u}^1 = \underset{\approx}{\mathfrak{u}}^K : \underset{\sim}{K}^{-1} - zU^0_3 \otimes \underset{\sim}{\nabla}_Y + U^1_3 \underset{\sim}{e}_3 \tag{1.35}$$

More precisely, the localization related to the curvature is:

$$\mathfrak{u}^K_{3\alpha\beta} = -\left[\int_{-\frac{1}{2}}^{z} y \frac{\mathbb{C}_{33\alpha\beta}}{\mathbb{C}_{3333}} dy\right]^* \quad \text{and} \quad \mathfrak{u}^K_{\alpha\beta\gamma} = 0 \tag{1.36}$$

where $[\bullet]^*$ denotes the averaged-out distribution: $[\bullet]^* = \bullet - \langle\bullet\rangle$. Finally U^1_3 is an integration constant which will load the next auxiliary problem. (There are no in-plane integration constants because of the symmetry already invoked with lower orders). The stress localization writes as:

$$\underset{\sim}{\sigma}^0 = \underset{\approx}{\mathfrak{s}}^K : \underset{\sim}{K}^{-1} \tag{1.37}$$

where the fourth-order stress localization tensor is:

$$\mathfrak{s}^K_{\alpha\beta\gamma\delta} = z\mathbb{C}^\sigma_{\alpha\beta\gamma\delta} \quad \text{and} \quad \mathfrak{s}^K_{i3\gamma\delta} = 0 \tag{1.38}$$

and $\mathbb{C}^\sigma_{\alpha\beta\gamma\delta} = \mathbb{C}_{\alpha\beta\gamma\delta} - \mathbb{C}_{\alpha\beta 33}\mathbb{C}_{33\gamma\delta}/\mathbb{C}_{3333}$ denotes the plane-stress elasticity tensor. Hence the plate is under pure plane-*stress* at this order.

The strain is derived using the local constitutive equation:

$$\varepsilon^0_{\alpha\beta} = zK^{-1}_{\alpha\beta}, \quad \varepsilon^0_{\alpha 3} = 0 \quad \text{and} \quad \varepsilon^0_{33} = -\frac{z\mathbb{C}_{33\alpha\beta}}{\mathbb{C}_{3333}} K^{-1}_{\alpha\beta} \tag{1.39}$$

This confirms Kirchhoff's assumption regarding the in-plane strain. The reader's attention is drawn to the fact that the out-of-plane strain is not zero, as already mentioned in several works [4, 5] in contrast to the original assumption from Kirchhoff.

Hence, for given macroscopic fields U_3^{-1} and its derivatives, the microscopic strain and stress are fully determined at this order. However, we also need U_3^0 and U_3^1 for estimating the displacement field. This requires solving higher-order problems.

At this order, there remains to derive the macroscopic problem which enables the derivation of U_3^{-1}.

1.3.2.2 Macroscopic problem

The Macroscopic equilibrium is derived integrating the first two components of $z \times (1.21)$ for $p = 0$. This gives after integrating by parts over z:

$$M^0_{\alpha\beta,\beta} - Q^1_\alpha = 0 \tag{1.40}$$

where the zeroth-order bending moment is defined as:

$$M^0_{\alpha\beta}(Y_1, Y_2) = \left\langle z\sigma^0_{\alpha\beta} \right\rangle, \tag{1.41}$$

and the first-order shear force is:

$$Q^1_\alpha(Y_1, Y_2) = \left\langle \sigma^1_{3\alpha} \right\rangle. \tag{1.42}$$

It can be easily established that $\left\langle \sigma^0_{3\alpha} \right\rangle = 0$ because of the equilibrium (1.20) and the boundary condition (1.22). Therefore, averaging the third component of Equation (1.21) for $p = 0$ leads to a trivial equation. Using the second order boundary condition (1.23, $p = 2$) and averaging the third component of the first-order equilibrium equation (1.21), for $p = 1$ gives:

$$Q^1_{\alpha,\alpha} + F_3 = 0. \tag{1.43}$$

We obtain also the constitutive equation by plugging the local stress derived in Equation (1.37) into the definition of $\underset{\sim}{M}^0$. This leads to the well-known Kirchhoff-Love constitutive equation:

$$\underset{\sim}{M}^0 = \underset{\approx}{D} : \underset{\sim}{K}^{-1} \quad \text{where:} \quad \underset{\approx}{D} = \left\langle z^2 \underset{\approx}{\mathbb{C}}^\sigma \right\rangle \tag{1.44}$$

