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Aurélien Alfonsi. Strong convergence of some drift implicit Euler scheme. Application to the CIR process.. Statistics and Probability Letters, 2013, 83 (2), pp.602-607. 10.1016/j.spl.2012.10.034 . hal-00709202

**HAL Id: hal-00709202**

**<https://enpc.hal.science/hal-00709202>**

Submitted on 18 Jun 2012

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# Strong convergence of some drift implicit Euler scheme. Application to the CIR process.

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June 18, 2012

## Abstract

We study the convergence of a drift implicit scheme for one-dimensional SDEs that was considered by Alfonsi [1] for the Cox-Ingersoll-Ross (CIR) process. Under general conditions, we obtain a strong convergence of order 1. In the CIR case, Dereich, Neuenkirch and Szpruch [2] have shown recently a strong convergence of order 1/2 for this scheme. Here, we obtain a strong convergence of order 1 under more restrictive assumptions on the CIR parameters.

*Keywords:* Discretization scheme, Cox-Ingersoll-Ross model, Strong error, Lamperti transformation.

*AMS Classification (2010):* 65C30, 60H35

This paper analyses the strong convergence error of a discretization scheme for the Cox-Ingersoll-Ross (CIR) process and complements a recent paper by Dereich, Neuenkirch and Szpruch [2]. The CIR process, which is widely used in financial modelling, follows the SDE:

$$dX_t = (a - kX_t)dt + \sigma\sqrt{X_t}dW_t, \quad X_0 = x. \quad (1)$$

Here,  $W$  denotes a standard Brownian motion,  $a \geq 0$ ,  $k \in \mathbb{R}$ ,  $\sigma > 0$  and  $x \geq 0$ . This SDE has a unique strong solution that is nonnegative. It is even positive when  $\sigma^2 \leq 2a$  and  $x > 0$ , which we assume in this paper. It is well-known that the usual Euler-Maruyama scheme is not defined for (1). Different ad-hoc discretization schemes have thus been proposed in the literature (see references in [2]). Here, we focus on a drift implicit scheme that has been proposed in Alfonsi [1]. We consider a time horizon  $T > 0$  and a regular time grid:

$$t_k = \frac{kT}{n}, \quad 0 \leq k \leq n.$$

By Itô's formula,  $Y_t = \sqrt{X_t}$  satisfies :

$$dY_t = \left( \frac{a - \sigma^2/4}{2Y_t} - \frac{k}{2}Y_t \right) dt + \frac{\sigma}{2}dW_t, \quad Y_0 = \sqrt{x}. \quad (2)$$

We consider the following drift implicit Euler scheme

$$\hat{Y}_0 = \sqrt{x}, \quad \hat{Y}_t = \hat{Y}_{t_k} + \left( \frac{a - \sigma^2/4}{2\hat{Y}_t} - \frac{k}{2}\hat{Y}_t \right) (t - t_k) + \frac{\sigma}{2}(W_t - W_{t_k}), \quad t \in (t_k, t_{k+1}]. \quad (3)$$

The equation (3) is a quadratic equation that has a unique positive equation:

$$\hat{Y}_t = \frac{\hat{Y}_{t_k} + \frac{\sigma}{2}(W_t - W_{t_k}) + \sqrt{\left( \hat{Y}_{t_k} + \frac{\sigma}{2}(W_t - W_{t_k}) \right)^2 + 2 \left( 1 + \frac{k}{2}(t - t_k) \right) \left( a - \frac{\sigma^2}{4} \right) (t - t_k)}}{2 \left( 1 + \frac{k}{2}(t - t_k) \right)},$$

provided that the time-step is small enough ( $T/n \leq 2/\max(-k, 0)$  with the convention  $2/0 = +\infty$ ). Last, we set  $\hat{X}_t = (\hat{Y}_t)^2$ ,  $t \in (t_k, t_{k+1}]$ . It is shown in [1] that this scheme has uniformly bounded moments. We recall now the main result of Dereich, Neuenkirch and Szpruch [2] that gives a strong error convergence of order 1/2.

**Theorem 1.** *Let  $x > 0$ ,  $2a > \sigma^2$  and  $T > 0$ . Then, for all  $p \in [1, \frac{2a}{\sigma^2})$ , there is a constant  $K_p > 0$  such that for any  $n \geq \frac{T}{2} \max(-k, 0)$ ,*

$$\left( \mathbb{E} \left[ \max_{t \in [0, T]} |\hat{X}_t - X_t|^p \right] \right)^{1/p} \leq K_p \sqrt{\frac{T}{n}}.$$

Let us remark that, contrary to [2], we do not consider a linear interpolation between  $t_k$  and  $t_{k+1}$  here for  $\hat{X}_t$ . This removes the logarithm term of Theorem 1.1 in [2].