Gathering the preceeding results leads to the definition of the Kirchhoff-Love plate problem:

$$\begin{cases} \underset{\sim}{M}^0 : \left(\underline{\nabla}_Y \otimes \underline{\nabla}_Y\right) + F_3 = 0, \quad \text{on} \quad \omega & \text{(1.45a)} \\ \underset{\sim}{M}^0 = \underset{\approx}{D} : \underset{\sim}{K}^{-1}, \quad \text{on} \quad \omega & \text{(1.45b)} \\ \underset{\sim}{K}^{-1} = U_3^{-1} \underline{\nabla}_Y \otimes \underline{\nabla}_Y, \quad \text{on} \quad \omega & \text{(1.45c)} \\ U_3^{-1} = 0 \quad \text{and} \quad \left(U_3^{-1} \otimes \underline{\nabla}_Y\right) \cdot \underline{n} = 0 \quad \text{on} \quad \partial\omega & \text{(1.45d)} \end{cases}$$

Finally, solving this macroscopic problem enables the derivation of the macroscopic displacement fields U_3^{-1}. However U_3^0 and U_3^1 remain unknown.

The well-known limitation of Kirchhoff-Love plate model is that it does not incorporate the effect of shear forces. In order to bring out the contribution of transverse shear, we need to go further in the expansion.

1.3.3 First-order plate model

1.3.3.1 First-order auxiliary problem

Gathering equilibrium equation for order 0, compatibility equation, boundary conditions and constitutive equations of order 1 we get the first-order auxiliary problem for $z \in [-\frac{1}{2}, \frac{1}{2}]$:

$$\begin{cases} \sigma^0_{i\alpha,\alpha} + \sigma^1_{i3,3} = 0 & (1.46a) \\ \sigma^1_{ij} = \mathbb{C}_{ijkl}\varepsilon^1_{kl} & (1.46b) \\ \varepsilon^1_{\alpha\beta} = \frac{1}{2}\left(u^1_{\alpha,\beta} + u^1_{\beta,\alpha}\right), \quad \varepsilon^1_{\alpha 3} = \frac{1}{2}\left(u^2_{\alpha,3} + u^1_{3,\alpha}\right) \quad \text{and} \quad \varepsilon^1_{33} = u^2_{3,3} & (1.46c) \\ \sigma^1_{i3}\left(z = \pm\frac{1}{2}\right) = 0 & (1.46d) \end{cases}$$

In this auxiliary problem, the zeroth-order displacement field $\underset{\sim}{u}^1$ (Equation (1.35)) and stress field $\underset{\sim}{\sigma}^0$ (Equation (1.37)) are local fields which depend linearly on $\underset{\sim}{K}^{-1}$, $U^0_{3,\alpha}$ and U^1_3. Hence, the first-order solution $\underset{\sim}{u}^2$ (as well as $\underset{\sim}{\varepsilon}^1$ and $\underset{\sim}{\sigma}^1$) will be a linear superposition of localization fields which depend on the gradient of those macroscopic fields.

The displacement field solution of this problem writes as:

$$\underset{\sim}{u}^2 = \underset{\approx}{\mathbb{u}}^{K\nabla} \vdots \left(\underset{\sim}{K}^{-1} \otimes \underset{\sim}{\nabla}_Y\right) + \underset{\approx}{\mathbb{u}}^K : \underset{\sim}{K}^0 - zU^1_3 \otimes \underset{\sim}{\nabla}_Y + U^2_3 e_3 \tag{1.47}$$

where the displacement localization tensor related to the curvature gradient writes as:

$$\mathbb{u}^{K\nabla}_{\alpha\beta\gamma\delta} = -\left[\int_{-\frac{1}{2}}^{z}\left(4\mathbb{S}_{\alpha 3\eta 3}\int_{-\frac{1}{2}}^{y} v\,\mathbb{C}^{\sigma}_{\eta\beta\gamma\delta}\,dv + \delta_{\alpha\beta}\,\mathbb{u}^K_{3\gamma\delta}\right)dy\right]^* \quad \text{and} \quad \mathbb{u}^{K\nabla}_{3\beta\gamma\delta} = 0 \tag{1.48}$$

The first order stress writes as:

$$\underset{\sim}{\sigma}^1 = \underset{\approx}{\mathbb{s}}^{K\nabla} \vdots \left(\underset{\sim}{K}^{-1} \otimes \underset{\sim}{\nabla}_Y\right) + \underset{\approx}{\mathbb{s}}^K : \underset{\sim}{K}^0 \tag{1.49}$$

where we defined the fifth-order localization tensor as:

$$\mathbb{s}^{K\nabla}_{\alpha\beta\gamma\delta\eta} = 0, \quad \mathbb{s}^{K\nabla}_{\alpha 3\beta\gamma\delta} = -\int_{\frac{1}{2}}^{z} y\,\mathbb{C}^{\sigma}_{\alpha\beta\gamma\delta}\,dy \quad \text{and} \quad \mathbb{s}^{K\nabla}_{33\beta\gamma\delta} = 0 \tag{1.50}$$

Hence, this order involves only transverse shear effects.

1.3.3.2 Higher-order macroscopic problem

Exactly as for the zeroth-order, it is possible to derive the macroscopic equilibrium equation as:

$$M^1_{\alpha\beta,\beta\alpha} = 0 \tag{1.51}$$

which holds also for higher orders ($p \geq 1$). For the constitutive equation, we have again:

$$\underset{\sim}{M}^1 = \left(\left\langle z\sigma^1_{\alpha\beta}\right\rangle\right) = \underset{\approx}{D} : \underset{\sim}{K}^0 \tag{1.52}$$

Finally, U^0_3 is solution of the same Kirchhoff-Love problem as with the zeroth-order case (Equation (1.45)), without external loads. Thus the solution is trivially zero everywhere. This is due to the monoclinic symmetry of the local constitutive equation. It is the analogue of the centro-symmetric assumption in the case of asymptotic expansion of a 3D medium (see [20] for instance). Thus, if we want to capture transverse shear effects following the asymptotic expansion procedure, we have to go one order higher. At this order, the macroscopic problem will not be trivial. However, it will require the derivation of the second gradient of the curvature $\underset{\sim}{K}^{-1}$ and consequently the fourth derivative of the deflection. This raises an issue in terms of physical meaning of this variable as well as of numerical implementation.

In contrast, it is remarkable that transverse shear effects are included in the localization field already at this order. Hence we suggest to stop at this order the asymptotic expansion and switch to variational arguments for deriving the Bending-Gradient theory.

1.3.4 Additional remarks on the asymptotic expansion approach

Before going further in the derivation of the Bending-Gradient theory, let us point out some useful remarks regarding the asymptotic expansion procedure.

In the present paper, we performed the asymptotic expansion up to the very next order after the classical homogenization procedure. However, this formalism has already been studied up to "infinite order" in other elasticity problems (see Smyshlyaev and Cherednichenko [21] for instance) and convergence results were derived [22]. Those works show that the fully reconstructed field $\underset{\sim}{u}$ is actually a double sum: a sum over orders, as expected because of the expansion, but also over degrees of derivative of the macroscopic displacement field. This is also the case in the present plate problem. If we gather all the fields derived in the cascade resolution we get the following:

$$\underset{\sim}{u} = \left(\frac{U_3^{-1}}{\eta} + U_3^0 + \eta U_3^1 + \eta^2 U_3^2 + \dots\right) \underset{\sim}{e}_3 - z\left(U_3^{-1} + \eta U_3^0 + \eta^2 U_3^1 + \dots\right) \otimes \underset{\sim}{\nabla}_Y$$
$$+ \eta \left(\underset{\sim}{\mathfrak{u}}^K : \left(\underset{\sim}{K}^{-1} + \eta \underset{\sim}{K}^0 + \dots\right)\right) + \eta^2 \left(\underset{\approx}{\mathfrak{u}}^{K\nabla} \vdots \left(\underset{\sim}{K}^{-1} \otimes \underset{\sim}{\nabla}_Y + \dots\right)\right) + \dots \tag{1.53}$$