The strong convergence rate of  $\hat{X}$  is studied numerically in Alfonsi ([1], Figure 2). This numerical study shows that the strong convergence rate depends on the parameters  $\sigma^2$  and  $a$ . When  $\sigma^2/a$  is small enough, a strong convergence of order 1 is observed. The scope of the paper is to prove the following result.

**Theorem 2.** *Let  $x > 0$ ,  $a > \sigma^2$  and  $T > 0$ . Then, for all  $p \in [1, \frac{4a}{3\sigma^2})$ , there is a constant  $K_p > 0$  such that for any  $n \geq \frac{T}{2} \max(-k, 0)$ ,*

$$\left( \mathbb{E} \left[ \max_{t \in [0, T]} |\hat{X}_t - X_t|^p \right] \right)^{1/p} \leq K_p \frac{T}{n}.$$

Thus, we get a strong convergence of order 1 under more restrictive conditions on  $\sigma^2/a$ . Both theorems are complementary and are compatible with the numerical study of [1], which indicates that the strong convergence order downgrades as long as  $\sigma^2/a$  increases.

The paper is structured as follows. We first prove that  $\left( \mathbb{E} \left[ \max_{t \in [0, T]} |\hat{Y}_t - Y_t|^p \right] \right)^{1/p} \leq K_p \frac{T}{n}$  under a general framework for  $Y$  and  $\hat{Y}$  that extends (2) and (3). Then, we deduce Theorem 2 from this result. Also, we construct an analogous drift implicit scheme for general one-dimensional diffusion, and get a strong convergence of order one under suitable assumptions on the coefficients. This scheme has the advantage to be naturally defined in the diffusion domain like  $\mathbb{R}_+^*$  for the CIR case.

## A general framework for $Y$ and $\hat{Y}$

Let  $c \in [-\infty, +\infty)$ ,  $I = (c, +\infty)$  and  $d \in I$ . We consider in this section the following SDE defined on  $I = (c, +\infty)$ :

$$dY_t = f(Y_t)dt + \gamma dW_t, \quad t \geq 0, Y_0 = y \in I, \quad (4)$$

with  $\gamma > 0$ . We make the following monotonicity assumption on the drift coefficient  $f$ :

$$f : I \rightarrow \mathbb{R} \text{ is } \mathcal{C}^2, \text{ such that } \exists \kappa \in \mathbb{R}, \forall y, y' \in I, \quad y \leq y', \quad f(y') - f(y) \leq \kappa(y' - y). \quad (5)$$

Besides, we assume

$$v(x) = \int_d^x \int_d^y \exp\left(-\frac{2}{\gamma^2} \int_z^y f(\xi) d\xi\right) dz dy \text{ satisfies } \lim_{x \rightarrow c+} v(x) = -\infty. \quad (6)$$

The Feller's test (see e.g. Theorem 5.29 p. 348 in [5]) ensures that  $Y$  never reaches  $c$  nor  $+\infty$  by (5), and the SDE (4) admits a unique strong solution on  $I$ .

Let us now define the drift implicit scheme. Let us first observe that for  $h > 0$  such that  $\kappa h < 1$ ,  $y \mapsto y - hf(y)$  is a bijection from  $I$  to  $\mathbb{R}$ . Indeed, it is continuous and we have from (4):

$$y \leq y', \quad y' - y - h(f(y') - f(y)) \geq (1 - \kappa h)(y' - y).$$

This shows the claim for  $c = -\infty$ . For  $c > -\infty$ , we first remark that  $\lim_{c+} f$  exists from (5), and is necessarily equal to  $+\infty$  from (6). Thus, for  $n$  such that  $\kappa T/n < 1$ , the following drift implicit Euler scheme is well defined

$$\hat{Y}_0 = y, \quad \hat{Y}_t = \hat{Y}_{t_k} + f(\hat{Y}_t)(t - t_k) + \gamma(W_t - W_{t_k}), \quad t \in (t_k, t_{k+1}], \quad 0 \leq k \leq n-1, \quad (7)$$

and satisfies  $\hat{Y}_t \in I$ , for any  $t \in [0, T]$ . From a computational point of view, let us remark here that in cases where  $\hat{Y}_{t_{k+1}}$  cannot be solved explicitly like in the CIR case,  $\hat{Y}_{t_{k+1}}$  can still be quickly computed from  $\hat{Y}_{t_k}$  and  $W_{t_{k+1}} - W_{t_k}$  thanks to the monotonicity of  $y \mapsto y - (T/n)f(y)$  by using for example a dichotomic search.