Assuming that this double sum converges, it is legitimate to define:

$$U_3 = \sum_{p=-1}^{\infty} \eta^{p+1} U_3^p \tag{1.54}$$

and rewrite the total displacement field as:

$$\underset{\sim}{u} = \frac{U_3}{\eta} \underset{\sim}{e}_3 - z U_3 \otimes \underset{\sim}{\nabla}_Y + \eta\, \underset{\sim}{\mathfrak{u}}^K : \underset{\sim}{K} + \eta^2\, \underset{\approx}{\mathfrak{u}}^{K\nabla} \vdots \underset{\sim}{K} \otimes \underset{\sim}{\nabla}_Y + \dots \tag{1.55}$$

where $\underset{\sim}{K} = U_3 \underset{\sim}{\nabla}_Y \otimes \underset{\sim}{\nabla}_Y$. This was suggested by Boutin [23] and further justified in [21]. We have also for the stress field:

$$\underset{\sim}{\sigma} = \underset{\approx}{\mathfrak{s}}^K : \underset{\sim}{K} + \eta\, \underset{\approx}{\mathfrak{s}}^{K\nabla} \vdots \underset{\sim}{K} \otimes \underset{\sim}{\nabla}_Y + \dots \tag{1.56}$$

Finally, this reasoning also holds true for the equilibrium equation and we formally get:

$$\underset{\sim}{M} : \left(\underset{\sim}{\nabla}_Y \otimes \underset{\sim}{\nabla}_Y\right) + F_3 = 0 \tag{1.57}$$

where $M_{\alpha\beta} = \left\langle z \sigma_{\alpha\beta} \right\rangle$. Hence, it seems that going higher-order in the asymptotic expansion only involves higher gradients of the displacement inside the constitutive equation. However, as already pointed out in these papers, the problem remains ill-posed as it stands here. Some caution must be taken when considering the constitutive equation as well as the boundary conditions if one wants to derive a mathematically sound problem.

First, in order to derive the constitutive equation it seems straightforward to take directly the elastic energy of the infinite order stress or strain (Equation (1.56)) and to truncate this energy up to a given order *afterward*. However, this will lead to a non-positive quadratic form and makes the higher-order problem unstable. Hence, as pointed out by [21] it is critical to truncate the expansion of the stress or strain *before* taking the related energy to ensure positivity.

Second, whereas the boundary conditions are set at each order in the cascade resolution of the asymptotic expansion (here Equation (1.45d) at each order), in the format presented here, it is not possible to make distinction between orders and then the problem is not well-posed anymore. Here, variational tools will enable the derivation of consistent boundary conditions with the choice of macroscopic degrees of freedom.

1.4 The Bending-Gradient theory

Keeping in mind the difficulties mentioned regarding the asymptotic expansion, the Bending-Gradient theory is derived as follows. First, instead of keeping the first gradient of the curvature as higher-order unknown, we introduce the gradient of the bending moment. This will relax the compatibility condition between $\underset{\sim}{K}$ and $\underset{\sim}{K} \otimes \underline{\nabla}_Y$. After this change of variable, we define the stress localization as the truncation of the infinite order stress. Then we introduce the set of statically compatible macroscopic fields. Finally, using variational arguments, the kinematics as well as the boundary conditions of the plate model are derived. Once the plate model is solved, we are able to reconstruct an approximation of the 3D displacement field.

We select first the bending moment and its gradient instead of the curvature and its gradient for carrying the energy. Hence we define the bending gradient as:

$$\underset{\frown}{R} = \underset{\sim}{M} \otimes \underline{\nabla}_Y \tag{1.58}$$

Using Kirchhoff-Love constitutive equation and the following change of variable,

$$\underset{\frown}{R} = \underset{\approx}{D} : \underset{\sim}{K} \otimes \underline{\nabla}_Y \tag{1.59}$$

it is possible to rewrite the strain and stress localization fields derived with the asymptotic expansion (Section 1.3) only in terms of $\underset{\sim}{M}$ and $\underset{\frown}{R}$:

$$\underset{\sim}{\sigma}^{BG} = \underset{\approx}{\mathbb{S}}^{M} : \underset{\sim}{M} + \eta \, \underset{\approx}{\mathbb{S}}^{R} \vdots \underset{\frown}{R} \tag{1.60}$$

where:

$$\underset{\approx}{\mathbb{S}}^{M} = \underset{\approx}{\mathbb{S}}^{K} : \underset{\approx}{d}, \quad \underset{\frown}{\mathbb{S}}^{R} = \underset{\frown}{\mathbb{S}}^{K\nabla} : \underset{\approx}{d} \quad \text{and} \quad \underset{\approx}{d} = \underset{\approx}{D}^{-1} \tag{1.61}$$

It is easy to check that this stress field satisfies the 3D equilibrium equation (1.15), as well as the $z = \pm 1/2$ face boundary conditions, up to the order η^1. Hence, even if it does not define properly a restriction of SC^{3D}, it remains a good approximation in the sense of the asymptotic expansion.

Now, based on the macroscopic equilibrium equations derived through the asymptotic expansion and the definition of $\underset{\frown}{R}$, we suggest the following set of statically compatible fields for the Bending-Gradient theory:

$$SC^{BG} : \begin{cases} \underset{\frown}{R} = \underset{\sim}{M} \otimes \underline{\nabla}_Y & (1.62a) \\ \left(\underset{\approx}{i} \vdots \underset{\frown}{R} \right) \cdot \underline{\nabla}_Y + F_3 = 0 & (1.62b) \end{cases}$$

where the shear forces were substituted and we used the following relation:

$$\underset{\approx}{i} \vdots \underset{\frown}{R} = \underset{\sim}{M} \cdot \underline{\nabla}_Y \tag{1.63}$$

where $i_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right)$ is the identity for in-plane fourth-order tensors following the symmetries of linear elasticity.

Plugging $\underset{\sim}{\sigma}^{BG}$ into the complementary energy of the full 3D problem leads to the following functional:

$$P^{*BG}\left(\underset{\sim}{M},\underset{\frown}{R}\right) = \int_{\omega} w^{*KL}\left(\underset{\sim}{M}\right) + \eta^2 w^{*BG}\left(\underset{\frown}{R}\right) d\omega \tag{1.64}$$

where the stress elastic energies are defined as:

$$w^{*KL}\left(\underset{\sim}{M}\right) = \frac{1}{2}\underset{\sim}{M} : \underset{\approx}{d} : \underset{\sim}{M} \quad \text{and} \quad w^{*BG}\left(\underset{\frown}{R}\right) = \frac{1}{2}{}^{T}\underset{\frown}{R} \vdots \underset{\widehat{\approx}}{h} \vdots \underset{\frown}{R} \tag{1.65}$$

with:

$$\underset{\widehat{\approx}}{h} = \left\langle {}^{T}\underset{\approx}{\mathbb{s}}^{R} : \underset{\approx}{\mathbb{S}} : \underset{\approx}{\mathbb{s}}^{R} \right\rangle \tag{1.66}$$

This sixth-order tensor is the compliance related to the transverse shear of the plate. It is strictly identical to the one derived in [1]. Let us recall here that it is positive, symmetric, but not definite. More details about $\underset{\widehat{\approx}}{h}$ properties were discussed in [1]

NB: There is no uncoupling in the complementary energy (1.64) between $\underset{\sim}{M}$ and $\underset{\frown}{R}$ because of the monoclinic symmetry of the local constitutive equation. In the auxiliary problems, this symmetry enforces the localization related to $\underset{\sim}{M}$ to be purely in-plane and the one related to $\underset{\frown}{R}$ to be pure transverse shear. Hence the cross terms in the 3D elastic energy vanish.

Now we define the generalized strains as:

$$\underset{\sim}{\chi} = \frac{\partial w^{*KL}}{\partial \underset{\sim}{M}} \quad \text{and} \quad \underset{\frown}{\Gamma} = \frac{\partial w^{*BG}}{\partial \underset{\frown}{R}} \tag{1.67}$$

which leads to the following constitutive equations:

$$\begin{cases} \underset{\sim}{\chi} = \underset{\approx}{d} : \underset{\sim}{M} & (1.68a) \\ \underset{\frown}{\Gamma} = \underset{\widehat{\approx}}{h} \vdots \underset{\frown}{R} & (1.68b) \end{cases}$$