The drift implicit Euler scheme (also known as backward Euler scheme) has been studied by Higham, Mao and Stuart [4] for SDEs on  $\mathbb{R}^d$  with a Lipschitz condition on the diffusion coefficient and a monotonicity condition on the drift coefficient that extends (5). They show a strong convergence of order 1/2 in this general setting.

**Proposition 3.** *Let  $p \geq 1$  and  $n > 2\kappa T$ . Let us assume that*

$$\mathbb{E} \left[ \left( \int_0^T |f'(Y_u)f(Y_u) + \frac{\gamma^2}{2} f''(Y_u)| du \right)^p \right] < \infty \text{ and } \mathbb{E} \left[ \left( \int_0^T (f'(Y_u))^2 du \right)^{p/2} \right] < \infty. \quad (8)$$

*Then, there is a constant  $K_p > 0$  such that:*

$$\left( \mathbb{E} \left[ \max_{t \in [0, T]} |\hat{Y}_t - Y_t|^p \right] \right)^{1/p} \leq K_p \frac{T}{n}.$$

Before proving this result, let us recall that the same result holds for the usual (drift explicit) Euler-Maruyama scheme when  $I = \mathbb{R}$  (i.e.  $c = -\infty$ ), under some regularity assumption on  $f$ . Said differently, the Euler-Maruyama scheme ( $\bar{Y}_{t_{k+1}} = \bar{Y}_{t_k} + f(\bar{Y}_{t_k})T/n + \gamma(W_{t_{k+1}} - W_{t_k})$ ) coincides with the Milstein scheme when the diffusion coefficient is constant, and its order of strong convergence is thus equal to one. The main advantage of the drift implicit scheme is that it is well defined when  $c > -\infty$  while the Euler-Maruyama is not, since the Brownian increment may lead outside  $I$ .

*Proof.* We may assume without loss of generality that  $\kappa \geq 0$ . For  $t \in [0, T]$ , we set  $e_t = \hat{Y}_t - Y_t$ . From (5), there is  $\beta_t \leq \kappa$ , such that  $f(\hat{Y}_t) - f(Y_t) = \beta_t e_t$ . For  $0 \leq k \leq n-1$ , we have

$$e_{t_{k+1}} = e_{t_k} + [f(\hat{Y}_{t_{k+1}}) - f(Y_{t_{k+1}})]\frac{T}{n} + \int_{t_k}^{t_{k+1}} f(Y_{t_{k+1}}) - f(Y_s)ds,$$

and then, by using Itô's formula:

$$\left(1 - \beta_{t_{k+1}}\frac{T}{n}\right) e_{t_{k+1}} = e_{t_k} + \int_{t_k}^{t_{k+1}} (u - t_k)[f'(Y_u)f(Y_u) + \frac{\gamma^2}{2}f''(Y_u)]du + \gamma \int_{t_k}^{t_{k+1}} (u - t_k)f'(Y_u)dW_u. \quad (9)$$

For  $u \in [0, T]$ , we denote by  $\eta(u)$  the integer such that  $t_{\eta(u)} \leq u < t_{\eta(u)+1}$ . We set  $\Pi_0 = 1$ ,  $\Pi_k = \prod_{l=1}^k (1 - \beta_{t_l}\frac{T}{n})$ ,  $\tilde{e}_k = \Pi_k e_{t_k}$ ,  $\tilde{\Pi}_k = \Pi_k / (1 - \kappa T/n)^k$  and

$$M_t = \int_0^t (1 - \kappa T/n)^{\eta(u)} (u - t_{\eta(u)}) \gamma f'(Y_u) dW_u.$$