Introducing respectively $\Phi_{\alpha\beta\gamma}$, U_3 as Lagrange multipliers of Equations (1.62a) and (1.62b) and taking the variations with respect to the static variables leads to the following definition for the strains:

$$KC^{BG} : \begin{cases} \underset{\sim}{\chi} = \underset{\frown}{\Phi} \cdot \underset{\sim}{\nabla}_Y & (1.69a) \\ \eta^2 \underset{\frown}{\Gamma} = \underset{\frown}{\Phi} + \underset{\approx}{i} \cdot \underset{\sim}{\nabla}_Y U_3 & (1.69b) \end{cases}$$

where both $\underset{\frown}{\Phi}$ and $\underset{\frown}{\Gamma}$ are third-order tensors which follows the same index symmetry as $\underset{\frown}{R}$. Setting $\eta^2 = 0$ in those definitions leads exactly to Kirchhoff-Love strains. Hence, the Bending-Gradient curvature is slightly different from the one of the asymptotic expansion and Equation (1.69a) rewrites:

$$\underset{\sim}{\chi} = \underset{\sim}{K} + \eta^2 \underset{\frown}{\Gamma} \cdot \underset{\sim}{\nabla}_Y \tag{1.70}$$

Namely it is the sum of the conventional curvature and a small correction term which relaxes this compatibility relation.

Considering the variations of the Lagrangian on the edges leads also to the following clamped boundary conditions:

$$U_3 = 0 \quad \text{and} \quad \underset{\frown}{\Phi} \cdot \underline{n} = 0 \text{ on } \partial\omega \tag{1.71}$$

Finally we have a well-posed plate theory.

Once the exact solution of the macroscopic problem is derived, it is possible to reconstruct the local displacement field. We suggest the following 3D displacement field where U_3, $\underset{\frown}{\Phi}$ are the fields solution of the plate problem:

$$\underline{u}^{BG} = \frac{U_3}{\eta} \underline{e}_3 - zU_3 \otimes \underline{\nabla}_Y + \eta \, \underset{\approx}{\mathfrak{u}}^K : \underset{\approx}{\chi} + \eta^2 \, \underset{\approx}{\mathfrak{u}}^{K\nabla} \vdots \left(\underset{\approx}{\chi} \otimes \underline{\nabla}_Y \right) \tag{1.72}$$

Defining the strain as $\underset{\approx}{\varepsilon}^{BG} = \underset{\approx}{\mathbb{S}} : \underset{\approx}{\sigma}^{BG}$ it is possible to check that:

$$\varepsilon \left(\underline{u}^{BG} \right)_{(Y,z)} - \underset{\approx}{\varepsilon}^{BG} = \eta^2 \left(\left(\underset{\approx}{\delta} \otimes^s \underset{\approx}{\mathfrak{u}}^{K\nabla} \right) :: \left(\underset{\approx}{\chi} \otimes \underline{\nabla}_Y^2 \right) + z \underset{\frown}{\Gamma} \cdot \underline{\nabla}_Y \right) \tag{1.73}$$

which shows that the compatibility equation between the reconstructed displacement field $\underline{u}^{BG}$ and strain localization $\underset{\approx}{\varepsilon}^{BG}$ is satisfied up to the η^2 order.

1.5 Conclusion

Finally, we derived a plate model which enables the full description of local 3D fields ($\underline{u}^{BG}$, $\underset{\approx}{\varepsilon}^{BG}$ and $\underset{\approx}{\sigma}^{BG}$) including the effects of transverse shear. Compared to the classical theory from Reissner [2], we just add four macroscopic variables included into the generalized rotation $\underset{\frown}{\Phi}$ and which are related to transverse shear warping. Contrary to the asymptotic expansions approach or the approach suggested in [21], our theory does not require the derivation of the first or even the second gradient of the curvature. Actually, when looking at the definition of strains in Equation (1.69), only the first derivatives of U_3 and $\underset{\frown}{\Phi}$ are involved. Having low-order interpolation is a serious advantage compared to "strain-gradient-like" approaches given in [7, 21].

Now, let us recall that the derivation of the Bending-Gradient theory through asymptotic expansions was purely formal. The small parameter η was essentially used for discriminating between orders. More precisely, the 3D local fields chosen for the Bending-Gradient theory satisfy the 3D compatibility equation and the 3D equilibrium equation one order higher than the Kirchhoff-Love fields. However, this is not a proof of convergence even if the good results in [3] are clearly encouraging. Especially, it is broadly acknowledged that the boundary have a critical role on that matter when going in higher orders. This question raises already with asymptotic expansions: it was demonstrated that the approximation which is derived in the bulk is not compatible with the actual 3D boundary condition and can only be fulfilled

weakly (see [24, 25] for a clear illustration in the case of beams and also [26]). In the case of the Bending-Gradient theory the boundary conditions are different from the asymptotic expansions and requires further analysis which is out of the scope of this paper.

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