Let us remark that  $\Pi_k > 0$ ,  $\tilde{\Pi}_k \geq 1$  and  $\tilde{\Pi}_k$  is nondecreasing with respect to  $k$ . By multiplying equation (9) by  $\Pi_k$ , we get

$$\tilde{e}_{k+1} = \tilde{e}_k + \Pi_k \left( \int_{t_k}^{t_{k+1}} (u - t_k)[f'(Y_u)f(Y_u) + \frac{\gamma^2}{2}f''(Y_u)]du + \int_{t_k}^{t_{k+1}} (u - t_k)\gamma f'(Y_u)dW_u \right).$$

Then, we obtain  $\tilde{e}_k = \int_0^{t_k} \Pi_{\eta(u)} (u - t_{\eta(u)}) [f'(Y_u)f(Y_u) + \frac{\gamma^2}{2}f''(Y_u)]du + \sum_{l=0}^{k-1} \tilde{\Pi}_l (M_{t_{l+1}} - M_{t_l})$  by summing over  $k$  and finally get

$$e_{t_k} = \int_0^{t_k} \frac{\Pi_{\eta(u)}}{\Pi_k} (u - t_{\eta(u)}) [f'(Y_u)f(Y_u) + \frac{\gamma^2}{2}f''(Y_u)]du + \frac{1}{\Pi_k} \sum_{l=0}^{k-1} \tilde{\Pi}_l (M_{t_{l+1}} - M_{t_l}). \quad (10)$$

Since  $\frac{1}{1-x} \leq \exp(2x)$  for  $x \in [0, 1/2]$ , we have

$$0 \leq l \leq k \leq n, \quad 0 < \frac{\Pi_l}{\Pi_k} = \frac{1}{(1 - \kappa T/n)^{k-l}} \frac{\tilde{\Pi}_l}{\tilde{\Pi}_k} \leq \exp\left(2(k-l)\kappa\frac{T}{n}\right) \leq \exp(2\kappa T).$$

On the other hand, an Abel transformation gives  $\sum_{l=0}^{k-1} \tilde{\Pi}_l (M_{t_{l+1}} - M_{t_l}) = \tilde{\Pi}_{k-1} M_{t_k} + \sum_{l=1}^{k-1} (\tilde{\Pi}_{l-1} - \tilde{\Pi}_l) M_{t_l}$  and thus

$$\left| \sum_{l=0}^{k-1} \tilde{\Pi}_l (M_{t_{l+1}} - M_{t_l}) \right| \leq \tilde{\Pi}_{k-1} |M_{t_k}| + \sum_{l=1}^{k-1} (\tilde{\Pi}_l - \tilde{\Pi}_{l-1}) |M_{t_l}| \leq 2\tilde{\Pi}_k \max_{1 \leq l \leq k} |M_{t_l}|,$$

since  $\tilde{\Pi}_k$  is nondecreasing. From (10) and  $\frac{\tilde{\Pi}_k}{\Pi_k} = \frac{1}{(1-\kappa T/n)^k} \leq \exp(2\kappa T)$ , we get

$$|e_{t_k}| \leq \exp(2\kappa T) \left( \frac{T}{n} \int_0^{t_k} |f'(Y_u)f(Y_u) + \frac{\gamma^2}{2} f''(Y_u)| du + 2 \max_{0 \leq l \leq k} |M_{t_l}| \right).$$

Since the right hand side is nondecreasing with respect to  $k$ , we can replace the left hand side by  $\max_{0 \leq l \leq k} |e_{t_l}|$ . Burkholder-Davis-Gundy inequality gives that

$$\mathbb{E} \left[ \max_{0 \leq l \leq n} |M_{t_l}|^p \right] \leq C_p \gamma^p (T/n)^p \mathbb{E} \left[ \left( \int_0^T (f'(Y_u))^2 du \right)^{p/2} \right],$$

since  $0 \leq (1 - \kappa T/n)^{\eta(u)} \leq 1$ . Thus, there is a positive constant  $K$  depending on  $\kappa$ ,  $T$  and  $p$  such that:

$$\begin{aligned} \mathbb{E} \left[ \max_{0 \leq l \leq n} |e_{t_l}|^p \right] &\leq K \left( \frac{T}{n} \right)^p \left( \mathbb{E} \left[ \left( \int_0^T |f'(Y_u)f(Y_u) + \frac{\gamma^2}{2} f''(Y_u)| du \right)^p \right] \right. \\ &\quad \left. + \gamma^p \mathbb{E} \left[ \left( \int_0^T (f'(Y_u))^2 du \right)^{p/2} \right] \right) \end{aligned} \quad (11)$$

It remains to show the analogous upper bound for  $\mathbb{E}[\max_{t \in [0, T]} |e_t|^p]$ . Similarly to (9), we have for  $t \in [t_k, t_{k+1}]$ :

$$(1 - \beta_t(t - t_k)) e_t = e_{t_k} + \int_{t_k}^t (u - t_k) [f'(Y_u)f(Y_u) + \frac{\gamma^2}{2} f''(Y_u)] du + \gamma \int_{t_k}^t (u - t_k) f'(Y_u) dW_u.$$

Since  $(1 - \beta_t(t - t_k)) \geq 1/2$ , we get:

$$\begin{aligned} \max_{t \in [t_k, t_{k+1}]} |e_t| &\leq 2 \left( |e_{t_k}| + \frac{T}{n} \int_{t_k}^{t_{k+1}} |f'(Y_u)f(Y_u) + \frac{\gamma^2}{2} f''(Y_u)| du \right. \\ &\quad \left. + \gamma \max_{t \in [t_k, t_{k+1}]} \left| \int_{t_k}^t (u - t_k) f'(Y_u) dW_u \right| \right), \end{aligned}$$

and thus

$$\begin{aligned} \max_{t \in [0, T]} |e_t|^p &\leq 2^p 3^{p-1} \left( \max_{0 \leq k \leq n} |e_{t_k}|^p + \left( \frac{T}{n} \right)^p \left( \int_0^T |f'(Y_u)f(Y_u) + \frac{\gamma^2}{2} f''(Y_u)| du \right)^p \right. \\ &\quad \left. + \gamma^p \max_{0 \leq s \leq t \leq T} \left| \int_s^t (u - t_{\eta(u)}) f'(Y_u) dW_u \right|^p \right). \end{aligned}$$

Since  $\left| \int_s^t (u - t_{\eta(u)}) f'(Y_u) dW_u \right|^p \leq 2^p \left( \left| \int_0^t (u - t_{\eta(u)}) f'(Y_u) dW_u \right|^p + \left| \int_0^s (u - t_{\eta(u)}) f'(Y_u) dW_u \right|^p \right)$ , we conclude by using once again Burkholder-Davis-Gundy inequality, (11) and (8).  $\square$

## Application to the CIR process

For the CIR case, we have  $c = 0$  (i.e.  $I = \mathbb{R}_+^*$ ),  $f(y) = \frac{a-\sigma^2/4}{2y} - \frac{k}{2}y$  and  $\gamma = \sigma/2$ . When  $2a \geq \sigma^2$ , we can check that both (5) and (6) are satisfied. By Jensen inequality, (8) holds if we have

$$\int_0^T \mathbb{E}[|f'(Y_u)f(Y_u)|^p + |f''(Y_u)|^p + |f'(Y_u)|^{2\vee p}]du < \infty. \quad (12)$$

The moments of the CIR process can be uniformly bounded on  $[0, T]$  under the following condition (see [2] equation (7)):

$$\sup_{t \in [0, T]} \mathbb{E}[X_t^q] < \infty \text{ for } q > -\frac{2a}{\sigma^2}. \quad (13)$$

Condition (12) will hold as soon as  $\sup_{t \in [0, T]} \mathbb{E}[Y_t^{-(4\vee 3p)}] = \sup_{t \in [0, T]} \mathbb{E}[X_t^{-(2\vee \frac{3}{2}p)}] < \infty$ . This is satisfied when  $\sigma^2 < a$  and  $p < \frac{4}{3}\frac{a}{\sigma^2}$ , and we have  $\left(\mathbb{E}\left[\max_{t \in [0, T]} |\hat{Y}_t - Y_t|^p\right]\right)^{1/p} \leq K_p \frac{T}{n}$ .

From now on, we assume that  $\sigma^2 < a$  and consider  $1 \leq p < \frac{4}{3}\frac{a}{\sigma^2}$ . Let  $\varepsilon > 0$  such that  $p(1 + \varepsilon) < \frac{4}{3}\frac{a}{\sigma^2}$ . Since  $\hat{X}_t - X_t = (\hat{Y}_t - Y_t)(\hat{Y}_t + Y_t)$ , we have by Hölder's inequality:

$$\mathbb{E}\left[\max_{t \in [0, T]} |\hat{X}_t - X_t|^p\right]^{\frac{1}{p}} \leq \mathbb{E}\left[\max_{t \in [0, T]} |\hat{Y}_t - Y_t|^{p(1+\varepsilon)}\right]^{\frac{1}{p(1+\varepsilon)}} \mathbb{E}\left[\max_{t \in [0, T]} |\hat{Y}_t + Y_t|^{p\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{p(1+\varepsilon)}}.$$

The moment boundedness of  $\hat{Y}$  is checked in [1] and [2], and the second expectation is thus finite. Proposition 3 gives Theorem 2.

## Application to $dX_t = (a - kX_t)dt + \sigma X_t^\alpha dW_t$ , with $1/2 < \alpha < 1$

We consider this SDE starting from  $X_0 = x > 0$  with parameters  $a > 0$ ,  $k \in \mathbb{R}$  and  $\sigma > 0$ . This SDE is known to have a unique strong positive solution  $X$ , which can be checked easily by Feller's test for explosions. We set

$$Y_t = X_t^{1-\alpha}.$$

It is defined on  $I = \mathbb{R}_+^*$  and satisfies (4) with

$$f(y) = (1 - \alpha) \left( ay^{-\frac{\alpha}{1-\alpha}} - ky - \alpha \frac{\sigma^2}{2} y^{-1} \right) \text{ with } \gamma = \sigma(1 - \alpha).$$

Since  $a > 0$  and  $\frac{\alpha}{1-\alpha} > 1$ ,  $f$  is decreasing on  $(0, \varepsilon)$ , for  $\varepsilon > 0$  small enough. It is also clearly Lipschitz on  $[\varepsilon, +\infty)$ , and (5) is thus satisfied. Also, we check easily that (6) holds. The drift implicit scheme  $(\hat{Y}_t, t \in [0, T])$  given by (7) is thus well defined for large  $n$  and we set:

$$\hat{X}_t = (\hat{Y}_t)^{\frac{1}{1-\alpha}}.$$

To apply Proposition 3, it is enough to check that (12) holds. To do so, we have the following lemma.

**Lemma 4.** *We have:  $\forall q \in \mathbb{R}, \sup_{t \in [0, T]} \mathbb{E}[X_t^q] < \infty$ .*

*Proof.* For  $q \geq 0$ , it is well known that we even have  $\mathbb{E}[\max_{t \in [0, T]} X_t^q] < \infty$  from the sublinear growth of the SDE coefficients (see e.g. Karatzas and Shreve [5], p 306). Let  $q < 0$ . We set  $Z_t = X_t^{2(1-\alpha)}$  and have:

$$dZ_t = b(Z_t)dt + 2(1-\alpha)\sigma\sqrt{Z_t}dW_t, \text{ with } b(z) = 2(1-\alpha) \left[ az^{\frac{1-2\alpha}{2(1-\alpha)}} - kz + \sigma^2 \left( \frac{1}{2} - \alpha \right) \right].$$

Since  $\lim_{z \rightarrow 0^+} b(z) = +\infty$  and  $b$  is Lipschitz on  $[\varepsilon, +\infty)$  for any  $\varepsilon > 0$ , we can find for any  $M > 0$  a constant  $k_M \in \mathbb{R}$  such that  $b(z) \geq M - k_M z$  for all  $z > 0$ . We consider then the following CIR process:

$$d\xi_t^M = (M - k_M \xi_t^M)dt + 2(1-\alpha)\sigma\sqrt{\xi_t^M}dW_t, \quad \xi_0^M = x^{2(1-\alpha)}.$$

From a comparison theorem (Proposition 2.18, p 293 in [5]) we get that  $\forall t \geq 0, Z_t \geq \xi_t^M$  and thus  $\sup_{t \in [0, T]} \mathbb{E}[Z_t^q] \leq \sup_{t \in [0, T]} \mathbb{E}[(\xi_t^M)^q]$ . We conclude by using (13) and taking  $M$  is arbitrary large.  $\square$

We can then apply Proposition 3 and get, for any  $p \geq 1$  and  $n$  large enough, the existence of a constant  $K_p > 0$  such that  $\left( \mathbb{E} \left[ \max_{t \in [0, T]} |\hat{Y}_t - Y_t|^p \right] \right)^{1/p} \leq K_p \frac{T}{n}$ . In particular, we get  $\mathbb{E}[\max_{t \in [0, T]} \hat{Y}_t^p] < \infty$ . We have  $\hat{X}_t = (\hat{Y}_t)^{\frac{1}{1-\alpha}}$  and

$$\hat{y}, y > 0, \quad |\hat{y}^{\frac{1}{1-\alpha}} - y^{\frac{1}{1-\alpha}}| = \frac{1}{1-\alpha} \left| \int_y^{\hat{y}} z^{\frac{\alpha}{1-\alpha}} dz \right| \leq \frac{1}{1-\alpha} |\hat{y} - y| (\hat{y} \vee y)^{\frac{\alpha}{1-\alpha}}.$$

The Cauchy-Schwarz inequality leads then to

$$\mathbb{E} \left[ \max_{t \in [0, T]} |\hat{X}_t - X_t|^p \right]^{\frac{1}{p}} \leq \frac{1}{1-\alpha} \mathbb{E} \left[ \max_{t \in [0, T]} |\hat{Y}_t - Y_t|^{2p} \right]^{\frac{1}{2p}} \mathbb{E} \left[ \max_{t \in [0, T]} (\hat{Y}_t \vee Y_t)^{\frac{2p\alpha}{1-\alpha}} \right]^{\frac{1}{2p}} \leq \tilde{K}_p \frac{T}{n}.$$

## Strong convergence towards $X$ in a general framework

Let us now consider a one-dimensional SDE with Lipschitz coefficients  $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ :

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x.$$

We will consider the Lamperti transformation of this SDE. We assume that there exist  $0 < \underline{\sigma} < \bar{\sigma}$  such that  $\underline{\sigma} \leq \sigma(x) \leq \bar{\sigma}$ , so that

$$\varphi(x) = \int_0^x \frac{1}{\sigma(z)} dz \text{ is bijective on } \mathbb{R},$$

Lipschitz and such that  $\varphi^{-1}$  is Lipschitz. Besides, we assume that  $\sigma \in \mathcal{C}^1$  and that  $f = (\frac{b}{\sigma} - \frac{\sigma'}{2}) \circ \varphi^{-1}$  satisfies (5), (6) and:

$$\exists K > 0, q > 0, \forall y \in \mathbb{R}, \quad |f'(y)| + |f''(y)| \leq C(1 + |y|^q).$$



Then  $Y_t = \varphi(X_t)$  satisfies  $dY_t = f(Y_t)dt + dW_t$ . The Lipschitz assumption on the coefficients  $b$  and  $\sigma$  ensures the boundedness of moments of  $X$  and thus of  $Y$ . The condition (8) is thus satisfied and the conclusion of Proposition 3 holds. Then, defining  $\hat{Y}$  by (7) and setting  $\hat{X}_t = \varphi^{-1}(\hat{Y}_t)$  for  $t \in [0, T]$ , we get that:

$$\exists K_p > 0, \left( \mathbb{E} \left[ \max_{t \in [0, T]} |\hat{X}_t - X_t|^p \right] \right)^{1/p} \leq K_p \frac{T}{n}.$$

Let us mention that the same result holds under suitable conditions on  $f$  for the scheme  $\bar{X}_t = \varphi^{-1}(\bar{Y}_t)$ , where  $\bar{Y}$  denotes the Euler-Maruyama scheme  $d\bar{Y}_t = f(\bar{Y}_{t_{\eta(t)}})dt + dW_t$ . The weak convergence of this scheme has been studied by Detemple, Garcia and Rindisbacher [3].

**Remark 5.** Let  $\gamma > 0$ ,  $\varphi_\gamma(x) = \gamma\varphi(x)$  and  $f_\gamma(y) = \gamma f(y/\gamma)$ . Then,  $Y'_t = \varphi_\gamma(X_t)$  solves  $dY'_t = f_\gamma(Y'_t)dt + \gamma dW_t$ . The associated drift implicit scheme

$$\hat{Y}'_0 = \varphi_\gamma(X_0), \hat{Y}'_t = \hat{Y}'_{t_k} + f_\gamma(\hat{Y}'_t)(t - t_k) + \gamma(W_t - W_{t_k}), \quad t \in (t_k, t_{k+1}], \quad 0 \leq k \leq n-1,$$

clearly satisfies  $\hat{Y}'_t = \gamma \hat{Y}_t$ . Thus,  $\hat{X}_t = \varphi_\gamma^{-1}(\hat{Y}'_t)$ : the scheme  $\hat{X}$  is unchanged when the transformation between  $X$  and  $Y$  is multiplied by a positive constant.

**Acknowledgments.** The author acknowledges the support of the Eurostars E!5144-TFM project and of the “Chaire Risques Financiers” of Fondation du Risque.

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